Representations of Lie groups
Four lectures for physicists, Erlangen, October 2013

Bent Ørsted, Aarhus University

October 2, 2013

Overview and motivation

In these lectures we shall explain some of the basic mathematical theory of group representations, with connections to the way these arise as symmetries in quantum mechanics. Understanding the symmetries of a physical system is (as for mathematical structures) an important part of our description of the system; this is for example the case for special relativity, where the Poincaré group acts by symmetries, or for the classification of quarks, mesons, and baryons, where the compact unitary groups play a similar role. We base some of the presentation on [11] and [12] for the the mathematical structure of quantum mechanics and the way Lie groups enter; for the mathematical background and in particular harmonic analysis on locally compact groups we refer to [3]. See also the lecture notes [1] and [13] for the role played by Mackey's Imprimitivity Theorem. At the end we will mention some more advanced topics from representation theory of semisimple Lie groups, namely the so-called *minimal representations*, see [6], [7], and [8] as well as [5]. Among these we find the metaplectic representation (= the oscillator representation) and the representation corresponding to the quantized Kepler problem, i.e. the Hydrogen atom.

Lecture I: Locally compact groups and Lie groups

We start by the general notions of locally compact groups and Lie groups, and study their role as symmetries of Hilbert spaces, formalized by the definition of *unitary group representations*. The text we follow here is [3] (p. 67–73, p. 84, p. 90–93), and also [1] (p. 3, p. 30, p. 32), and [11] (p. 5, p. 76–85); the topics will be:

- Basic notions of locally compact groups, Lie groups, and representations
- Examples of groups and representations

- Symmetries of physical systems as representations

Exercise 1. Exercise 7.1 p. 31 in [1].

Lecture II: The Imprimitivity Theorem with applications

This theorem is one of the cornerstones of the theory of unitary group representations in the general setting; we shall follow [3] and [1] (p. 37, p. 42, p. 50 and the key Theorem 11.5) - see also [14] - and treat

- Homogeneous spaces and induced representations
- Projection-valued measures
- The imprimitivity Theorem
- Representations of semi-direct product groups
- Examples

Exercise 2. One (perhaps less well-known) application of The Imprimitivity Theorem is to the following version of abstract scattering theory - discuss how a system imprimitivity arises in a natural way from the scattering axioms below. Hint: Consider the closed subspace

$$\mathcal{D}_{t_0} = \overline{\bigcup_{t \leq t_0} U(t)\mathcal{D}}$$

and the orthogonal projection $P((-\infty, t_0])$ onto this space. Then P gives rise to a projection-valued measure, transforming in a certain way under the one-parameter group $U(t)$, namely for every Borel set $B \subset \mathbb{R}$ we have

$$U(t)P(E)U(-t) = P(t + E).$$

The following is the abstract scattering theory as formulated by Lax and Phillips in [9], forming the basis of concrete scattering theory both for wave equations and automorphic functions.

Theorem 3. *Let $\mathcal{H}$ be a separable Hilbert space, $U(t)$ a one-parameter group of unitary operators, $\mathcal{D}$ a closed subspace satisfying*

1. $U(t)\mathcal{D} \subset \mathcal{D}$ *for* $t \leq 0$
2. $\bigcap_{t<0} U(t)\mathcal{D} = \{0\}$
3. $\overline{\bigcup_{t \in \mathbb{R}} U(t)\mathcal{D}} = \mathcal{H}$

Then $\mathcal{H}$ can be represented as a space $L^2(\mathbb{R}, \mathcal{N})$, where $\mathcal{N}$ is an auxiliary Hilbert space, such that

1. $U(t)$ corresponds to right tranlation by t

2. the correspondence between $\mathcal{H}$ and $L^2(\mathbb{R}, \mathcal{N})$ is unitary

3. $\mathcal{D}$ correponds to $L^2(\mathbb{R}_-, \mathcal{N})$ where $\mathbb{R}_- = (-\infty, 0)$.

This is called *the incoming translation representation* of the unitary group; similarly we talk about *the outgoing translation representation* if there is another closed subspace $\mathcal{D}_+$ with the same properties for the reversed inequalities in t; this will then represent $\mathcal{H}$ as $L^2(\mathbb{R}, \mathcal{N}_+)$ with $\mathcal{D}_+$ corresponding to $L^2(\mathbb{R}_+, \mathcal{N}_+)$. Since we can identify $\mathcal{N}$ and $\mathcal{N}_+$ (using e.g. the Fourier transform) we may now define the *scattering operator*

$$S : L^2(\mathbb{R}, \mathcal{N}) \mapsto L^2(\mathbb{R}, \mathcal{N}) \tag{1}$$

by taking the incoming representer to the outgoing; then we have

Proposition 4. *S is a unitary operator, in particular onto, commuting with translation.*

The point we want to make here, is that the Lax-Phillips Translation Representation Theorem may be thought of as an instance of the Imprimitivity Theorem. Now scattering theory deals (among other things) with the Fourier transform of the scattering operator S. This operator is of major importance in physics.

Lecture III: Representations of the Poincaré group

Here we use the Mackey machine to find all (up to unitary equivalence) unitary irreducible representations of the Poincaré group (modulo known lists of representations of certain subgroups); and we find the list of physically relevant representations, essentially labeled by mass and spin, corresponding to the relativistic description of elementary particles. See [1] (p. 62–65, and the comments p. 66), [13] (p. 9), [12] (p. 189–193, p. 202–217), [11] (p. 56–67). We will discuss

- The structure of orbits and little groups
- Realizations of the unitary irreducible representations
- Wave equations for relativistic particles
- Conformal symmetries for mass zero particles

Exercise 5. Exercise 15.1 p. 62 in [1]

Lecture IV: Minimal representations

Here we refer to [5] as well as the special case of $G = O(p, q)$ as treated in [6], [7], and [8]. See also [13] (p. 19) for the metaplectic representation as well as [4] and [10]. One point to be made is that not all unitary irreducible representations of semisimple Lie groups are induced representations; some are quite isolated and need special constructions. Even some important ones from the point of physics arise here, and we shall look at

- The metaplectic representation and the connection to the harmonic oscillator
- The Kepler problem and the wave equation: A representation of the conformal group
- Minimal representations for $O(p, q)$ and isomophism groups of tube domains.

Time permitting, we shall also mention a way of connecting by deformation the metaplectic representation and the Kepler representation, see [2].

References

[1] Erik van den Ban: *Applications of representation theory in classical quantum mechanics*, Lecture Notes for the MRI Spring School Lie groups in Analysis, Geometry and Mechanics. Utrecht, June, 2004. Updated version, 13/6 - 2004. http://www.staff.science.uu.nl/~ban00101/lecnotes/repq.pdf

[2] Ben Saïd, Salem; Kobayashi, Toshiyuki; Ørsted, Bent: Laguerre semigroup and Dunkl operators, in *Compos. Math.* **148** (2012), no. 4, 1265–1336.

[3] Folland, Gerald B. : *A course in abstract harmonic analysis*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995. x+276 pp.

[4] Folland, Gerald B. : *Harmonic analysis in phase space*, Annals of Mathematics Studies, **122**. Princeton University Press, Princeton, NJ, 1989. x+277 pp.

[5] Hilgert, Joachim; Kobayashi, Toshiyuki; Möllers, Jan; Ørsted, Bent: Fock model and Segal-Bargmann transform for minimal representations of Hermitian Lie groups, in *J. Funct. Anal.* **263** (2012), no. 11, 3492–3563.

[6] Kobayashi, Toshiyuki; Ørsted, Bent: Analysis on the minimal representation of O(p,q). I. Realization via conformal geometry. *Adv. Math.* **180** (2003), no. 2, 486–512.

[7] Kobayashi, Toshiyuki; Ørsted, Bent: Analysis on the minimal representation of O(p,q). II. Branching laws, in *Adv. Math.* **180** (2003), no. 2, 513–550.

[8] Kobayashi, Toshiyuki; Ørsted, Bent: Analysis on the minimal representation of O(p,q). III. Ultrahyperbolic equations on $\mathbb{R}^{p-1,q-1}$, in *Adv. Math.* **180** (2003), no. 2, 551–595. Also on http://de.arxiv.org/pdf/math/0111086.pdf

[9] Peter D. Lax and Ralph Phillips: Scattering Theory for Automorphic Functions, Annals of Mathematics Studies **87**, Princeton 1976.

[10] Lion, Gérard; Vergne, Michéle: *The Weil representation, Maslov index and theta series.* Progress in Mathematics, **6**. Birkhäuser, Boston, Mass., 1980. vi+337 pp.

[11] D. J. Simms: *Lie Groups and Quantum Mechanics*, Lecture Notes in Mathematics **52**, Springer 1968.

[12] V. S. Varadarajan: *Geometry of Quantum Theory, Vol II*, The University Series in Higher Mathematics, Van Nostrand 1970.

[13] V. S. Varadarajan: *George Mackey and his work on representation theory and foundations of physics*, Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey, 417–446, Contemp. Math., 449, Amer. Math. Soc., Providence, RI, 2008. Also at http://www.math.ucla.edu/˜vsv/mackey.pdf

[14] Bent Ørsted: Induced representations and a new proof of the imprimitivity theorem, in *J. Funct. Anal.* **31** (1979), no. 3, 355–359.