Quantum Reduced Loop Gravity II

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Plan of the Talk

- The Idea
- Quantization and Symmetry Reduction
- Reduced Kinematical Hilbert Space: Cosmological LQG
- Constraints
- Hamiltonian
- News

Motivation

General Relativity in Ashtekar variables

$$A_a^i(x), E_i^a(x)$$

Symmetry reduction: homogeneity and isotropy

Cosmological models

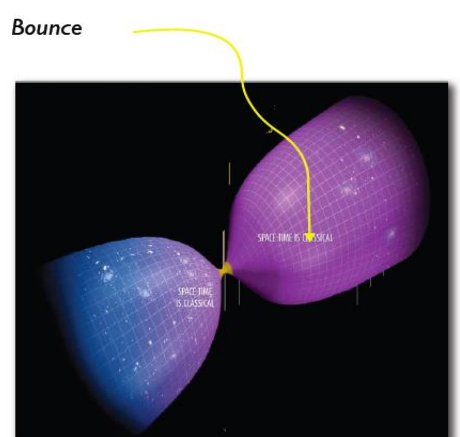
$$c, p$$

"LQG inspired" quantization

LQC

Only homogeneous Cosmologies

Ashtekar, Agullo, Barrau, Bojowald, Campiglia, Corichi, Giesel, Hofmann, Grain, Henderson, Kaminski, Lewandowski, Mena Marugan, Nelson, Pawlowski, Pullin, Singh, Sloan, Taveras, Thiemann,, Winkler, Wilson-Ewing



Motivation

General Relativity in
Ashtekar variables

$$A_a^i(x), E_i^a(x)$$

"LQG"
quantization

LQG

Symmetry reduction:
Homogeneity, isotropy,
inohomogeneities

?

Which cosmology we get from the
full theory?
Beyond homogeneity ?

The Idea

Look at the <u>inhomogeneous</u> line element
in the BKL conjecture Belinski-Khalatnikov-Lifshitz ‘70 :

$$ds^2 = N^2(t)dt^2 - e^{2\alpha(t,x)}(e^{2\beta(t,x)})_{ij}\,\omega^i \otimes \omega^j$$

α Describes the Volume

β (diagonal matrix, Tr β =0)
Describes local anisotropies

ω one forms corresponding to an homogeneous Bianchi model

GOAL:
find a quantum symmetry reduction of LQG compatible with this line element

If we remove the spatial dependence from α and β, we can recover generic Bianchi models

Reduced Ashtekar variables
in the hypotesis of the BKL conjecture

$\boxed{A_a^i = c_i(t,x)\omega_a^i}$ $\boxed{E_i^a = p^i(t,x)\omega\omega_i^a}$

The Poisson Brackets $\{A_a^i(x,t), E_k^b(y,t)\} \propto \delta_k^i \delta_a^b \delta^3(x,y)$ induce

$$\boxed{\{p^i(x,t), c_j(y,t)\} = 8\pi G\gamma\delta_j^i\delta^3(x-y)}$$

We don’t impose homogenity:
The SU(2) Gauss and the Vector constraint don’t vanish identically

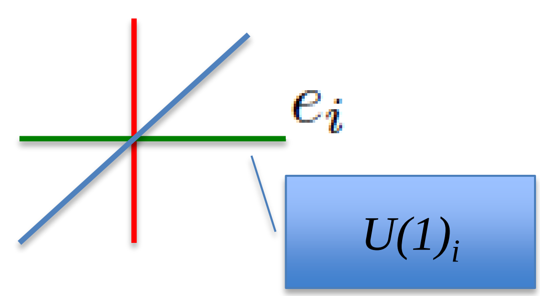
Residual Gauss constraint

$$\boxed{G_i = \omega_i^a \partial_a p^i}$$

$$\{G_i, G_j\} = 0$$

$${}^{\mathrm{red}}h_{e_i} = P(e^{i\int_{e_i} c_i\omega_a^i dx^a})$$

The loop quantization of the reduced theory, based on the reduced holonomy along the e_i paths lead to the product of three independent $U(1)_i$



Cosmological LQG

GOAL:

Implement on the SU(2) Kinematical Hilbert space of LQG the classical reduction:

$$\boxed{A_a^i = c_i(t,x)\omega_a^i} \qquad \boxed{E_i^a = p^i(t,x)\omega\omega_i^a}$$

$$\boxed{\{p^i(x,t), c_j(y,t)\} = 8\pi G\gamma\delta_j^i\delta^3(x-y)}$$

First truncation: we restrict the holonomies to curves along edges e_i parallel to fiducial ω_i^a vectors

The SU(2) classical holonomies associated to the reduced variables are

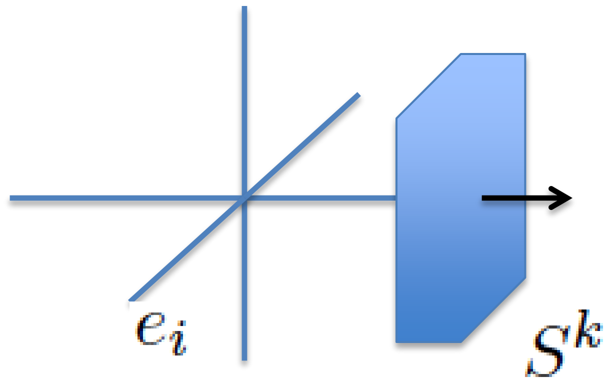
$$^R h_{e_i}^j = P(e^{i\int_{e_i} c^i\omega_a^i dx^a(s)\tau_i})$$

$$^R h_{e_i}^j = \exp\left(i\alpha^i\tau_i\right)$$

NO sum over *i*

Holonomy belong to the U(1) subgroup generated by τ_i

Consider fluxes across surfaces $x^a(u,v)$ with normal vectors parallel to the fiducial ones



The classical reduction implies

$$E_i(S^k) = \int E_i^a \frac{1}{\omega} \omega_a^k du dv = \delta_i^k \int p_i \frac{1}{\omega} du dv$$

For consistency only the diagonal part of the matrix $E_i(S^j)$ is non vanishing

$$\chi_i = \sum_{l,k} \epsilon_{il}{}^{k} E_k(S^l) = 0$$

Second class with the Gauss constraint

How to implement the reduction on the holonomies and consistently impose χ_i=0 ?

Strategy: Mimic the spinfoam procedure

Impose the second class constraint weakly to find a “Physical Hilbert space”

Engle, Pereira, Rovelli,Livine ‘07- ‘08

Imposing a Master constraint strongly on the SU(2) holonomies:

$$\boxed{\chi^2 = \sum_i \chi_i \chi_i = \sum_{i,m,k,l} [\delta^{im}\delta_{kl} E_i(S^k) E_m(S^l) - E_i(S^k) E_k(S^i)]}$$

$$\boxed{\chi^2 h^j_{e_i} = (8\pi\gamma l_P^2)^2 (\tau^2 - \tau_i \tau_i) h^j_{e_i}} = 0$$

Different i for each direction

To solve it is convenient to introduce SU(2) coherent states

SU(2) coherent states

$$|j,\vec{u}>= D^j(\vec{u})|j,j>=\sum_m |j,m> D^j(\vec{u})_{mj}$$

The Master constraint condition acting at the endpoint
(the conjugate condition at the starting point):

$$\chi^2 D^j(g)|j,\vec{u}>= D^j(g)(\tau^2-(\vec{e_i}\cdot\vec{\tau})^2)|j,\vec{u}>= D^j(g)(j(j+1)-(\vec{e_i}\cdot\vec{\tau})^2)|j,\vec{u}>$$

Using the property $\vec{v}\cdot\vec{\tau}|j,\vec{v}>= j|j,\vec{v}>$

If $\vec{e_i}=\vec{u}$ in the large j limit up to L_p corrections the basis element will satisfy:

$$\chi^2 D^j(g)|j,\vec{u}>= 0$$

Reduced basis Elements

$$\boxed{\langle j,\vec{e_i}|D^j(g)|j,\vec{e_i}\rangle}$$

There is a natural way of embedding U(1) cylindrical functions in SU(2) ones:

Projected spinnetworks (Alexandrov, Livine '02)
with the Dupuis-Livine map (Dupuis Livine '10)

$$f : U(1) \to SU(2)$$

$$\tilde{\psi}(g) = \int_{U(1)} dh\, K(g,h)\psi(h), \qquad g \in SU(2)$$

$$K(g,h) = \sum_n \int_{U(1)} dk\, \chi^{j(n)}(gk)\chi^n(kh)$$

SU(2) trace

U(1) trace

These SU(2) functions have the remarkable property that they are completely determined by their restriction to their U(1) subgroup

$$\tilde{\psi}(g)|_{U(1)} = \psi$$

If we consider projected functions defined over the edge e_i choosing the subgroup

$U(1)_i$ as the one generated by τ_i

$$\tilde{\psi}(g)_{e_i} = \sum_{n_i} {}^{i}D^{j(n_i)}_{m=n_i\, r=n_i}(g)\psi^{n_i}_{e_i}$$

$< j, \vec{e_i}|D^j(g)j, \vec{e_i} >$

U(1) quantum number

The Master constraint equation selects the degree of the map: $|n_i| = j(n)$

The strong quadratic condition implies the linear one weakly (restriction to symmetric matrix) !

$$< \tilde{\psi}'_i|E_k(S^l)|\tilde{\psi}_i >= 8\pi\gamma l_P^2 \sum_{j,j'} \psi^{j'}_{e_i} \int dg\, {}^{i}D^{j'}_{j'j'}(g)\tau_k\, {}^{i}D^{j}_{jj}(g)\psi^{j}_{e_i} = 0, \qquad k \neq i$$

The quantum states associated with an edge e_i are entirely determined by their projection into the subspace with maximum magnetic numbers along the internal direction i

$$\psi_{e_i} = \tilde{\psi}(g)_{e_i}|_{U(1)_i} = \sum_j e^{i\theta^i j}\psi^j_{e_i} = \sum_j {}_i< j,j|^R h^j_{e_i}|j,j>_i \psi^j_{e_i}$$

The action of fluxes $E_l(S^k)$ on the reduced space is nonvanishing only for $l = k = i$

$$\boxed{E_i(S^i)\tilde{\psi}_{e_i} = 8\pi\gamma l_P^2 \sum_j j D^j_{jj}\psi^j_{e_i}}$$

This is how we find in the SU(2) quantum theory the classical reduction

$$\boxed{A^i_a = c_i(t,x)\omega^i_a} \qquad \boxed{E^a_i = p^i(t,x)\omega\omega^a_i} \qquad \boxed{\{p^i(x,t), c_j(y,t)\} = 8\pi G\gamma\delta^i_j\delta^3(x-y)}$$

Analogy with Spinfoam Quantization:

SL(2,C) basis elements

SU(2) basis elements

$$\langle g|p,k,j,m,j',m'\rangle = D^{p,k}_{jm,j'm'}(g) \longrightarrow \langle g|j,m,r\rangle = D^{j}_{mr}(g)$$

$$\tilde{\psi}(g) = \sum_{jmn} d_j\, \psi_{jmn}\, D^{p(j),j}_{jm,jn}(g) \longrightarrow \tilde{\psi}(g)_{e_i} = \sum_{n_i} {}^{i}D^{j(n_i)}_{m=n_i\, r=n_i}(g)\psi^{n_i}_{e_i}$$

Linear simplicity constraint

$$\vec{K} + \gamma\vec{L} = 0 \longrightarrow \tau^k h^j_{e_i} = 0 \quad \forall k \neq i$$

Quadratic part of the constraint imposed strongly:

$$(2\gamma C_1 - (\gamma^2 - 1)C_2)|\tilde{\psi}\rangle = 0 \longrightarrow (\tau^2 - \tau_i\tau_i)h^j_{e_i} = 0$$

Select j(n)=n

$$p = \gamma k$$

Only one condition (SU(2)
has only one parameter label for the irrep.)

Weakly satisfied in the large limit

$$\langle\tilde{\psi}|\vec{K} + \gamma\vec{L}|\tilde{\psi}'\rangle = 0$$

Weakly satisfied in the large limit

$$< \tilde{\psi}'_i|E_k(S^l)|\tilde{\psi}_i > = 0$$

Select $\quad k = j$

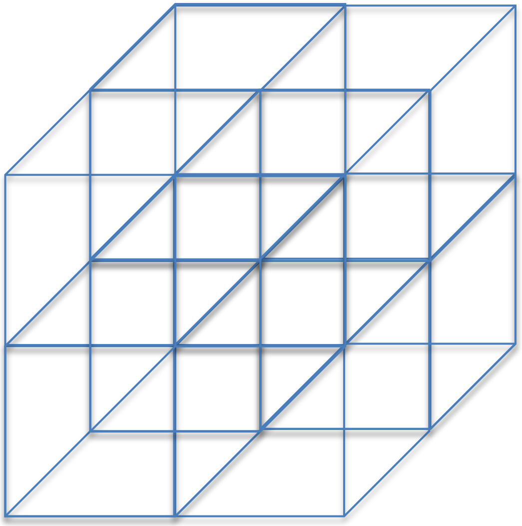
$$g|_K = D^{\gamma j,j}_{jm,jn}(g) = \int_{SU(2)} dh\; K(g,h)\; D^j_{mn}(h)$$

$$g|_K = {}^{i}D^{|j|}_{jj}(g)$$

If we define a Projector P_χ on Physical reduced states:

The projector P_χ acting on ψ_Γ SU(2) cylindrical functions defined on general Graphs Γ :

- Restrict the Graphs to be part of a cubical lattice



- Select the states belonging to the SU(2) subspace where our constraint conditions hold weakly:

$$\tilde{\psi}(g)_{e_i} = \sum_{n_i} {}^i D^{j(n_i)}_{m=n_i\, r=n_i}(g) \psi^{n_i}_{e_i}$$

What is the fate of the GR constraints ?

Gauss Constraint

$$\hat{G}_i(A,E) \quad \Rightarrow \quad P_\chi^\dagger \hat{G}_i P_\chi$$

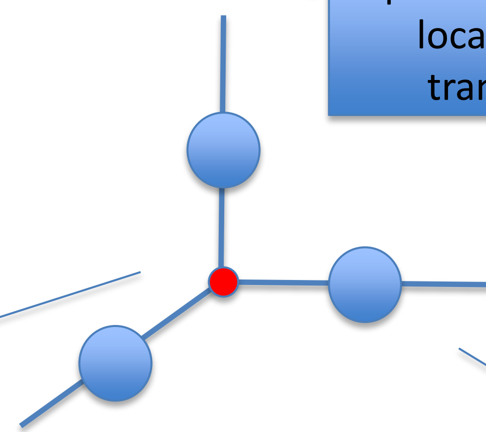
The Gauss constraint of the full theory is implemented by group averaging

$$P_{\mathcal{G}} = \int dg\, U_{\mathcal{G}}(g) \qquad U_{\mathcal{G}}(g) D^j_{mn}(h_e) = D^j_{mn}(g_{s(e)} h_e g_{t(e)})$$

Operator that generates local SU(2) gauge transformations

Spinnetwork states:

$$< h|\Gamma, \{j_e\}, \{x_v\} >= \prod_{v\in\Gamma} \prod_{e\in\Gamma} x_v \cdot D^{j_e}(h_e)_{mn}$$



SU(2) intertwiner

SU(2) holonomy

Implementing $\boxed{P_\chi^\dagger \hat{G}_i P_\chi}$

The reduced states will be of the form :

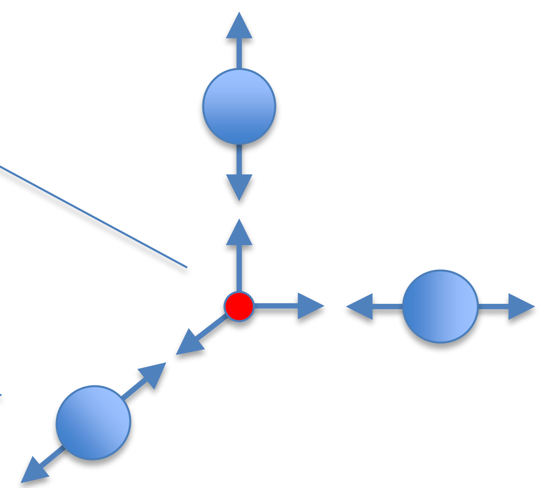
$$< h|\Gamma, j_e, x_v\rangle_R = \prod_{v\in\Gamma}\prod_{e\in\Gamma} < \mathbf{j_i}, \mathbf{x}|\mathbf{j_i}, \vec{\mathbf{u}}_i > \cdot\, {}^{i}D^{j_{e_i}}(h_{e_i})_{j_i j_i}$$

Projection on the intertwiner base of the Livine Speziale Intertwiner: Livine, Speziale '07

$$|\mathbf{j_i}, \vec{\mathbf{u}}_i >= |j_1, \cdots, j_i, \vec{u_1}, \cdots, \vec{u_i} >= \int dg \prod_i |j_i, \vec{u_i} >$$

SU(2) intertwiner
projected on coherent states:
Reduced intertwiner

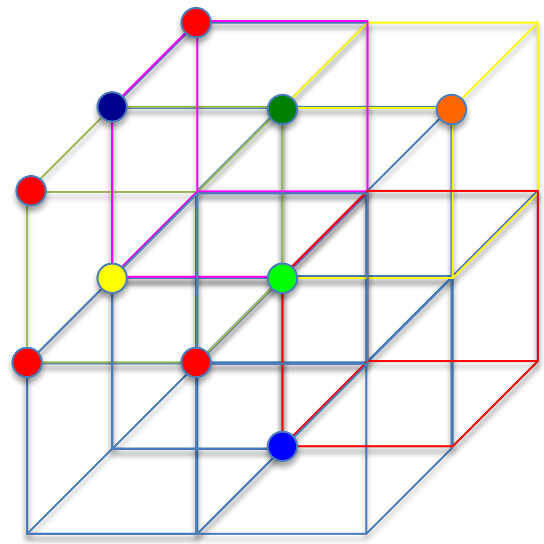
SU(2) holonomy
Projected on coherent states
Reduced holonomy



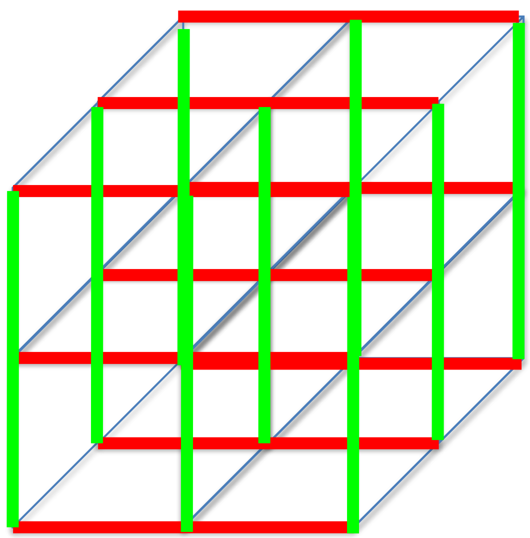
The Inhomogenous sector

Different
Reduced SU(2)
intertwiners:
inhomogeneities

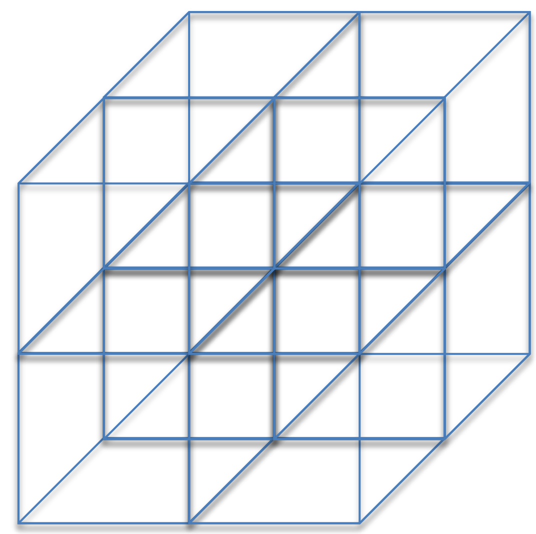
Different
Spin labels:
Anisotropies



Homogeneous and anisotropic sector



Homogeneous and Isotropic sector



Diff Constraint

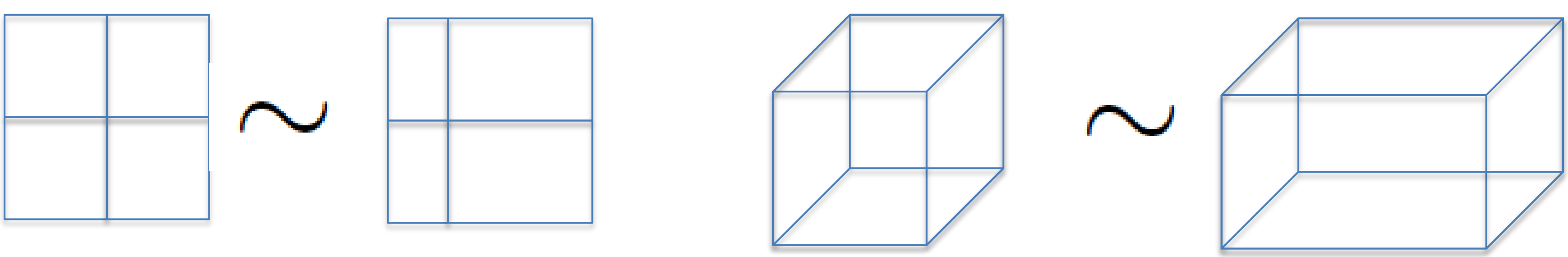
$$\boxed{\hat{V}_a(A,E)} \Rightarrow \boxed{P^{\dagger}_{\chi}\hat{V}_a P_{\chi}}$$

Full Theory: s-knot state

Ashtekar, Lewandowski, Marolf, Mourao, Thiemann

On the reduced space: Reduced s-knot states

Equivalence class of graphs that preserve the cellular structure:



Hamiltonian Constraint

$$\hat{H}(A,E) \quad \longrightarrow \quad P_\chi^\dagger \hat{H} P_\chi$$

The regularized Euclidean constraint in the full theory reads:

T. Thiemann ’96-’98

$$H^m_{\square}\,[N] := \frac{N(\mathfrak{n})}{N_m^2}\, \epsilon^{ijk}\, \mathrm{Tr}\Big[h^{(m)}_{\alpha_{ij}} h^{(m)}_{s_k} \{h^{(m)-1}_{s_k}, V\}\Big]$$

We regularize à la Thiemann, but using only elements of the reduced space:

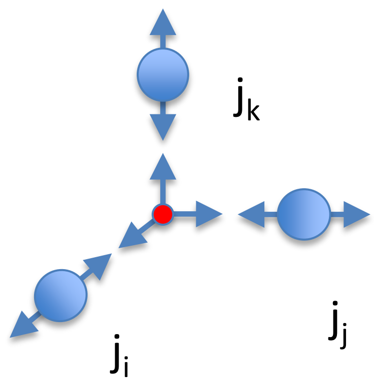
$$^{R}H^m_{\square}\,[N] := \frac{N(\mathfrak{n})}{N_m^2}\, \epsilon^{ijk}\, \mathrm{Tr}\Big[{}^{R}h^{(m)}_{\alpha_{ij}}\, {}^{R}h^{(m)}_{s_k} \{{}^{R}h^{(m)-1}_{s_k}, V\}\Big]$$

Action of the operator
on a tri-valent node:

Antonia's talk

Computed with recoupling theory adapted to the reduced case

$${}^{R}H^{m}_{\square}\ [N]$$



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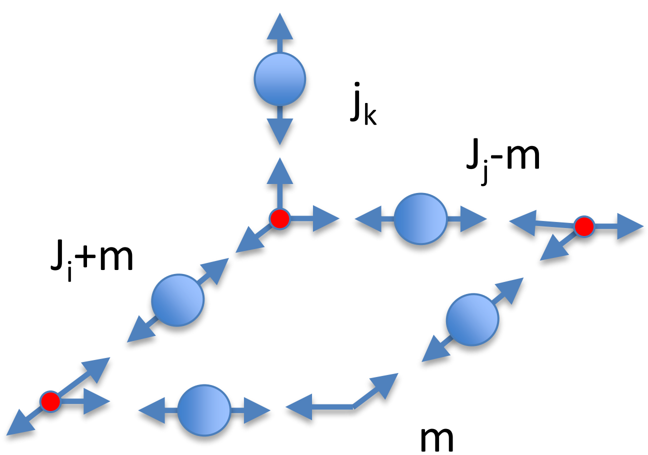
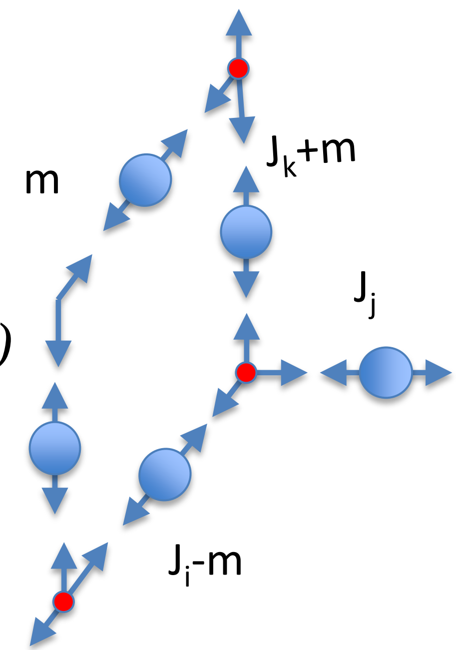
$A_{ij}(j_i, j_j, j_k, m)$



+ $A_{ki}(j_i, j_j, j_k, m)$



+ Permutations

$$\boxed{A_{ij}(j_i, j_j j_k, m)} =$$

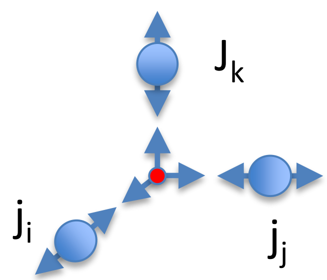
$$\sqrt{j_i j_j j_k + 1}\left[\begin{Bmatrix} j_i+m & j_j & j_k+m \\ j_k & m & j_i \end{Bmatrix}\begin{Bmatrix} j_i+m & j_j-m & j_k \\ m & j_k+m & j_j \end{Bmatrix} - \begin{Bmatrix} j_j+m & j_i & j_k+m \\ j_k & m & j_j \end{Bmatrix}\begin{Bmatrix} j_i+m & j_i-m & j_k \\ j_k+m & m & j_i \end{Bmatrix}\right]$$

Diagonal volume

Remarkably this expression for m=1 and large values simplify to

$$\boxed{A_{ij}(j_i, j_j j_k, 1) = \frac{j_i + j_j}{j_k}\,\frac{(1+\cos\theta_{ij})}{j_k}}$$

$${}^{R}H^{m}_{\square}\;[N]$$



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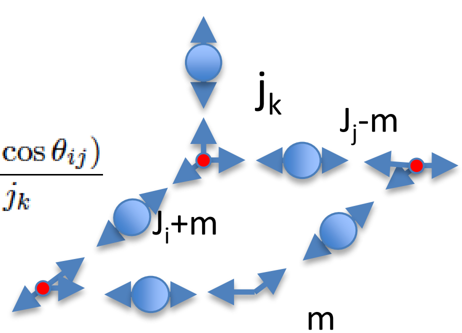
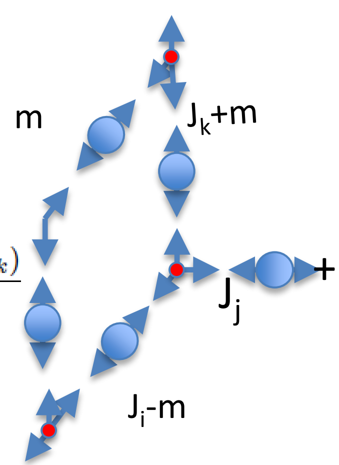
$\frac{j_i + j_j}{j_k}\,\frac{(1+\cos\theta_{ij})}{j_k}$



$+\ \frac{j_i + j_k}{j_j}\,\frac{(1+\cos\theta_{ik})}{j_j}$



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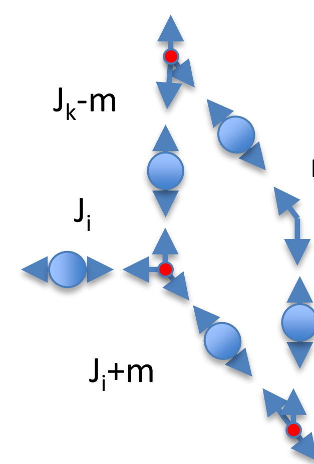
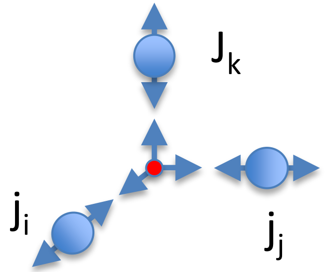


$$^{R}H^{m}_{\square}\,[N]$$



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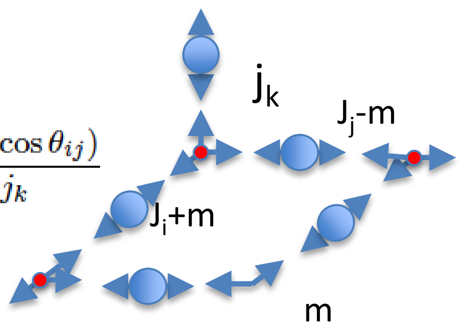
$$\frac{j_i + j_j}{j_k}\;\frac{(1+\cos\theta_{ij})}{j_k}$$



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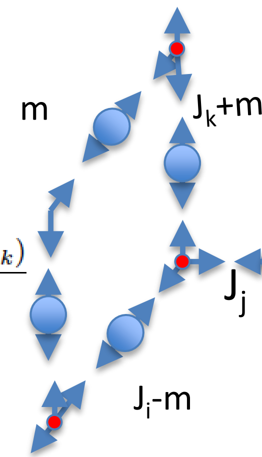
$$\frac{j_i + j_k}{j_j}\;\frac{(1+\cos\theta_{ik})}{j_j}$$



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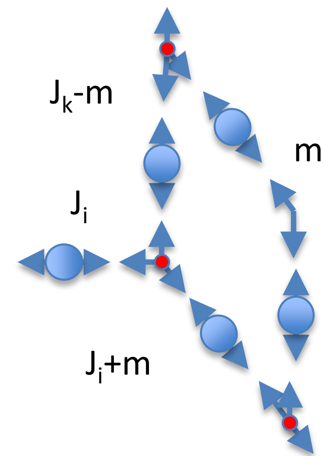
$$\frac{j_j + j_k}{j_i}\;\frac{(1+\cos\theta_{jk})}{j_i}$$



Large j limit “seems”:

$$\boxed{\frac{c_1c_2}{p_3}+\frac{c_1c_3}{p_2}+\frac{c_2c_3}{p_1}=0}$$

News

Semiclassical limit

$$\Psi_{\Gamma,H_l}(h_l) = \int \prod_n dg_n \prod_l K_{\alpha_l}(h_l,\, g_{s(l)}\, H_l\, g_{t(l)}^{-1})$$

Heat Kernel coherent states

$$H_l = \; h_l \; \exp(i \frac{\alpha_l E_l}{8\pi G\hbar\gamma})$$

SL(2,C) element coding classical data

Hall, Thiemann, Winkler, Sahlmann, Bahr

$$\Psi_{H_l}(h_l) = \sum_{j_l, i_n} \psi_{H_l}(j_l, i_n) \Psi_{j_l, i_n}(h_l)$$

intertwiner base

Large distance asymptotic behaviour Bianchi Magliaro Perini

$$\Psi_{H_l}(h_l) \simeq \sum_{j_l, i_n} \prod_l e^{-\frac{(j_l - j_l^0)^2}{2\sigma_l^2}} e^{-i\xi_l j_l} (\prod_n \Phi_{i_n}) \Psi_{j_l, i_n}(h_l)$$

Codes the intrinsic geometry

Codes the extrinsic curvature

Livine-Speziale Intertwiners

$$j_0 = \frac{|E|}{8\pi G\hbar\gamma}$$

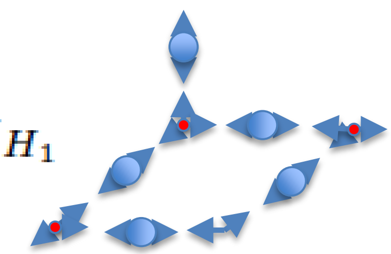
$$\xi \sim K = c$$

Project in our reduced space the coherent states

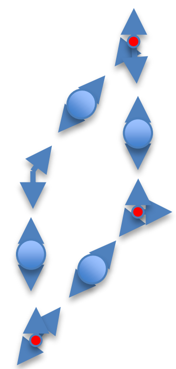
$$P_\chi |\Psi_{H_l}\rangle = |\Psi_{H_l}\rangle_R$$

Single cell:

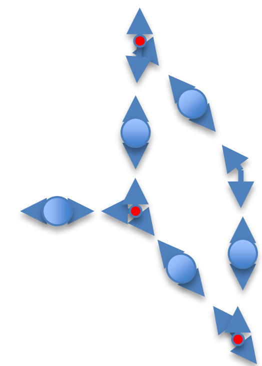
$|\Psi_{H,\square}\rangle_R = {}^R\Psi_{H_1}$



$+ \; {}^R\Psi_{H_2}$



$+ \; {}^R\Psi_{H_3}$



Expectation value of the Hamiltonian on coherent states for a single cell:

$${}_R\langle\Psi_{H,\square}|\,{}^R\hat{H}^m_\square\,|\Psi_{H,\square}\rangle_R =$$

$$\sqrt{\frac{p^1p^2}{p^3}}c_1c_2 + \sqrt{\frac{p^2p^3}{p^1}}c_2c_3 + \sqrt{\frac{p^3p^1}{p^2}}c_3c_1$$

Classical Bianchi I Hamiltonian

Perspectives

This analysis opens the way to

- Study the Physical solutions on the Dual Diff invariant Space and eventually construct a Physical Scalar Product

- Add matter as a clock: Big Bounce ? QFT on quantum spacetime ?

- Link to LQC ?

- Spinfoam Cosmology? *Bianchi, Krajewski, Rovelli, Vidotto*

- Something Different ?

(In the homogeneous anisotropic case the scale factors are not independent (Clebsh conditions) and even less dynamically) Isotropization mechanism?

- Arena for the canonical theory:

  AQG, Master constraint, deparametrized theories.. Computable!