Deparametrized gravity with a scalar field & Beyond:

A new quantum Hamiltonian operator for LQG

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Plan of the talk

1. Gravity coupled to a scalar field
   - Overview of the classical framework
2. Quantum theory
   - Kinematic Hilbert space
   - The vertex Hilbert space $\mathscr{H}_{\text{vtx}}$
   - Implementation of the Hamiltonian operator
     - The Euclidean operator
     - The curvature operator
     - Symmetric Hamiltonian operator
3. Vacuum theory
4. Outlooks

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The theory of 3+1 gravity (Lorentzian) minimally coupled to a free massless scalar field[1] $\phi(x)$ is described by the action

$$S = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \ \left[2\dot{A}^i_a \, E^a_i + \pi\dot{\phi} - \left(\Lambda^j \, G_j + N^a \, C_a + N \, C\right)\right]$$

where

$$G_j = D_a \, E^a_i$$

$$C_a = \underbrace{F^i_{ab} E^b_i}_{C^{gr}_a} + \pi \, \phi_{,a}$$

$$C = \underbrace{-\left[\frac{\epsilon^{ij}{}_k F^k_{ab} E^a_i E^b_j}{\sqrt{\det(E^a_i)}} + \left(1 + \frac{1}{\beta^2}\right)\sqrt{\det(E^a_i)} R\right]}_{C^{\rm gr}} + \frac{1}{2\sqrt{\det(E^a_i)}}\left(\pi^2 + E^a_l E^b_l \phi_{,a}\phi_{,b}\right)$$

[1] [K.V. Kuchar 93'], [L. Smolin 89'], [C. Rovelli, L. Smolin 93'], [K.V. Kuchar, J.D. Romano 95'], [M. Domagala, K. Giesel, W. Kaminski, J. Lewandowski 10'], [M. Domagala, M. Dziendzikowski, J. Lewandowski 12']

Overview

Assuming that

$$C_a = 0 \qquad C \;= 0$$

The Hamiltonian constraint is solved for $\pi(x)$ using the diff. constraint

$$\pi = \pm\sqrt{-\sqrt{\det(E^a_i)}C_{\text{gr}} \pm \sqrt{\det(E^a_i)}\sqrt{C^2_{\text{gr}} - E^a_l E^b_l C^{\text{gr}}_a C^{\text{gr}}_b}}$$

$(\pm, \pm)$ select different regions of the phase space.

Overview

$\longrightarrow$ ϕ becomes the emergent time.

In the region $(+, +)$, an equivalent model could be obtained by keeping the Gauss and Diff. constraints and reformulating the scalar constraints

$$C' := \pi - h$$

where

$$h := \sqrt{-\sqrt{\det(E_i^a)}C_{\text{gr}} + \sqrt{\det(E_i^a)}\sqrt{C_{\text{gr}}^2 - E_l^a E_l^b C_a^{\text{gr}} C_b^{\text{gr}}}}$$

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ϕ no longer occurs in the function h $\Leftrightarrow$ the scalar constraints deparametrized

The scalar constraints strongly commute

$$\{C'(x), C'(y)\} = 0$$

As a consequence

$$\{h(x), h(y)\} = 0$$

[2] [M. Domagala, K. Giesel, W. Kaminski, J. Lewandowski 10'], [M. Domagala, M. Dziendzikowski, J. Lewandowski 12']

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For a Dirac observable[2] O

$$\{O, G^i_a\} = \{O, C_a\} = \underbrace{\{O, C\}}_{\frac{\partial O}{\partial \phi} = \{O,\pi\} = \{O,h\}} = 0$$

[2] [M. Domagala, K. Giesel, W. Kaminski, J. Lewandowski 10'], [M. Domagala, M. Dziendzikowski, J. Lewandowski 12']

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- The physical Hilbert space $\mathcal{H}$ is the space of the quantum states of the *matter free* gravity which satisfy the quantum vector constraint and the quantum Gauss constraint.

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- The quantum Hamiltonian

$$\hat{H} = \int d^3x \sqrt{-2\widehat{\sqrt{q(x)}C^{\rm gr}(x)}}$$

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- Given a quantum observable, the evolution is generated by

$$\frac{d}{dt}\hat{O}_t = -i\left[\hat{H}, \hat{O}_t\right]$$

Kinematic Hilbert space

The kinematic Hilbert space of vacuum gravity $\mathscr{H}_{\text{kin}}$ is the space of cylindrical functions.

The gauge invariant subspace

$$\mathscr{H}_{\text{kin}}^{G} = \overline{\mathscr{D}_{\text{kin}}^{G}}$$

It is the space of solutions of the Gauss constraint obtained by group averaging with respect to the YM gauge transformations.

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The vertex Hilbert space $\mathscr{H}_{\rm vtx}$

To construct the Hilbert space of gauge and Diffeomorphism invariant states, an averaging procedure is performed w.r. to the group

$$\mathrm{Diff}/\mathrm{TDiff}_\Gamma \quad \to \quad \mathscr{H}^G_{\Gamma,\mathrm{Diff}} \subset \mathrm{Cyl}^*$$

[3] [M. Domagala, M. Dziendzikowski, J. Lewandowski 12'],[J. Lewandowski, H. Sahlmann 14']

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However, another idea is to introduce a Hilbert space that contains solutions of a partial vector constraint[3], by averaging each of the sub-spaces $\mathscr{D}^G_\Gamma$ with respect to the diffeomorphisms $\text{Diff}(\Sigma)_{\text{vtx}}$ which act trivially in the set of the vertices of Γ. This new Hilbert space admits the operator $\hat{C}$, and in general the operator $\hat{C}(N)$, and it is preserved by those operators.

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This averaging is achieved through the "rigging" maps η_Γ

$$\mathscr{D}^G_\Gamma \ni \psi \mapsto \frac{1}{n_\Gamma} \sum_{[\varphi]\in\text{Diff}/\text{TDiff}_\Gamma} \langle U(\varphi)\psi| =: \eta_\Gamma(\psi)$$

Combined we get the map η_{vtx}

$$\eta_{\text{vtx}} \;:\; \mathscr{D}^G_{\text{kin}} \mapsto \left(\mathscr{D}^G_{\text{kin}}\right)^*$$

Then the space of the Gauss and partially diff. invariant states is defined as

$$\mathscr{H}_{\text{vtx}} = \overline{\eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right)} \subset \left(\mathscr{D}^G_{\text{kin}}\right)^* \subset \text{Cyl}^*$$

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Plan of the talk

We want to implement a quantum operator corresponding to the classical quantity

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Implementation of the Hamiltonian operator

We want to implement a quantum operator corresponding to the classical quantity

$$\int d^3x \sqrt{-2\sqrt{\det(E^a_i)}C_{gr}(x)} = \lim_{\epsilon\to 0} \sum_{\Delta\in\mathscr{C}} \sqrt{-2\sqrt{\det(E^a_i)_\Delta}C^{(\epsilon')}_{gr,\Delta}(h,E)}$$

Δ - cell of the cellular decomposition $\mathscr{C}$.

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Δ - cell of the cellular decomposition $\mathscr{C}$.

Explicitly

$$\begin{aligned} C^{(\epsilon')}_\Delta &:= -2\sqrt{\det(E^a_i)_\Delta}C^{(\epsilon')}_{gr,\Delta}(h,E) \\ &= 2\epsilon^{ij}{}_k Tr\left[\frac{2\cdot 3}{W(l)^2} h^{(l)}_{\alpha^{(\epsilon')}_{IJ}}(\Delta)\tau^{(l)k}\right]\eta^{ab}E_i(S^I_a\subset\Delta)E_j(S^J_b\subset\Delta) \\ &+ 2\left(1+\frac{1}{\beta^2}\right)V(\Delta)\int_\Delta \sqrt{\det(E^a_i)}R\, d^3x \\ &=: C^{E\ (\epsilon')}_\Delta + \left(1+\frac{1}{\beta^2}\right)C^L_\Delta \end{aligned}$$

ϵ' - regulator that is the coordinates size of the loop $\alpha^{(\epsilon')}_{IJ}$.

Euclidean part:

$$\epsilon^{ij}{}_{k} Tr\left[\tfrac{2\cdot 3}{W(l)^2} h^{(l)}_{\alpha^{(\epsilon')}_{IJ}} \tau^{(l)k}\right] \eta^{ab} E_i(S^I_a) E_j(S^J_b) \longrightarrow \epsilon^{ij}{}_{k} Tr\left[\tfrac{2\cdot 3}{W(l)^2} \widehat{h^{(l)}_{\alpha^{(\epsilon')}_{IJ}} \tau^{(l)k}}\right] \eta^{ab} E_i(S^I_a) E_j(S^J_b)$$

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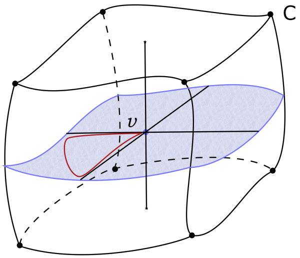
$h_{\alpha^{(\epsilon')}_{IJ}}$:

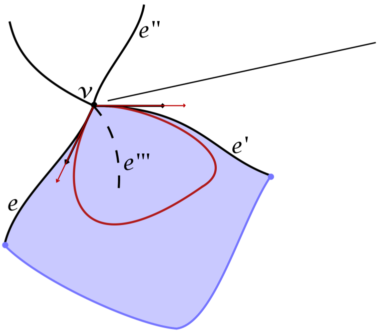
Consider a pair of edges $\{e_1, e_2\}$ incident at node v with a loop α associated to it and a set of local coordinates (x^a) such that

$$x^a(e_1(t_1)) = (t_1, 0, 0)$$
$$x^a(e_2(t_2)) = (0, t_2, 0)$$
$$x^a(e_i(t_i)) = f^a_i(t_i);\ \ldots$$
$$x^a(\alpha(t)) = \alpha^a(t)$$

The loop α is lying in the surface spanned by e_1 and e_2.

$$(x'^a): \qquad \varphi: \ x'^a \longrightarrow x^a$$
$$\alpha'(t) \longrightarrow \alpha(t)$$





The loop is tangent to the two edges e and e' at the node up to the $k+1$ and $k'+1$ orders respectively, where k and k' are respectively the tangentiality orders of e and e' with respect to the rest of edges at the node.

This prescription makes a loop assigned to a given pair of edges perfectly distinguishable from any other loop at the same node. It also allows to distinguish a loop, the outer loop, from the inner loop associated to it when those loops carry the same representation label.

A class of Diff. equivalent surfaces[4] $\longrightarrow$ Diff. invariant prescription

[4] [Contribution of J. Lewandowski in T. Thiemann "Quantum Spin Dynamics (QSD)" 96']

The resulting operator:

$$\lim_{\epsilon'\to 0}\sum_{\Delta} 2\epsilon^{ij}{}_k Tr\left[\frac{2\cdot 3}{W(l)^2} h^{(l)}_{\alpha^{(\epsilon')}_{IJ}}(\Delta)\tau^{(l)k}\right]\eta^{ab}E_i(S^I_a\subset\Delta)E_j(S^J_b\subset\Delta)$$

$$:= \sum_{\{v,e_I,e_J\}\in\Gamma}\frac{6}{W(l)^2}\kappa_1(v)\,\epsilon\left(\dot{e}_I,\dot{e}_J\right)\epsilon^{ij}{}_k\, Tr\left[\hat{h}^{(l)}_{\alpha_{IJ}}\,\tau^{(l)\,k}\right]\hat{J}_{i,v,e_I}\hat{J}_{j,v,e_J}$$

$$= \sum_{\{v,e_I,e_J\}\in\Gamma}\widehat{C^E}_{v,e_I,e_J}$$

$$=: \widehat{C^E}$$

$\kappa_1(v)$ - Coefficient resulting from the averaging over relevant background structure.

$\epsilon\left(\dot{e}_I,\dot{e}_J\right)$ - 0 , 1 .

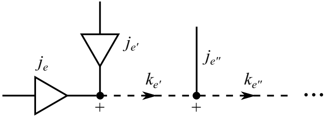
$$\widehat{C^E}\ :\ \eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right)\to\eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right)\subset\mathscr{H}_{\text{vtx}}$$

The Euclidean operator

The action of the Euclidean operator on a node of a spin network

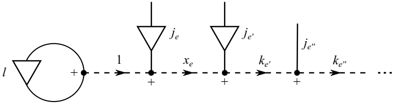
The action of the Euclidean operator on a node of a spin network

$$\widehat{C^E}_{v,e,e'}\,\langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots| = \widehat{C^E}_{v,e,e'}$$



$$= \; 3\sqrt{6}(-)^{j_e - j_{e'} - k_{e'} + 1}\, \frac{W_{j_e} W_{j_{e'}}}{W_l}\, \kappa_1(v)\, \epsilon\left(\dot{e}, \dot{e}'\right) \times$$

$$\times \sum_{x_e} d_{x_e} \begin{Bmatrix} j_{e'} & j_{e'} & 1 \\ j_e & x_e & k_{e'} \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ j_e & x_e & j_e \end{Bmatrix}$$



The curvature operator

Lorentzian part

$$C^L_\Delta = V(\Delta)\left(\int_\Delta \sqrt{\det(E^a_i)}R\, d^3x\right) \to \hat{V}(\Delta)\left(\int_\Delta \widehat{\sqrt{\det(E^a_i)}R\, d^3x}\right)$$

$\hat{V}(\Delta)$ - Volume operator[5]

$\widehat{\left(\int_\Delta \sqrt{\det(E^a_i)}R\, d^3x\right)}$ - Curvature operator[6]

[5] [C. Rovelli, L. Smolin 95'], [A. Ashtekar, J. Lewandowski 95']
[6] [E. Alesci, M. A., J. Lewandowski. Phys. Rev. D 124017 (2014)]

Regularization of the curvature term:

$$\left(\int_{\Delta} \sqrt{\det(E_i^a)} R \, d^3x \right) = \sum_{h \in \Delta} L_h^{\Delta} \left(\frac{2\pi}{\alpha_h} - \theta_h^{\Delta} \right)$$

Regularization of the curvature term:

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Length operator

$$L_\Delta = \sqrt{\delta_{ij} G_\Delta^i G_\Delta^j} \quad ; \quad G_\Delta^i = \frac{\frac{\Delta s}{2} \sum_{\alpha,\beta} \frac{1}{\Delta s^4} Y^i_{\Delta\alpha\beta}}{\frac{1}{\Delta s^3} V(\Delta)} = \frac{\frac{\Delta s}{2} \sum_{\alpha,\beta} \frac{1}{\Delta s^4} T^{ijk}_{x\Delta} E_j(S^1_{\Delta\alpha}) E_k(S^2_{\Delta\beta})}{\frac{1}{\Delta s^3} V(\Delta)}$$

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Angle operator

$$\theta_{\Delta\alpha\beta}^{12} = \pi - \arccos \left[\frac{\delta'^{ik} E_i(S_{\Delta\alpha}^1) E_k(S_{\Delta\beta}^2)}{\sqrt{\delta'^{ij} E_i(S_{\Delta\alpha}^1) E_j(S_{\Delta\alpha}^1)} \sqrt{\delta'^{kl} E_k(S_{\Delta\beta}^2) E_l(S_{\Delta\beta}^2)}} \right]$$

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Regularized expression:

$$\int_\Delta \sqrt{\det(E_i^a)} R\, d^3x = \frac{\sqrt{\sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} \delta_{ij} Y^i_{\Delta,\alpha_1\beta_1}(E)\, Y^j_{\Delta,\alpha_2\beta_2}(E)}}{V_\Delta} \left(\frac{2\pi}{\alpha_h} - \sum_{\alpha_3,\beta_3} \theta_{\Delta,\alpha_3\beta_3}(E)\right)$$

The curvature operator

The final quantum operator corresponding to the Lorentzian part:

$$\widehat{C^L} \;:\; \eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) \;\to\; \eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) \subset \mathscr{H}_{\text{vtx}}$$

$$\widehat{C^L} := \sum_{v,e,e'} \frac{\kappa_2(v)}{2}\left(\hat{\mathbb{I}}_v - \hat{P}_{\hat{V}_v=0}\right)\sqrt{\delta^{ij}(\epsilon_{ikl}\hat{J}^k_{v,e}\hat{J}^l_{v,e'})(\epsilon_{jk'l'}\hat{J}^{k'}_{v,e}\hat{J}^{l'}_{v,e'})}\left(\frac{2\pi}{\alpha_{v,e,e'}}\hat{\mathbb{I}}_v - \hat{\Theta}_{v,e,e'}\right)$$

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$$\begin{aligned}
\widehat{C^L}_{v,e,e'} \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots| &= \frac{\kappa_2(v)}{2} \left(\hat{\mathbb{I}}_v - \hat{P}_{\hat{V}_v=0}\right) \sqrt{\delta^{ij}(\epsilon_{ikl}\hat{J}^k_{v,e}\hat{J}^l_{v,e'})(\epsilon_{jk'l'}\hat{J}^{k'}_{v,e}\hat{J}^{l'}_{v,e'})} \times \\
&\times \left(\frac{2\pi}{\alpha_{v,e,e'}}\hat{\mathbb{I}}_v - \hat{\Theta}_{v,e,e'}\right) \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots| \\
&= \frac{\kappa_2(v)}{2} \sqrt{c(j_e, j_{e'}, k_{e'})} \left(\frac{2\pi}{\alpha_{v,e,e'}} - \theta(j_e, j_{e'}, k_{e'})\right) \times \\
&\times \left(\hat{\mathbb{I}}_v - \hat{P}_{\hat{V}_v=0}\right) \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots|
\end{aligned}$$

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$$\begin{aligned}\widehat{C^L}_{v,e,e'} \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots| &= \frac{\kappa_2(v)}{2} \left(\hat{\mathbb{I}}_v - \hat{P}_{\hat{V}_v=0}\right) \sqrt{\delta^{ij}(\epsilon_{ikl}\hat{J}^k_{v,e}\hat{J}^l_{v,e'})(\epsilon_{jk'l'}\hat{J}^{k'}_{v,e}\hat{J}^{l'}_{v,e'})} \times \\ &\times \left(\frac{2\pi}{\alpha_{v,e,e'}}\hat{\mathbb{I}}_v - \hat{\Theta}_{v,e,e'}\right) \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots| \\ &= \frac{\kappa_2(v)}{2} \sqrt{c(j_e, j_{e'}, k_{e'})} \left(\frac{2\pi}{\alpha_{v,e,e'}} - \theta(j_e, j_{e'}, k_{e'})\right) \times \\ &\times \left(\hat{\mathbb{I}}_v - \hat{P}_{\hat{V}_v=0}\right) \langle v; j_e, j_{e'}, \ldots; k_{e'}, \ldots|\end{aligned}$$

where

$$\begin{aligned}c(j_e, j_{e'}, k_{e'}) &:= j_e(j_e+1)j_{e'}(j_{e'}+1) - \left(\frac{k_{e'}(k_{e'}+1) - j_e(j_e+1) - j_{e'}(j_{e'}+1)}{2}\right)^2 \\ &- \frac{k_{e'}(k_{e'}+1) - j_e(j_e+1) - j_{e'}(j_{e'}+1)}{2},\end{aligned}$$

$$\theta(j_e, j_{e'}, k_{e'}) := \ \pi - \arccos\left[\frac{k_{e'}(k_{e'}+1) - j_e(j_e+1) - j_{e'}(j_{e'}+1)}{2\sqrt{j_e(j_e+1)j_{e'}(j_{e'}+1)}}\right].$$

Symmetric Hamiltonian operator

At this level we defined two operators $\widehat{C^E}$ and $\widehat{C^L}$ on the space $\eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) \subset \mathscr{H}_{\text{vtx}}$.
To construct a symmetric Hamiltonian operator, we choose to use the adjoint operator $\hat{C}^\dagger$

$$\hat{C}^\dagger \,:\, \mathscr{D}\left[\hat{C}^\dagger\right] \subset \mathscr{H}_{\text{vtx}} \longrightarrow \mathscr{H}_{\text{vtx}}$$

such that

$$\forall\, \psi,\ \psi' \in \eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) : \qquad \left\langle \Psi' \right| \hat{C}^\dagger \left|\Psi\right\rangle := \overline{\left\langle \Psi \right| \hat{C} \left|\Psi'\right\rangle},$$

Notice that $\eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) \subset \mathscr{D}\left[\hat{C}^\dagger\right]$.

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Notice that $\eta_{\text{vtx}}\left(\mathscr{D}^G_{\text{kin}}\right) \subset \mathscr{D}\left[\hat{C}^\dagger\right]$.

Now we can introduce different expression to built a symmetric operator

$$\hat{C}^{\text{Sym}} := \text{Sym}(\widehat{C^E}, \widehat{C^{E\dagger}}, \widehat{C^L}, \widehat{C^{L\dagger}})$$

Typical examples:

- $\frac{1}{2}(\hat{C} + \hat{C}^\dagger)$ — Self-adjoint extensions?
- $\sqrt{\hat{C}\hat{C}^\dagger}$
- $\sqrt{\hat{C}^\dagger\hat{C}}$
- $\frac{1}{2}\left(\sqrt{\hat{C}^E\hat{C}^{E\dagger}} \pm \sqrt{\hat{C}^L\hat{C}^{L\dagger}}\right)$
- ...

(the last four items:) admit self-adjoint extensions

We can finally define the physical Hamiltonian for this deparametrized model

$$\hat{H} := \sqrt{\hat{C}^{\mathrm{Sym}}} = \sum_{v \in \Gamma} \sqrt{\hat{C}_v^{\mathrm{Sym}}}$$

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$\hat{H}$ has a domain that contains $\eta_{\text{vtx}}\left(\mathscr{D}_{\text{kin}}^{G}\right)$ & has the following properties:

- Gauge invariant & diffeomorphism invariant;
- Cylindrically consistent;
- Locality problem can be solved & Self-adjoint extensions exist (both depending on the choice of symmetrization);
- Identification of orthogonal stable subspaces;

Plan of the talk

Hamiltonian constraint operator

$$C^{\text{gr}}(x) = \left[\frac{\epsilon^{ij}{}_k F^k_{ab} E^a_i E^b_j}{\sqrt{\det(E^a_i)}} + \left(1 + \frac{1}{\beta^2}\right) \sqrt{\det(E^a_i)} R \right] = 0.$$

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- Construct $\mathscr{H}^G_{\rm kin} = \overline{\mathscr{D}^G_{\rm kin}}$;
- Construct $\mathscr{H}_{\rm vtx}$ by averaging w.r.t. $\text{Diff}(\Sigma)_{\rm vtx}$;
- Implement the Hamiltonian constraint:

$$C(N) \;=\; \int d^3x\, N(x)\, C^{\rm gr}(x) = 0, \qquad \forall\, N$$

by adopting the regularization shown above, define the operator:

$$\hat{C}^{\rm Sym}(N) := \text{Sym}(\widehat{C^E}(N), \widehat{C^{E\dagger}(N)}, \widehat{C^L}(N), \widehat{C^{L\dagger}}(N))$$

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  on $\mathscr{H}_{\rm vtx}$;
- Solve $\hat{C}^{\rm Sym}(N) = 0$ in $\mathscr{H}_{\rm vtx}$;
- Average w.r.t. to the remaining diffeomorphisms to obtain the physical space.

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$$C \ = \int d^3x \, \frac{C^{\text{gr}}(x)^2}{\sqrt{q(x)}} = 0$$

following the same regularization procedure as for $C(N)$, then define the operator:

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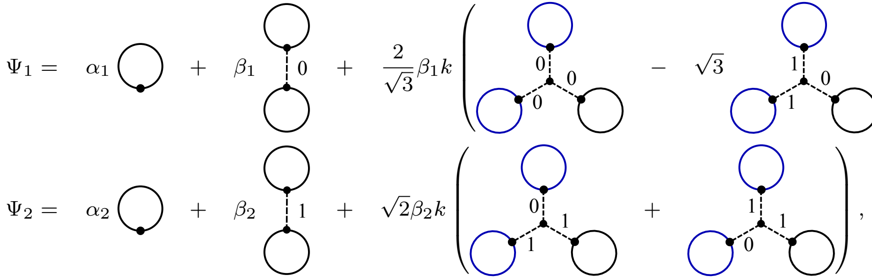
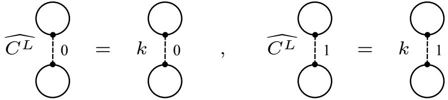


(Blue loops are tangent loops.)

where $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in \mathbb{C}$, and



$\Psi_1, \Psi_2 \in \mathscr{H}_{\text{phys}}$

Plan of the talk

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- Computing commutators of $\hat{C}(N)$'s & $\hat{C}^\dagger(M)$'s (algebra anomalies?);
- Construct (just enough!) Kernel elements & test coherent states approach;
- Investigate self-adjointness for other symmetrizations of $\hat{C}^{\text{Sym}}(N)$;
- Calculate states (approximate/truncated) evolution in presence of a scalar field.

Thank you!