A quantum reduction to Bianchi I models in LQG

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Plan of the talk

Outline

Proposals for a symmetry reduced quantum theory in LQG

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- Mini / midi-superspace quantisation
  - LQC [Bojowald '99-; Ashtekar, Bojowald, Lewandowski '03; ...]
  - Schwarzschild black hole [Kastrup, Thiemann '93; Kuchař '94, Gambini, Pullin '13]
  - Spherical symmetry [Bojowald, Kastrup '99, ..., Bojowald, Swiderski '04, ...]
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- Approximately symmetric spin networks
  - Weave states [Ashtekar, Rovelli, Smolin '92; Bombelli '00]
  - Spinfoam cosmology [Bianchi, Rovelli, Vidotto '10-; Kisielowski, Lewandowski, Puchta '12]
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- Code symmetry as $f(p, q) = 0$, impose $\widehat{f(p, q)} \, |\Psi\rangle_{\text{sym}} = 0$ at quantum level
  - Bianchi I models [NB '14] $\leftarrow$ this talk
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Outline

General strategy for the symmetry reduction

1. **Suitable classical starting point**
   - Gauge fix spatial diffeomorphisms adapted to the symmetry reduction
   - Go to the reduced phase space, i.e. solve constraints or employ Dirac bracket
   - Find new connection variables on Γ_{red} (not Ashtekar-Barbero variables)

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   - Find phase space functions $f_i(p, q) = 0$ in the symmetric subspace
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4. **Impose reduction conditions** $f_i = 0$ as operator equations: $\hat{f}_i \left|\Psi\right\rangle_{\text{sym}} = 0$
   - Find subspace of quantum reduced states $\left|\Psi\right\rangle_{\text{sym}}$
   - Find observables $\hat{\mathcal{O}}_{\text{sym}}$ w.r.t. reduction constraints: $[\hat{\mathcal{O}}_{\text{sym}}, \hat{f}_i] = 0$

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5. **Relate observables** $\hat{\mathcal{O}}_{\text{sym}}$ to the parameters of a $\mathbb{T}^3$ Bianchi I model
   - Can be related naturally to Bianchi I LQC?
   - Can support improved LQC dynamics?

Outline

Classical preparations I: Phase space

Gauge fixing to obtain suitable coordinates

1. Start with ADM phase space $\{q_{ab}(\sigma), P^{cd}(\sigma')\} = \delta^{(3)}(\sigma, \sigma')\delta^c_{(a}\delta^d_{b)}$
2. Impose **diagonal metric** gauge $q_{a \neq b} = 0 \quad \Leftrightarrow \quad q = \text{diag}(q_{xx}, q_{yy}, q_{zz})$

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4. Coordinatise the reduced phase space via $q_{xx}, q_{yy}, q_{zz}, P^{xx}, P^{yy}, P^{zz}$
5. Solve $C_a = 0$ for $P^{a\neq b} \quad \Rightarrow \quad P^{a\neq b}(q_{aa}, P^{bb}) \quad \Rightarrow \quad$ insert in Hamiltonian

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Choose connection type variables

1. Define $e_a e_a = q_{aa}$, $e_a e^a = 1$, without summation, and $E^a = \sqrt{\det q}\, e^a$
2. Define $K_a = K_{ab} e^b$ with K_{ab} being the extrinsic curvature constructed form P^{ab}
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At this stage, only Hamiltonian constraint and reduced spatial diffeomorphisms left.

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1. All spatial diffeomorphisms: $\tilde{C}_a[N^a] = \int_\Sigma d^3\sigma\, E^a \mathcal{L}_{\vec{N}} K_a = 0$ (incorporates also reduced ones)
2. Abelian Gauß law: $G[\omega] = \int_\Sigma d^3\sigma\, \omega\, \partial_a E^a = 0$

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Result:

Direct consequences of a Bianchi I reduction can be imposed as spatial diffeomorphisms and a Gauß law on the (quantised) reduced phase space (as operator equations).

Classical preparations III: Summary

Phase space: (full GR admitting diagonal metric gauge)

1. $K_a(\sigma)$, $E^b(\sigma)$ are $3+3$ canonical variables per spatial point σ
2. Remaining constraints are
   1. reduced spatial diffeomorphisms (preserving the diagonal gauge)
   2. Hamiltonian constraint

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Strategy:

1. Quantise full phase space via LQG techniques
2. Impose symmetry reduction by imposing $\tilde{C}_a = 0 = G$ at the quantum level

Outline

Quantisation I: Full theory in diagonal gauge

Standard LQG type quantisation

1. Compute holonomies $h_\gamma^\lambda(K) := \exp\left(i\lambda \int_\gamma K_a \, ds^a\right)$ and fluxes $E(S) = \int_S E^a \, d^2 s_a$

   γ path, S surface, $\lambda \in \mathbb{Z}$ for U(1), or $\lambda \in \mathbb{R}$ for $\mathbb{R}_{\text{Bohr}}$ see e.g. [Corichi, Krasnov '97] for U(1)

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2. Define positive linear Ashtekar-Lewandowski functional on holonomy-flux algebra
3. Representation follows from the GNS construction: Hilbertspace $= L^2(\bar{\mathcal{A}}, d\mu_{\text{AL}})$
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Remarks

- For $\mathbb{R}_{\text{Bohr}}$: $\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} dx \, f(x) = \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{H}} \, f(x)$ provides **normalised** and translation invariant Haar measure $\Rightarrow$ per edge: $\mathcal{H} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{H}})$

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- Choosing $\lambda \in \mathbb{Z}$ over $\lambda \in \mathbb{R}$ (i.e. compactifying $\int_\gamma K_a ds^a$) has no justification at this stage (also not later)

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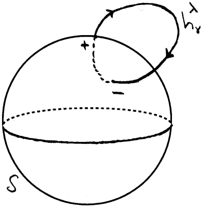
- For $\mathbb{R}_{\text{Bohr}}$: $\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} dx\, f(x) = \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{H}}\, f(x)$ provides **normalised** and translation invariant Haar measure $\Rightarrow$ per edge: $\mathcal{H} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{H}})$
- Choosing $\lambda \in \mathbb{Z}$ over $\lambda \in \mathbb{R}$ (i.e. compactifying $\int_\gamma K_a ds^a$) has no justification at this stage (also not later)

Quantisation II: Area operator

Area operator for Abelian theory

- $A(S) = |E(S)| = |\int_S E^a \, d^2 s_a|$ is analogous to (absolute value of) electric flux
- Important difference to non-Abelian, e.g. SU(2), area op. $\int_S \sqrt{|E^i E_i|}$:
  - \- Absolute value is **outside** of the integral
  - \- $E(S)$ does not detect closed contractible loops for closed S

    While one can also define "non-Abelian like" area operator here, the Abelian one will turn out to be most useful.



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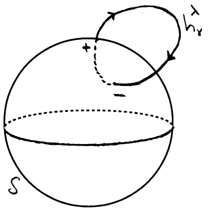
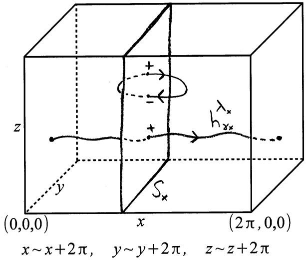
    While one can also define "non-Abelian like" area operator here, the Abelian one will turn out to be most useful.
  - \- Non-trivial topology: $A(S)$ can detect Wilson loops even for closed S





Quantisation III: Imposing the symmetry reduction

Reduction constraints are very familiar from full theory

1. All spatial diffeomorphisms: $\tilde{C}_a[N^a] = \int_\Sigma d^3\sigma \, E^a \mathcal{L}_{\vec{N}} K_a = 0$
2. Abelian Gauß law: $G[\omega] = \int_\Sigma d^3\sigma \, \omega \, \partial_a E^a = 0$

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Observables w.r.t. the reduction constraints

1. Area of closed surfaces $\rightarrow$ 3 non-trivial areas $A(\mathbb{T}^2_x), A(\mathbb{T}^2_y), A(\mathbb{T}^2_z)$
2. Diff-equiv. classes of Wilson loops $\rightarrow$ 3 non-trivial closed loops along $\mathbb{T}^1_x, \mathbb{T}^1_y, \mathbb{T}^1_z$

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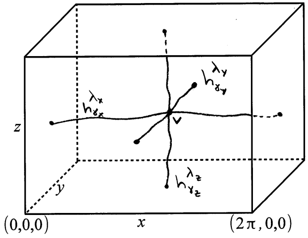
Simplest choice of quantum state

Consider spin network made from 3 Wilson loops wrapping around $\mathbb{T}^1_x, \mathbb{T}^1_y, \mathbb{T}^1_z$, meeting in a single vertex v.

Mapping to Bianchi I LQC states of

[Ashtekar, Wilson-Ewing '09]

$|\lambda_x, \lambda_y, \lambda_z\rangle \mapsto |p_1, p_2, p_3\rangle$



Quantisation IV: Dynamics

Hamiltonian constraint / true Hamiltonian (via deparametrisation)

Take original Hamiltonian:

- Evaluate at $q_{a \neq b} = 0$ because of gauge fixing
- Discard $P^{a \neq b}$, $\partial_a e_b$, and $\partial_a K_b$ terms because of reduction constraints

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- Substitute e_a either by fluxes or Thiemann's trick $e_a = 2\{K_a, V\}$
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  $\Rightarrow$ "old" LQC dynamics [Ashtekar, Bojowald, Lewandowski '03]
- $\mathbb{R}_{\text{Bohr}}$ allows arbitrarily small $\lambda \in \mathbb{R}$ for better approximation.
  "improved" LQC choice: $1/\lambda_x = \sqrt{|E^y E^z / E^x|}$ = size of universe in x-direction
  $\Rightarrow$ "new" LQC dynamics [Ashtekar, Pawlowski, Singh '06; Ashtekar, Wilson-Ewing '09]

Outline

Further application:

The reduction technique works also more generally.

Main steps in the derivation:

1. Choose gauge fixing $q_{??} = 0$ adapted to the symmetry reduction
2. Proceed to the reduced phase space
3. Construct new connection variables
4. Evaluate the reduction constraints $P^{??} = 0$ on the reduced phase space

   $\Rightarrow$ Reduction generates spatial diffeomorphisms (due to linearity of C_a in P^{ab})
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Works also for spherical symmetry

- (Ongoing) work with J. Lewandowski and J. Świeżewski for radial gauge
- Dynamical equivalence more challenging, ongoing work with A. Zipfel

Outline

Conclusion: Proposed reduction programme successful

1. **Suitable classical starting point**
   - ADM in diagonal metric gauge in terms of q_{aa}, P^{bb}
   - Rewritten as Abelian gauge theory in terms of K_a, E^b
2. **Identification of constraints** imposed by symmetry reduction
   - $P^{a \neq b} = 0$ and $\partial_a e_b = 0 = \partial_a K_b$ consistent with Bianchi I
   - First class subset: spatial diffeomorphisms and Abelian Gauß law
3. **Quantise** the reduced phase space via LQG techniques
   - Like Maxwell theory plus additional reduced diffeomorphisms and Hamiltonian constraint
4. **Impose reduction conditions** $f_i = 0$ as operator equations: $\hat{f}_i \left|\Psi\right\rangle_{\text{sym}} = 0$
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   - Observables are closed surfaces and diff-equivalent Wilson loops in x, y, z
5. **Relate observables** $\hat{\mathcal{O}}_{\text{sym}}$ to the parameters of a $\mathbb{T}^3$ Bianchi I model
   - Can be related naturally to Bianchi I LQC ✓
   - Can support improved LQC dynamics for $\mathbb{R}_{\text{Bohr}}$ choice ✓
6. Future work: **perturbations to Bianchi I**, **coarse graining**, ...

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Thank you for your attention!