The Status of Quantum Reduced Loop Gravity

Francesco Cianfrani*

Alesci, FC, Rovelli, Phys. Rev. D 88, 104001
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*Institute of Theoretical Physics, University of Wrocław.
Plan of the talk

_Quantum-reduced Loop Gravity
_Semiclassical state
_Semiclassical dynamics
_Perspectives
Quantum Reduced Loop Gravity
QRLG can be seen as the gauge-fixed quantization of LQG, in the frame in which the metric tensor and the triads are both diagonal.

1) one can always fix a frame in which the three-metric tensor is diagonal

\[ dl^2 = a_1^2(dx^1)^2 + a_2^2(dx^2)^2 + a_3^2(dx^3)^2 \]

2) one can always the gauge in which the triads is diagonal

\[ E_i^a = p^i \delta_i^a \quad \quad A_i^a = c_i \delta_i^a + \ldots \]

Let us implement 1) and 2) **weakly** in the gauge-invariant kinematical Hilbert space of LQG:
A little bit of notation:

let us call $i=1,2,3$ the principal direction, $l_i$ the associated links and $S^i$ the dual surfaces
1) gauge-fixing condition in terms of fluxes

\[ \eta_{x}^{km} = \delta^{ij} E_{i}(S_{x}^{k}) E_{j}(S_{x}^{m}) = 0, \quad k \neq m, \quad \forall x \in \Sigma \]

\[ \langle \psi | \eta_{x}^{km} | \phi \rangle = 0, \quad k \neq m, \quad \forall x \in \Sigma \]

only nontrivial case: states based at the same graph \( \Gamma \) and \( x \) belongs to \( \Gamma \)

\( x \) is the internal point of some link in \( \Gamma \)

\[ \langle l_{x}, j_{l_{x}} | \eta_{x}^{km} | l_{x}, j_{l_{x}} \rangle = (8\pi \gamma l_{P}^{2})^{2} o(S_{x}^{k}, l_{x}) o(S_{x}^{m}, l_{x}) j_{l_{x}}(j_{l_{x}} + 1) \delta_{i, j_{l_{x}}} \]

only choice: they cannot simultaneously be non vanishing for \( k \neq m \).

\[ l_{x} = l_{i} \] for some \( i \)

Restriction to reduced graphs.
Let us consider the basis of Livine-Speziale coherent states

\[ \langle h | \Gamma, j_1, \mathbf{u}_1 \rangle = \sum_{x_n} \langle h | \Gamma, j_1, x_n \rangle \langle j_1, x_n | j_1, \mathbf{u}_1 \rangle \]

In the large j limit one has

\[ \langle \Gamma, j_1, \mathbf{u}_1 | \vec{E}(S_n^k) \cdot \vec{E}(S_n^m) | \Gamma, j_1, \mathbf{u}_1 \rangle \approx (8\pi\gamma l_P^2)^2 \sum_{l_k} j_{l_k} \mathbf{u}_{l_k} \cdot \sum_{l_m} j_{l_m} \mathbf{u}_{l_m} \]

and it vanishes if the normals along k and m are orthogonal.

Without loss of generality we can take

\[ \mathbf{u}_1 = (\pm 1, 0, 0) \quad \mathbf{u}_2 = (0, \pm 1, 0) \quad \mathbf{u}_3 = (0, 0, \pm 1) \]
a generic quantum state can thus be written as

$$\langle \Gamma, j_1, x_n | \psi \rangle = \prod_{n \in \Gamma} \langle j_1, \bar{u}_1 | j_1, x_n \rangle \prod_{l} \psi_{j_l, \bar{u}_l}$$

Reduced intertwiners

Alesci, FC, Rovelli
2) gauge-fixing condition in terms of fluxes

\[ \chi_i(S) = \epsilon_{ij} E_k(S^j) = 0 \]

We mimic the imposition of simplicity constraints in the EPRL model: we impose strongly

\[ \chi^2 = \sum_i \chi_i^2 \]

\[ \hat{\chi}^2 (S_x)^i D^j_{mn}(h) = (8\pi\gamma l_P^2) (j(j+1) - m^2) D^j_{mn}(h) \]

In the basis in which \( \tau_i \) is diagonal (coherent states in the direction \( i \))

A solution in the large \( j \) limit is obtained for \( m = \pm j \)
We can define a projector to such subrepresentations

\[ P_l^\pm = |j_l, \vec{u}_l^\pm\rangle \langle j_l, \vec{u}_l^\pm| \]

such that the solutions of \( \chi^2 = 0 \) are obtained by

\[ P_\chi : D^{jl}(h_l) \mapsto P_l D^{jl}(h_l) P_l \]

and basis states in the gauge-invariant space are projected to

\[ \langle h|\Gamma, j_1, x_n \rangle = \prod_{n \in \Gamma} \langle j_1, x_n|j_1, \vec{u}_1 \rangle \prod_l lD^{jl}_{m_l m_l}(h_l) \]

\[ m_l = \pm jl \]

Fluxes acts diagonally

\[ R \hat{E}_i(S^i) lD^{jl}_{m_l m_l}(h_l) = 8\pi\gamma l_P^2 m_l lD^{jl}_{m_l m_l}(h_l) \]

\[ l = l_i \cap S^i \neq \emptyset \]
To summarize: Quantum-reduced Hilbert space is obtained by

- restricting to reduced graphs

- projecting SU(2) representations and operators as follows

\[ D^j_{mn}(g) = g \rightarrow R_l R_l^{-1} g R_l R_l^{-1} \]

Fluxes \( E_i(S^l) \) read the magnetic numbers of states based at links \( l \).
The scalar constraint can be defined by using the elements of the reduced Hilbert space:

**ITS MATRIX ELEMENTS CAN BE EXPLICITLY COMPUTED**, since the volume operator is diagonal

**IT CAN BE REGULARIZED**, thanks to reduced diffeo-invariance

Euclidean part:

\[ R \hat{H}_E[N] = \sum R \hat{H}^m_{E \square}[N] \]

\[ R \hat{H}^m_{E \square}[N] := N(n)C(m) \epsilon^{ijk} \text{Tr} \left[ R \hat{h}^{(m)}_{\alpha ij} R \hat{h}^{(m)}_{sk}^{-1} \left[ R \hat{h}^{(m)}_{sk}, R \hat{V} \right] \right] \]

obtained by projecting SU(2) elements on reduced elements

diagonal operator
Semiclassical states
Semiclassical limit in LQG

Complexifier:

\[ H' = h' \exp \left( \frac{\alpha}{8\pi\gamma l^2 P} E_i' \tau_i \right) \]

Heat-Kernel:

\[ K_\alpha(h_l, h'_l) = e^{- \frac{\alpha}{2} \Delta h_l} \delta(h_l, h'_l) \]

\[ K_\alpha(h_l, h'_l) = \sum_{j_l} (2j_l+1) e^{-j_l(j_l+1)\frac{\alpha}{2}} Tr\{D^{j_l}(h_l^{-1}h'_l)\} \]

Semiclassical state on the link:

\[ \psi^\alpha_{H'}(h_l) = K_\alpha(h_l, H') \]

Peaked around the classical configuration \((h', E')\)
Semiclassical state on multi-link:

\[ \Psi_{H',\Gamma}(\{h_l\}) = \prod_{l \in \Gamma} \psi_{H'}^\alpha(h_l) \]

under gauge transformation:

\[ \Psi'_{H',\Gamma}(\{h_l\}) = \prod_{l \in \Gamma} \psi_{H'}^\alpha(g_{t_l} h_l g_{s_l}^{-1}) \]

Projection on the gauge invariant state (insertion invariant intertwiners)

Semiclassical state in the gauge-invariant Hilbert space

Thiemann, Winkler
Magliaro, Marcianò, Perini
Semiclassical limit in QRLG

Complexifier:

\[ H'_{i} = h'_{i} \exp \left( \frac{\alpha}{8 \pi \gamma l_{P}^{2}} E'_{i} \tau_{i} \right) \]

\[ E'_{i} \sim \bar{p}^{i} \delta_{i}^{2} \]

not summed

Area of smearing surface

reduced holonomy:

\[ h'_{i} = e^{i \theta \tau_{i}} \in U(1)_{i} \]

\[ \theta \sim \pm e_{l} c_{i} \]

link length

Heat-Kernel:

\[ l = l' \]

\[ K_{\alpha}(h_{l}, h'_{i}) = \sum_{m_{l} = -\infty}^{+\infty} (2j_{l} + 1)e^{-j_{l}(j_{l+1})^{\alpha}/2} lD_{m_{l}m_{l}}^{j_{l}}(h_{l}^{-1}h'_{i}) \]

sum over magnetic indices

Semiclassical state on the link:

\[ \psi^{\alpha}_{H'_{i}}(h_{l}) = K_{\alpha}(h_{l}, H'_{i}) = \sum_{m_{l} = -\infty}^{\infty} \psi^{\alpha}_{H'_{i}}(m_{l}) lD_{m_{l}m_{l}}^{j_{l}}(h_{l}^{-1}) \]
\[ \psi_{H_l}^\alpha(m_l) = (2j_l + 1) e^{-j_l(j_l+1)\frac{\alpha}{2}} e^{i\theta_l m_l} e^{\frac{\alpha}{8\pi\gamma l_P^2} E'_l m_l} \]

\[ \psi_{H_l}^\alpha(h_l) \sim \sum_{m_l=-\infty}^{\infty} (2j_l + 1) e^{-\frac{\alpha}{2} \left( m_l - \frac{E'_l}{8\pi\gamma l_P^2} \right)^2} e^{i\theta_l m_l} lD_{m_l m_l}^{j_l} (h_l^{-1}) \]

\[ \frac{E'}{8\pi\gamma l_P^2} \gg 1 \]

Semiclassical state on multi-link in the gauge-invariant space:

\[ \psi_{\Gamma H'}^\alpha = \sum_{m_l} \prod_{n \in \Gamma} \langle j_l, \bar{u}_l | j_l, x_n \rangle \prod_{l \in \Gamma} \psi_{H_l}^\alpha(m_l) \langle h | \Gamma, j_l, x_n \rangle \]

- reduced intertwiner
- reduced basis elements
Semiclassical dynamics
Inhomogeneous Bianchi I dynamics

Let us evaluate the expectation value of the following operator on proper states:

\[ R \hat{H} = \frac{1}{\gamma^2} R \hat{H}_E \]

which in the classical limit described the dynamics of a spacetime made of a collection of local Bianchi I patches (BKL conjecture)

\[ H[N] = \frac{1}{\gamma^2} \sum_x V(x)N(x) \left[ \sqrt{ \frac{p^1 p^2}{p^3} c_1 c_2 } + \sqrt{ \frac{p^2 p^3}{p^1} c_2 c_3 } + \sqrt{ \frac{p^3 p^1}{p^2} c_3 c_1 } \right](x) \]
We describe each homogeneous patch via: states which already contains the loop added by one of the three terms into the Euclidean scalar constraint (dressed node):

\[ |n^z \rangle_R = \]
Action of the Euclidean scalar constraint:

\[ R \hat{H}_E^m[N] := N(n)C(m) \epsilon^{ijk} \text{Tr} \left[ R \hat{h}_{\alpha ij} R \hat{h}_{\alpha k}^{-1} \left[ R \hat{h}_{\alpha}^{(m)}, R \hat{V} \right] \right] \]

\[ R \hat{h}_{s z}^{(m)}|n^z\rangle_R = \]
volume is diagonal

\[ R_{s_{z}}^{(m)-1} R_{V}^{\gamma} R_{s_{z}}^{(m)} | n^z \rangle_R = (8\pi \gamma l_P^2)^{3/2} \sqrt{j_x j_y (j_z + \mu)} \]

\[
= (8\pi \gamma l_P^2)^{3/2} \sqrt{j_x j_y (j_z + \mu)}
\]
Loop trick:

\[ h_{\alpha[ij]} = (h_{\alpha ij} - h_{\alpha ji}) = \sum_{\tilde{m} \in (2\mathbb{N}+1)} (-)^{2m} \]

\[
\text{Tr} \left[ \left( R^\dagger \hat{h}(m) - R^\dagger \hat{h}(m) \right) R^\dagger \hat{h}(m) - 1 R^\dagger \hat{h}(m) \right] |n^z\rangle_R =
\]

\[
= (8\pi \gamma l_P^2)^{3/2} \sum_{\tilde{m}} \sum_{\mu, \mu'} \sqrt{j_x j_y (j_z + \mu)} \]

Alesci, Liegener, Ziepfel
\[
(8\pi \gamma l_p^2)^{3/2} \sum_{\tilde{m}} \sum_{\mu=\pm m} \sqrt{j_x j_y (j_z + \mu)} \ s(\mu) C_{\tilde{m}m\tilde{m}0}^{m\tilde{m}m}
\]

\[
(8\pi \gamma l_p^2)^{3/2} \sum_{k} \sum_{\tilde{m}} \sum_{\mu_x, \mu_y, \mu} \sqrt{j_x j_y (j_z + \mu)} \ s(\mu) C_{\tilde{m}m\tilde{m}0}^{mm\tilde{m}0}
\]
\[(8\pi\gamma l_P^2)^{3/2} \sum_k \sum_{\tilde{m}} \sum_{\mu_x,\mu_y,\mu} \sqrt{j_x j_y (j_z + \mu)} s(\mu) C_{mm\tilde{m}0}^{m}\left\{\begin{array}{c} k \\
\tilde{m} \\
j_z \\
j_y - \mu_y \quad m \\
j_x + \mu_x \quad m \\
j_x \end{array}\right\}\]
At other nodes the situation is similar:
The final result:

\[
\text{Tr} \left[ R \hat{h}(m) R \hat{h}_s(m) - 1 R \hat{h}(m) \right] |n^z \rangle_R = 
\]

\[
= \sum_{\mu_x',\mu_y',\mu_x,\mu_y = \pm m} H^m_{\mu_x,\mu_x',\mu_y,\mu_y'}(j_z, j_i')
\]
\[ H_{\mu_x \mu'_x \mu_y \mu'_y}^{m, j_x j'_x j_y j'_y}(j_z, j_l) = (8\pi \gamma l_P^2)^{3/2} \sum_{k} \sum_{\tilde{m}} \sum_{\mu = \pm m} \sqrt{j_x \cdot j_y (j_z + \mu)} \]

\[ S(\mu) C_{m m \tilde{m} 0}^{\mu m m} \left\{ \begin{array}{c} k \tilde{m} j_z \\ j_y - \mu_y m j_y \\ j_x + \mu_x m j_x \end{array} \right\} \left\{ \begin{array}{c} j'_x j_l + \mu'_y j_x + \mu_x j' \end{array} \right\} \left\{ \begin{array}{c} j_l + \mu'_x j'_y j_y - \mu_y \end{array} \right\} \]

(Euclidean) scalar constraint matrix elements
Semiclassical dynamics

Semiclassical state

\[ |\Psi_H \ n\rangle = \sum_{k} \frac{1}{\sqrt{3}} |\Psi_H \ n^k\rangle \quad k = x, y, z \]

\[ |\Psi_H \ n^z\rangle = \sum_{j_x,j_y,j_z} |\Psi_{H_{1z}}(j_z)\rangle \]

\[ \approx \text{Gaussians} \]

\[ \langle \Psi_H \ n | R \hat{H}^m_{E \square} [N] |\Psi_H \ n\rangle = \sum_{k} \langle \Psi_H \ n^k | R \hat{H}^m_{E \square} [N] |\Psi_H \ n^k\rangle \]
\[ \hat{R}_{E}^{m,z}[N]\hat{\Psi}_{H}^{n\hat{z}} = -N(n)C(m) \] \[ \sum_{j_{x},j_{y},j_{z},j_{l}} \sum_{\mu_{x}^{l},\mu_{y}^{l},\mu_{x},\mu_{y}} H_{\mu_{x}^{l}}^{m} j_{x} j_{x}^{l} j_{y} j_{y}^{l} (j_{z},j_{l}) \]
\[ \langle \Psi_H \ n^z | R^m \hat{H}_{\bigotimes}^{m, z} | \Psi_H \ n^z \rangle = \]

\[ = -N(n)C(m) \sum_{j_x,j_y,j_z,j_l} \sum_{\mu_x,\mu_y,\mu_x,\mu_y = \pm m} \]

\[ \begin{pmatrix}
\Psi_H_{ix}(j_x) \\
\Psi_H_{iy}(j_y - \mu_y)
\end{pmatrix}
\begin{pmatrix}
\Psi_H_{lx}(j_l + \mu_x') \\
\Psi_H_{ly}(j_l + \mu_y')
\end{pmatrix} \]

\[ = H^m_{j_x,j_y,j_z,j_l} \]

\[ \begin{pmatrix}
\Psi_H_{ix}(j_x) \\
\Psi_H_{iy}(j_y)
\end{pmatrix}
\begin{pmatrix}
\Psi_H_{lx}(j_l) \\
\Psi_H_{lx}(j_l)
\end{pmatrix} \]
Asymptotic n-j symbols:

\( j \gg m \)

\[
\left\{ \begin{array}{c}
  j_1 \\
  j_2 + \mu_2 j_1 + \mu_1 m \\
  j_3 
\end{array} \right. 
\]

\[
\sum_{k} \left\{ \begin{array}{c}
  k \\
  m j_z \\
  j_x + \mu_x m j_y \\
  j_y - \mu_y m j_x \\
  \end{array} \right. 
\]
$$\langle \Psi_H \ n^z \mid \hat{R} \hat{H}^m_{E \Box} \mid \Psi_H \ n^z \rangle \approx -N(n)C(m)(8\pi r^2_P)^{3/2} \sum_{\tilde{m}} \sum_{\mu=\mp m} \sum_{\mu_x, \mu_y=\pm m} \sum_{\mu_x', \mu_y'=\pm m}$$

$$\Psi^*_{H_{1z}}(j_z) \Psi_{H_{1z}}(j_z)$$

$$\sum_{j_x, j_y, j_z, j_l} \sqrt{j_x j_y (j_z + \mu)} s(\mu) C_{m m \tilde{m} 0}^{m m}$$

$$\Psi^*_{H_{1x}}(j_x + \mu_x) \Psi_{H_{1x}}(j_x) \quad \Psi^*_{H_{1y}}(j_y - \mu_y) \Psi_{H_{1y}}(j_y)$$

$$\Psi^*_{H_{1y}}(j_l + \mu_y') \Psi_{H_{1y}}(j_l) \quad \Psi^*_{H_{1x}}(j_l + \mu_x') \Psi_{H_{1x}}(j_l)$$
By an expansion around the centers of the Gaussians:

$$\langle \Psi_H \ n^z | R^m \ H^{\mu_y} \ E^{\mu_x} | \Psi_H \ n^z \rangle \approx -N(n)C(m)(8\pi\gamma l_P^2)^{3/2}$$

$$\sum_{\tilde{m}} \sum_{\mu=\pm m} \sum_{\mu_x, \mu_y=\pm m} \sum_{\mu_x', \mu_y'=\pm m} \sqrt{\tilde{m}_x \tilde{m}_y (\tilde{m}_z + \mu)} \ s(\mu) C_{mm \tilde{m}}^{mm} \tilde{m}_0$$

$$\alpha = 1/(\tilde{j}^k) \quad k > 1$$

classical values
For $m=1/2$, we can rewrite the expectation value as a sum over SU(2) elements (in the fundamental representation):

$$\sum_{\mu=\pm1/2} R_{i\mu'}^\dagger \mu e^{-i\theta \mu} R_{i\mu''}^{-1} = (e^{-\frac{i}{2} \theta \sigma_i})_{\mu'\mu''} \equiv h_{\mu'\mu''}(\theta_i)$$

Pauli matrices

$$\langle \Psi_H \ n \hat{R} \hat{H}^{1/2} \hat{E} \Psi_H \ n \hat{z} \rangle \approx -N(n)(8\pi\gamma l_P^2)^{1/2}$$

$$\frac{2i}{3\sqrt{3}} \sum_{\mu=\pm1/2} \sqrt{\hat{j}_x \hat{j}_y (\hat{j}_z + \mu)} \ s(\mu)$$
the explicit expression reads:

\[
\langle \Psi_H^z | R^H_{E \square}^{1/2} | \Psi_H^z \rangle \approx -\frac{2i}{9} N(n)(8\pi \gamma l_P^2)^{1/2} \sum_{\mu = \pm 1/2} \sqrt{\bar{j}_x \bar{j}_y (\bar{j}_z + \mu)} \ s(\mu) \ 
 T_r \{ \sigma_3 h(\theta_{l_x}) h(\theta_{l_y}) h(\theta_{l_x}) h(\theta_{l_y}) \} 
\]

\[
\langle \Psi_H^z | R^H_{E \square}^{1/2} | \Psi_H^z \rangle \approx -\frac{2i}{9} N(n)(8\pi \gamma l_P^2)^{1/2} \sqrt{\frac{\bar{j}_x \bar{j}_y}{\bar{j}_z}} \ \sin (\epsilon l_x \bar{c}_x) \ \sin (\epsilon l_y \bar{c}_y) 
\]
The expectation value of the scalar constraint is obtained by:

- summing the analogous result for \( k = x, y \)
- multiplying times \( 1/\gamma^2 \)
- introducing reduced variables via the relation

\[
E'_i = 8\pi \gamma l^2 \tilde{p} j_i \sim \tilde{p}^i \delta_i^2
\]

Let us assume that \( \epsilon \)'s and \( \delta \)'s are equal, the resulting expression becomes for \( \epsilon \to 0 \):

\[
\langle R \hat{H}^{1/2} \rangle_n \to \frac{2}{9} \frac{1}{\gamma^2} N(n) \delta \epsilon^2 \left( \sqrt{\frac{p^x}{\bar{p}^x}} \tilde{c}_x \tilde{c}_y + \sqrt{\frac{p^y}{\bar{p}^y}} \tilde{c}_y \tilde{c}_z + \sqrt{\frac{p^z}{\bar{p}^z}} \tilde{c}_z \tilde{c}_x \right)
\]

and it coincides with the classical limit by identifying

\[
V(n) = \frac{2}{9} \delta \epsilon^2
\]
Generically, we can have different values for $\varepsilon$'s, but we must assume:

$$
\delta_x = \frac{9 V(n)}{2\varepsilon_l x \sqrt{\varepsilon_l y \varepsilon_l z}}, \quad \delta_y = \frac{9 V(n)}{2\varepsilon_l y \sqrt{\varepsilon_l z \varepsilon_l x}}, \quad \delta_z = \frac{9 V(n)}{2\varepsilon_l z \sqrt{\varepsilon_l x \varepsilon_l y}}
$$

And if we do not take the limit $\varepsilon \to 0$ we get

$$
\left\langle R \hat{H}^{1/2} \right\rangle_n \approx \frac{1}{\gamma^2} N(n) V(n) \left( \sqrt{\frac{p^x}{p^z}} \frac{\sin (\varepsilon_l x \bar{c}_x)}{\varepsilon_l x} \frac{\sin (\varepsilon_l y \bar{c}_y)}{\varepsilon_l y} + \right.
$$

$$
+ \sqrt{\frac{p^y}{p^x}} \frac{\sin (\varepsilon_l y \bar{c}_y)}{\varepsilon_l y} \frac{\sin (\varepsilon_l z \bar{c}_z)}{\varepsilon_l z} + \sqrt{\frac{p^z}{p^x}} \frac{\sin (\varepsilon_l z \bar{c}_z)}{\varepsilon_l z} \frac{\sin (\varepsilon_l x \bar{c}_x)}{\varepsilon_l x} \left. \right)
$$

This expression coincides with the analogous one found in LQC if

$$
\varepsilon_l i = \mu_0, \overline{\mu}_i
$$

Ashtekar, Wilson-Ewing, Martin-Benito, Mena-Marugan, Pawlowski
Perspectives
It's time to do physics.....

**Inhomogeneous Bianchi I model:**

- study of corrections: new effects from next-to-the-leading-order term in the large spin expansion?

- realistic description for the early Universe: 6 valence nodes.

- quantum fields on a quantum space, loop quantization in action: role of the fundamental fields composing the thermal bath (phenomenological implications, comparison with LQC).

**Full theory with a diagonal metric tensor:**

- proper characterization of the dynamics (computable matrix elements!)

- test of BKL conjecture on a QUANTUM LEVEL (role of inhomogeneities).
Thank you!!!
Semiclassical limit