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Statistical mechanics for general-covariant systems

Goffredo Chirco

Thibaut Josset and Carlo Rovelli in preparation…

C.Rovelli, arXiv:1209.0065; — M.Montesinos, C.Rovelli arXiv:0002024;
— G.Chirco, H.M.Haggard, C.Rovelli, arXiv:1309.0777v1…

Third EFI winter conference on quantum gravity

UNIWERSYTET WARSZAWSKI

EMERGING FIELDS INITIATIVE

FAU FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

motivation & logic

WHY?

1- GRAVITY $\longrightarrow$ describe fluctuations of the gravitational field (strong field regime, BHs, Big bang, new white holes scenario)

2- QUANTUM GRAVITY recently shared common picture: at small scales/high energies space-time $\neq$ continuum manifold

description in terms of discrete/pre-geometric d.o.f. $\longrightarrow$ How do continuum space-time and GR emerge ??

=> quantum statistical field theory language is expected to be useful to face the puzzle

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2- QUANTUM GRAVITY — recently shared common picture: at small scales/high energies space-time ≠ continuum manifold

↙

description in terms of discrete/pre-geometric d.o.f. → How do continuum space-time and GR emerge ??

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motivation & logic

PROBLEM

- GR is a general-covariant theory: dynamics is defined by a re-parametrisation invariant Lagrangian leading to a vanishing canonical hamiltonian

- Statistical mechanics is based on notions such as energy and preferred time which have no equivalent in a general covariant theory

=> how do we do statistical physics for a system without a hamiltonian ??

GOAL

"gravitizing statistical mechanics" : re-derive the founding notions of the theory in a relativistic/pre-symplectic hamiltonian formalism

STRATEGY

rationalise the problem at a formal level: identify the main conceptual issues of the approach and provide a set of basic definitions

outline: four main issues

so far…

1 - phase space & statistical state

2 - thermodynamics & notion of subsystem

top down approach

bottom up approach

3 - recover thermodynamics

4 - gauge fixing and interaction

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1 - phase space & statistical state

Statistical mechanics: deducing macroscopic properties of matter from the atomic hypothesis

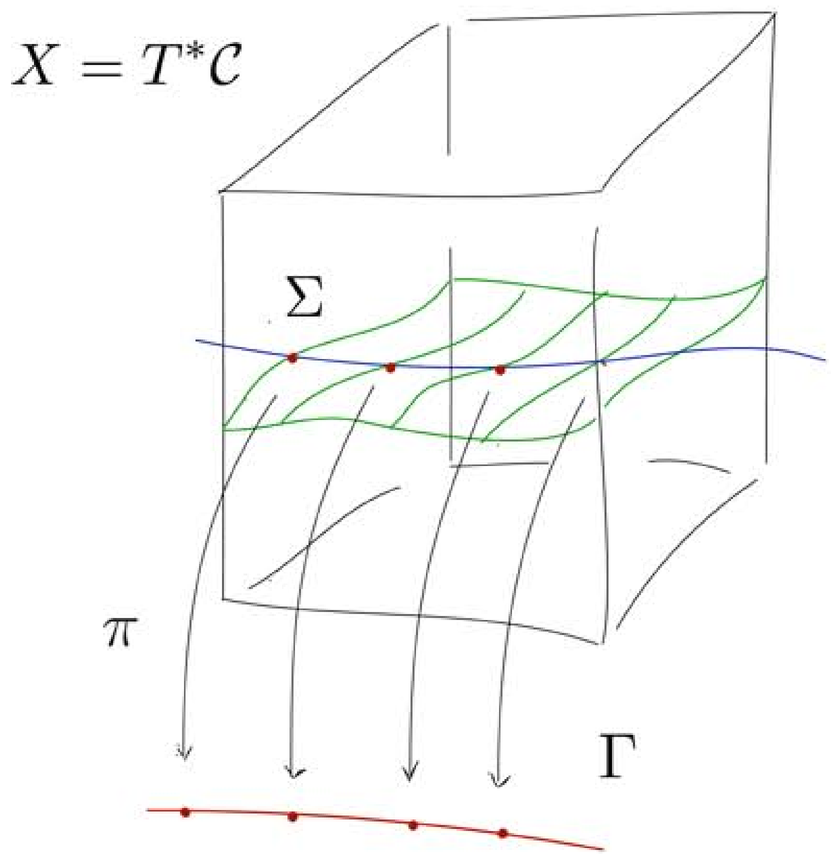
=> investigate the qualitative behavior of the ensemble of particles (atoms molecules) governed by a deterministic law of motion (classical mechanics or quantum mechanics)

PROBLEM

time plays a fundamental role: the notion of physical phase space is different for non-relativistic and relativistic systems

let's set a common language first

common description: hamiltonian formalism



kinematics $\mathcal{C} = \{q^a\}$ $\quad T^*\mathcal{C} = \{(q^a, p_a)\}$

dynamics information about the dynamics encoded in the surface Σ

$$\Sigma = \{p \in X |\, H(p) = 0\}$$

$H : X \to \mathbb{R}^k$ constraint hamiltonian

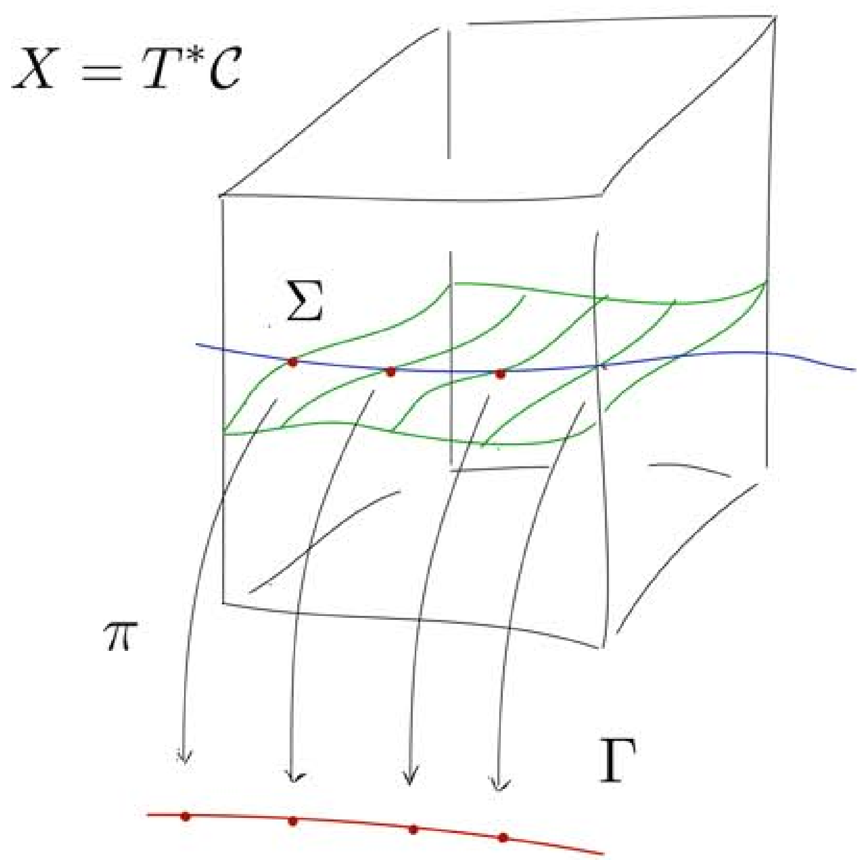
geometric approach!

- (X,ω_X) a symplectic space: X = T*C is a cotangent space

  $\theta = p_a dq^a$ Poincare' 1-form of the cotangent bundle

  => natural symplectic form $\omega_X = d\theta$

dynamics encoded in pre-symplectic space



the constraint surface $\Sigma = H^{-1}(0)$ is equipped with a pre-symplectic (closed degenerate) 2-form $\omega_\Sigma = \omega_X|_\Sigma$

=> orbits of ω_Σ = graphs of the physical motions

(integral surfaces of the null directions of ω_Σ defined by

the vector field $\underline{X}$ on Σ in the kernel of ω_Σ)

(Σ, ω_Σ) fully defines a gen. cov. dynamical system

$\omega_\Sigma(\underline{X}) = 0$ <=> HAMILTON EQNS

- Γ is the space of the solutions, or the physical phase space of the system: a point in Γ as a motion of the system, or a Heisenberg state

instantaneous measurement

- look at a non-relativistic systems in this terms

$$q^a = (t, q^i) \Rightarrow \mathcal{C} = I\!R \times Q$$

$$(q^a, p_a) = (t, q^i, p_t, p_i) \Rightarrow \Omega = T^*\mathcal{C}$$

hamiltonian constraint $H = p_t + H_0$ => $\boxed{\Sigma = I\!R \times \Gamma_{nr}}$

the coordinate on IR being the time t and $\Gamma_{nr} = T^*Q$ the usual phase space

on Σ

$$\omega = -dH_0 \wedge dt + \omega_{nr}$$

$$X = \frac{\partial}{\partial t} + X_{nr}$$

$\omega_\Sigma(\underline{X}) = 0 \Rightarrow \omega_{nr}(X_{nr}) = -dH_0$

- the space of the orbits Γ is in one to one correspondence with the cotangent space Γ_{nr}. the cotangent space $\Gamma_{nr} = T^*Q$ is the "natural arena" for non-relativistic hamiltonian mechanics and also the space of the motions: space of the instantaneous states

=> for a non relativistic system the statistical state is naturally defined as the probability distribution ρ such that dν = ρdμ where dμ is the Liouville's measure on Γ

instantaneous measurement

- indeed, consider the usual ergodic hypothesis

consider the generic (X,v) measured space, Ut : X -> X a "time" evolution that preserves v and f : X -> C and the ergodic assumption:

under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(U_t P_0) dt = \int_X f(P) d\nu(P) \quad , \; \nu - ae \; P_0$$

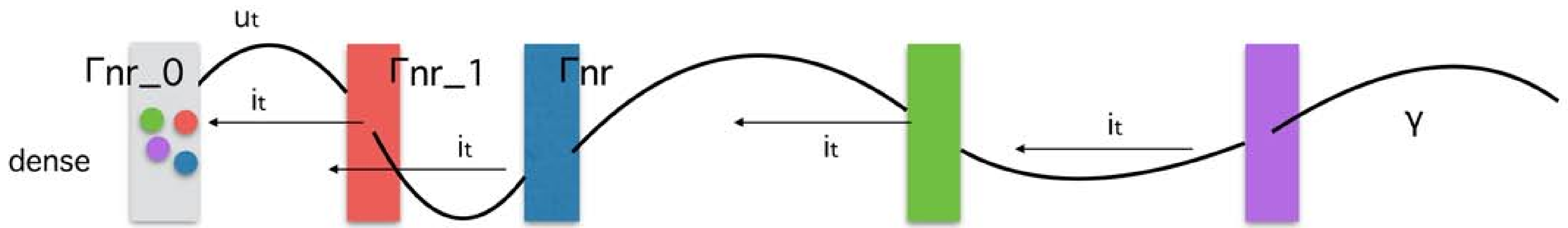
physically meaningful quantity

when the observable f is measured, the experiment take some finite time which is supposed to be large compared to the "mixing time" so that the measuring device see the time average of the measured quantity

theoretical quantity

we do not need to compute/solve the evolution in time (which is intractable for complex systems): independence from the microscopic hamiltonian dynamics

$$X = \Gamma \qquad U_t = i \circ i_t^{-1} \circ u_t \circ i^{-1} : \Gamma \to \Gamma \qquad f = \text{ complete observable}$$



instantaneous measurement on Σ

The same property does not hold in for the relativistic case: one has only

$$u_\tau : \Sigma \to \Sigma \qquad \pi : \Sigma \to \Gamma \quad \text{such that} \quad \pi \circ u_\tau = \pi$$

- the double role (motions + instantaneous states) is lost: a point of Γ is not seen as representing the instantaneous state, it represents a full solution (history) of the system

=> what is an instantaneous measurement for a relativistic system ?

=> on which space should we figure a statistical measurement ?

- so far this game was played on the physical phase space Γ, as in the non-rel case:

1. the statistical state is naturally defined over Γ
2. f is a complete observable
3. the “time” average is meaningless because f is constant over every orbit

GOAL
a more operational approach

deduce the statistical properties of a gen cov system from its pre-symplectic structure only

instantaneous measurement on Σ

Let's then work on Σ (analysis limited to one dim systems)

1. we want to do statistical mechanics without referring to the (unphysical) time variable used to parametrised the orbits, by using compatible measurements in a relational approach:

the notion of instantaneous measurement is replaced by the (observation of an) event, i.e. a point P in the pre-symplectic space Σ. a succession of instantaneous measurements gives a list of points P(1),...,P(N) ∈ Σ, all sitting on the same orbit γ

CLOCK $$\bar{f} = \frac{1}{T}\int_0^T f(P(t))dt$$ the average of a list of measurements requires the choice of a 1-form, playing the role of dt

- a clock is a 1-form θ over Σ such that $$0 < \int_{\gamma_U} \theta < +\infty$$

  for all γ_U , open subset of an orbit γ

- the θ-average of a partial observable f : Σ → R along an orbit γ is defined by $$\bar{f}(\theta, \gamma) = \frac{1}{\int_\gamma \theta}\int_\gamma f\theta$$

2. f is a partial observable

Σ, what about the measure?

statistical state on Σ

3. from $\rho : \Gamma \to \mathbb{R}_+$ one can be tempted by defining $\tilde{\rho} = \rho \circ \pi : \Sigma \to \mathbb{R}_+$ to get a statistical state over Σ. However, this is not a computable quantity: π requires to know the orbits

=> try to define a measure on Σ starting from the natural (regular and locally finite) measure μ_L on T^*C, induced by the symplectic form

- consider (X,ω_X) a symplectic space and $C : X \to R$, such that $\Sigma = C^{-1}(0)$ and $\omega_\Sigma = \omega_{X|\Sigma}$. Then X is naturally equipped with the Liouville measure μ_X, associated to the volume form $\omega^{\dim X/2}$

$$\Rightarrow \quad d\mu_{\Sigma,C} = \delta(C) d\mu_X \quad \text{defines a measure on } \Sigma$$

- $\mu_{\Sigma,C}$ is preserved under the (unphysical) time evolution generated by the constraint C

$$\frac{d}{d\tau} \cdot = \omega_X(C, \cdot) \implies \int_\Sigma f(P(P_0,\tau)) d\mu_{\Sigma,C}(P_0) = \int_\Sigma f(P_0) d\mu_{\Sigma,C}(P_0) \ \forall \tau, \forall f$$

but The measure $\mu_{\Sigma,C}$ depends crucially on the peculiar choice of constraint C. As the physics should be entirely contained in the pre-symplectic structure (Σ,ω_Σ), this measure has no physical meaning so far, as physics should not depend on C

ergodicity and statistical state on Σ

4. try a different strategy: look again at the ergodic hypothesis as a starting tool

- the θ-statistical state associated to an orbit γ is the linear functional

$$\mu_{\theta,\gamma} : f \mapsto \bar{f}(\theta,\gamma)$$

defining a measure $\mu_{\theta,\gamma}$ over Σ that satisfies

$$\int_\Sigma f d\mu_{\theta,\gamma} = \frac{1}{\int_\gamma \theta} \int_\gamma f\theta \qquad \forall f : \Sigma \to \mathbb{R}$$

=> $\mu_{\theta,\gamma}$ depends a priori on the state γ

- a partial observable f is called θ-ergodic if

$$\bar{f}(\theta,\gamma) = \bar{f}(\theta) \qquad \forall \gamma \in \Gamma$$

=> a system is called θ-ergodic if every partial observable is θ-ergodic

$$\mu_{\theta,\gamma} = \mu_\theta \qquad \forall \gamma \in \Gamma$$

is our statistical state physically meaningful?? ...the notion of θ-ergodicity do play a role!

ergodicity and statistical state on Σ

- if the system is θ-ergodic $$\int_{\gamma_{P_0}} f\theta = \bar{f}(\theta)\int_{\gamma_{P_0}} \theta \quad \forall f, \forall P_0$$

- use $\theta|_\gamma = \rho_{\theta,C} d\tau$, by integrating over P_0 with the measure $\mu_{\Sigma,C}$ and using the fact that it is preserved

$$\theta = \alpha_i dq^i + \beta^i dp_i$$

$$\rho_{\theta,C} = \alpha_i \frac{\partial C}{\partial p_i} - \beta^i \frac{\partial C}{\partial q^i}$$

=> $$\bar{f}(\theta) = \frac{1}{\int_\Sigma \rho_{\theta,C} d\mu_{\Sigma,C}} \int_\Sigma f \rho_{\theta,C} d\mu_{\Sigma,C} \quad \forall f$$

which defines the statistical state $$d\mu_\theta = \frac{\rho_{\theta,C} d\mu_{\Sigma,C}}{\int_\Sigma \rho_{\theta,C} d\mu_{\Sigma,C}}$$

the statistical state μ_θ is entirely determined by the pre-symplectic structure (Σ, ω_Σ) and the choice of a clock θ. Thus, it is physically meaningful (at least for ergodic systems)

summary

after choosing a clock θ, one can talk about statistical measurement for a general-covariant system, in particular define θ-average. Equality between θ-average and statistical average is used as a definition of the θ-statistical state. This θ-statistical state is very abstract but, assuming ergodicity, it can be written explicitly

2 - thermodynamics: notion of subsystem

- thermodynamics: we would need to consider multiple gen-cov systems

philosophy — world is not made of systems that couple to each other, it is made of "a big system" that we split (in our mind) into components

=> the goal becomes to find meaningful way of splitting systems and the approach to follow is somehow imposed by the system...

TWO MAIN CASES:

1. if orbits are 2-dimensional with two independent constraints, then the system can be seen as the coupling of two subsystems: bottom up approach

   e.g. relevant in GR, single point hamiltonian constraint in strong field limit; gas of relativistic particles; extension to fields...

2. if orbits are 1-dimensional with a hamiltonian constraint which is the sum of two independent ones, then the system can be seen as the coupling of two subsystems: top down approach

   e.g. relevant for the decoupling between the gravitational field and matter fields

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top down approach

define a generic splitting on Σ :

A mechanical system S, given by (Σ,ω_Σ), splits into two non-interacting subsystems, S^a and S^b, if it can be seen as a subspace of an extended phase space $(X^a \times X^b, \omega_{X^a} + \omega_{X^b})$, defined by a constraint of the form $C = C^a + C^b = 0$

such splitting is characterised by:

- a foliation of the pre-symplectic space $$\Sigma = \bigsqcup_{I^a+I^b=0} \Sigma^a_{I^a} \times \Sigma^b_{I^b}$$

  where $X^a = \bigsqcup_{I^a} \Sigma^a_{I^a}$ $\quad X^b = \bigsqcup_{I^b} \Sigma^b_{I^b}$

- the existence of a constant of motion $I = I^a = -I^b : \Sigma \to \mathbb{R}$.

remark

Each $\left(\Sigma^a_{I^a}, \omega_{X^a}\big|_{\Sigma^a_{I^a}}\right)$ (resp. b) is a pre-symplectic space, that can be considered as a mechanical system by itself, noted $\mathcal{S}^a_{I^a}$

top down approach

e.g.

non-relativistic systems naturally splits into a time part and a system foliated by surface of constant energy

$$\Sigma = \bigsqcup_{I^a+I^b=0} \Sigma^a_{I^a} \times \Sigma^b_{I^b} \qquad \Sigma^a_{I^a} = \{(t)\} = \mathbb{R} \qquad \Sigma^b_{I^b} = \{(q^i, p_i); H_0(q^i, p_i) = I^b\}$$

seen as subspaces of the symplectic space

$$X = \{(t, p_t)\} \times \{(q^i, p_i)\} \qquad \omega_X = dp_t \wedge dt + dp_i \wedge dq^i$$

with $\quad C = p_t + H_0(q^i, p_i) = 0$

and $\quad \Sigma = \{(t, q^i, p_i)\} \qquad \omega_\Sigma = -\frac{\partial H_0}{\partial q^i} dq^i \wedge dt - \frac{\partial H_0}{\partial p_i} dp_i \wedge dt + dp_i \wedge dq^i$

remark

Writing Γ the physical phase space associated to S and Γ^a_{Ia} (resp Γ^b_{Ib}) the one associated to $S_{I^a a}$ (resp. $S_{I^b b}$). The foliation of the pre-symplectic space induce a foliation of the physical phase space

$$\Gamma = \bigsqcup_{I^a+I^b} \Gamma^a_{I^a} \times \Gamma^b_{I^b} \times \Delta_I$$

where, for all I ∈ R, ΔI is a one-dimensional space.

ie. a degree of freedom (I,ΔI) lives on the boundary of the subsystems S^a and S^b and is lost if they are treated as independent mechanical systems.

3 - from stat mech to thermodynamics: microcanonical ensemble

reproduce now this structure for gen cov systems:

consider a general-covariant system S that splits into two non-interacting subsystems S^a and S^b. Imagine we are interested in measuring partial observables of the subsystem S^a (i.e. $f : X^a \to R$), using the subsystem S^b as a clock (i.e. θ is a 1-form over X^b)

assuming θ-ergodicity, one gets a family of statistical states, labeled by I_a

$$d\mu_{\Sigma^a_{I^a}} = \frac{\delta(C^a - I^a)d\mu_{X^a}}{\int \delta(C^a - I^a)d\mu_{X^a}}$$

remarks

- as long as the subsystem $S_{I^a}^a$ is θ-ergodic, the statistical states does not depend on the clock θ nor on the dynamics of the subsystem S^b => the system S^b is used as a clock without specifying the 1-form θ

- we have one probability distribution for each value of I^a. As I^a is not allowed to vary, it would not mean anything to think about a probability distribution for I^a itself.

  micro canonical ensemble => entropy and temperature cannot be defined yet

interaction and thermodynamics

in order to talk about equilibrium, entropy, temperature, etc. one needs (at least) one quantity that can be exchanged between two subsystems such that its total amount remains constant

- consider a general-covariant system S that splits into two weakly-interacting subsystems S^a, S^b and one non-interacting system S^c used as a clock:

$$C = C^a + C^b(+V^{ab}) + C^c = 0 \qquad \Sigma = \bigsqcup_{I^a+I^b+I^c=0} \Sigma^a_{I^a} \times \Sigma^b_{I^b} \times \Sigma^c_{I^c}$$

- assuming ergodicity

$$d\mu_{\Sigma^{a,b}_{-I^c}} = \frac{\delta(C^a + C^b + (V^{ab}) + I^c) d\mu_{X^a \times X^b}}{\int \delta(C^a + C^b + (V^{ab}) + I^c) d\mu_{X^a \times X^b}}$$

- assuming weak perturbation, the state factorises:

$$d\mu_{\Sigma^{a,b}_{-I^c}} = \frac{\delta(C^a - I^a) d\mu_{X^a} \delta(C^b - (I^{ab} - I^a)) d\mu_{X^b} dI^a}{\int \delta(C^a + C^b - I^{ab}) d\mu_{X^a \times X^b}}$$

where $I^{ab} = -I^c$ is the amount of "I" to be shared between S^a and S^b

general-covariant equilibrium thermodynamics !

given the probability distribution of I^a, I^b, with fixed I^{ab}

$$\rho(I^a, I^b; I^a + I^b = I^{ab}) = \frac{\int \delta(C^a - I^a)\delta(C^b - I^b) d\mu_{X^a} d\mu_{X^b}}{\int \delta(C^a + C^b - I^{ab}) d\mu_{X^a \times X^b}}$$

- define the I-entropy of the system (a,b) by the usual formula $S_I = k_B \log \rho$

additivity $\rho(I^a, I^b; I^a + I^b = I^{ab}) \propto \rho^a(I^a)\ \rho^b(I^{ab} - I^a)$

=> $S_I = S_I^a(I^a) + S_I^b(I^{ab} - I^a)$

- define the I-temperatures $\frac{1}{T_I^a} = \frac{dS_I^a}{dI^a} \qquad \frac{1}{T_I^b} = \frac{dS_I^b}{dI^b}$

- Finally, the equilibrium value of I is defined by maximizing the I–entropy S_I or, equivalently, by equal-temperatures $T_I^a = T_I^b$

formally these thermodynamical relation do not depend on the form of the clock!!

test the splitting structure: FRLW universe

C.Rovelli, Class Quantum Gravity vol.10 (1993)

consider a homogeneous and isotropic universe (FLRW metric) that contains a non-uniform electro-magnetic field

$$ds^2 = -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j$$

$$S = \frac{1}{16\pi G} \int dt(-6aV) \left(\frac{\dot{a}^2}{N} + Nk \right) + S_e$$

=> $$S = \int dt\, N \left(-\frac{2\pi G\, p_a^2}{3V a} - \frac{3V k a}{8\pi G} \right) + H_e$$

not generally covariant, most of the coordinate freedom has been fixed by using the homogeneity of the gravitational field: there is a residual invariance under re-parametrisation of the coordinate time t

pre-symplectic structure given by

$$\boxed{C = -\frac{2\pi G}{3V} \frac{p_a^2}{a} - \frac{3V k}{8\pi G} a + \frac{V}{a} \tilde{H} = 0}$$

plus the Gauss constraint...

with

$$H = \int d^3x \sqrt{g} T_{00}[g, E, A](x) = \frac{V}{a} \tilde{H} \quad , \quad V = \int_d x \sqrt{\tilde{g}}$$

$$\tilde{H} = V^{-1} \int d^3x \left(\tilde{g}^{-1/2} \tilde{g}_{ij} E^i E^j + \tilde{g}^{1/2} \tilde{g}^{ij} \tilde{g}^{kl} \partial_{[i} A_{j]} \partial_{[k} A_{l]} \right)$$

Let's restrict to the spatially flat case:

$$k = 0 \quad \text{and} \quad \tilde{H} = V^{-1} \int d^3x \left(\delta_{ij} E^i E^j + \delta^{ij} \delta_{kl} \partial_{[i} A_{j]} \partial_{[k} A_{l]} \right)$$

- we want to describe the thermodynamics of the e.m. field, using the gravitational d.o.f. as a clock: performing Fourier transformation, the constraint is equivalent to

{a, pa} (subsystem taken as a clock) $$-\frac{2\pi G}{3V^2} p_a^2 + \int_0^{+\infty} \tilde{H}_\nu d\nu = 0$$

I_a is fixed and has to be shared among the different modes ν of the electromagnetic field

$$I_a = -\frac{2\pi G}{3V^2} p_a^2 \qquad I_\nu = \tilde{H}_\nu$$

assume a weak interaction => allow exchange of energy between the different modes of the electro-magnetic field: the dynamical system fits exactly in the developed framework

- assuming ergodicity, the statistical state for the electromagnetic field is independent from the specific dynamics of {a, pa} (subsystem taken as a clock)

FRLW universe

exactly what happens!

- $H_e = \tilde{H}\, V\, a^{-1}$ and $e = H_e\, a^{-3} = \tilde{H}\, V\, a^{-4}$
- black body like radiation $\langle e(a)\rangle = 4\sigma T(a)^4 \implies T(a) = \frac{1}{a}\left(\frac{\langle\tilde{H}\rangle}{4\sigma}\right)^{1/4}$

$\tilde{H}$ is the hamiltonian of the electromagnetic field in euclidean space-time so the statistical state should be the same as the usual one, i.e. given by Bose-Einstein statistics (or Maxwell-Boltzmann for the high frequency modes)

$$\rho = e^{-H/k_b T} = e^{-(4\sigma)^{1/4}/k_b\, V\, \tilde{H}^{3/4}}$$

- key point: instead of considering the conserved quantity $\tilde{H}$ we choose to call the energy

$$E(a) = H = \frac{V}{a}\tilde{H} \quad \Rightarrow \quad e(a) = \frac{H}{a^3 V} = \frac{\tilde{H}}{a^4} \quad \text{energy density}$$

remark the rate of the clock is reflected in the scale of the energy

outline: four main issues

so far...

1 - phase space & statistical state

2 - thermodynamics: notion of subsystem

top down approach

bottom up approach

3 - thermodynamics

4 - gauge fixing and interaction

4 - bottom up approach

how do we couple two general covariant sys?

- consider two non-relativistic systems S^a and S^b described by

$$X^a = \{(t^a, p_{ta}, q^a, p_a)\} \ , \ \omega_{X^a} = dp_{ta} \wedge dt^a + dp_a \wedge dq^a \ , \ C^a = p_{ta} + H_a(q^a, p_a)$$

$$X^b = \{(t^b, p_{tb}, q^b, p_b)\} \ , \ \omega_{X^b} = dp_{tb} \wedge dt^b + dp_b \wedge dq^b \ , \ C^b = p_{tb} + H_b(q^b, p_b)$$

- couple them in a general-covariant way, into a single system with two time observables:

$$X = \{(t^a, p_{ta}, q^a, p_a, t^b, p_{tb}, q^b, p_b)\}$$
$$\omega_{X^a} = dp_{ta} \wedge dt^a + dp_a \wedge dq^a + dp_{tb} \wedge dt^b + dp_b \wedge dq^b$$
$$\begin{cases} C^a = p_{ta} + H_a(q^a, p_a) \\ C^b = p_{tb} + H_b(q^b, p_b) \end{cases}$$

- the surface of constraint is $\Sigma = \Sigma^a \times \Sigma^b$, the orbits are 2-dimensional and the physical phase space is $\Gamma = \Gamma^a \times \Gamma^b$

gauge fixing implies a choice of a subspace of co-dimension 1 in Σ. The induced 2-form generates 1-dimensional orbits which are curves in the previous 2-dimensional orbits. The physical phase space remains unchanged: from the mechanical point of view, no specific gauge fixing is preferred.

- In each situation, there is a "natural" choice for the gauge fixing. Is this special choice encoded in the physics of our system ?

- is there a preferred choice of gauge from the thermodynamical point of view?

the two questions are related...

gauge and coupling

Idea — the interaction between S^a and S^b contains the information about a preferred choice of gauge

no interaction:

- H_a and H_b are both conserved quantities => any linear combination of them is conserved. both system are at equilibrium independently so any statistical state that is a product of equilibrium state is allowed (no need for the temperatures to be equal)

interaction on:

- H_a and H_b are not conserved, only a well chosen combination $\alpha H_a + \beta H_b + V_{ab} \approx \alpha H_a + \beta H_b$ is
- in order to write (simply) a physically meaningful statistical state, it is crucial that the interaction does not depend on t (the splitting between the clock and the rest should be perfect).

gauge and coupling

an instantaneous interaction among two systems with clocks which do not run the same way (e.g. in different gravitational potentials) should contain a term of the form $\chi(t^a/\alpha - t^b/\beta)$.

=> the preferred gauge choice: $t^a = \alpha t$, $t^b = \beta t$ is the natural one making the interaction independent of t

e.g. choose the gauge fixing condition $\{t^a = f^a(t), t^b = f^b(t);\ t \text{ in } \mathbb{R}\}$, then the reduced system previously defined is described by the single constraint

$$p_t + \frac{df^a}{dt} H_a(q^a, p_a) + \frac{df^b}{dt} H_b(q^b, p_b) = 0$$

to get the specific form $p_t + H_0 = 0$, allowing to write a statistical state for the subsystem $\{q^a, p_a, q^b, p_b\}$, we need: $t^a = \alpha t + t^a_0$ and $t^b = \beta t + t^b_0$

summary

starting from a system with two times (i.e. two constraints, 2-dimensional orbits), there might be a gauge choice such that, in the single constraint, the interaction term does not depend on the clock variables. Only in that case it is meaningful to write a statistical state

discussion

1 - phase space & statistical state

- we can do statistical mechanics for general covariant systems over Σ, that is in terms of partial observables

=> the systems we are interested in are weakly coupled to a measuring device, sensitive to partial observable quantities, thus not predictable. However, the value displayed by the measuring device should be predictable

new: previous work only considered statistical states over Γ

- the identification of a clock θ plays a fundamental role => θ-ergodicity

after choosing a reasonable clock θ, one can talk about statistical measurement for a general-covariant system, in particular define θ-average. Equality between θ-average and statistical average is used as a definition of the θ-statistical state. This θ-statistical state is very abstract but, assuming ergodicity, it can be written explicitly

discussion

2 - ensemble: notion of subsystem

- we can consistently define subsystems in the pre-symplectic formalism

clock and system live on two independent subsystems of the pre-symplectic space

=> straightforward formal analogy with stat mech (micro canonical ensemble)

3 - thermodynamics

- two weakly-interacting subsystems S^a, S^b and one non-interacting system S^c used as a clock

=> we define the I-entropy and I-temperatures and define the equilibrium I value via entropy maximization

the usual definition of energy does not coincide with the conserved quantity I

discussion

- derivation meaningful only when the clock system is rigorously noninteracting (foliation preserving) somehow the equivalent of the tensorial structure characterising a non-rel systems

important: thermodynamics of non-relativistic systems is contained in the more general framework developed

=> general insight: as long as ergodicity is satisfied the specific form of the clock does not play any role in thermodynamics also in the non-relativistic case

4 - gauge fixing and interaction

- we can reduce multiple constrained systems in a single 1-d system by gauge fixing and successively apply the procedure developed for deriving thermodynamics

the interaction between subsystem can be used to naturally set a preferred choice of gauge

time independent interaction <=> thermodynamics

perspectives (= work in progress)

1 - define interesting physical examples

2 - extend to field theory

3 - extend to quantum mechanics

THANK YOU