1 What is (conformal) QFT all about?

Session 1.1

I start with some heuristic considerations, in order to emphasize how Quantum Field Theory is rooted in some of the most fundamental principles of Physics, and also in order to indicate the physical meaning of the quantities featuring in QFT.

In Quantum Theory, the time evolution of any dynamical quantity is given by the adjoint action of the Hamiltonian:

$$\phi(t) = e^{itH}\phi(0)e^{-itH}, \qquad \dot\phi = i[H,\phi].$$

As an operator on a Hilbert space, the Hamiltonian has the meaning of "energy", ie, its spectrum gives the possible energies of a system.

In Field Theory, energy is an integral over a density field, eg $\frac{1}{2}(\vec{E}^2+\vec{B}^2)$ in Maxwell theory, where $\vec{E}(t,\vec{x})$ and $\vec{B}(t,\vec{x})$ are the electric and magnetic fields. Conservation of the energy is expressed by a continuity equation

$$\partial_t\rho + \vec\nabla\vec{S} = 0,$$

where $\vec{S}$ is the energy flow, eg, $\vec{S} = \vec{E}\times\vec{B}$ in Maxwell theory.

In Special Relativity, energy is the zero component of a Lorentz four vector P^μ, $\vec{P} =$ momentum. The relativistic conservation law reads

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad x = (t,\vec{x})$$

where $T^{\mu\nu}$ is called the stress-energy tensor field (SET)[1].

1. Combining all these principles, one arrives at QFT, with a distinguished conserved "operator valued quantum field" $T^{\mu\nu}(x)$, such that

   $$P^\mu = \int_t d^3x\, T^{\mu 0}(t,\vec{x})$$

   are independent of t and generate the translations in time and space:

   $$U(a)\phi(x)U(a)^* = \phi(x+a), \qquad i[P_\mu,\phi(t,x)] = \partial_\mu\phi(t,x) \tag{1}$$

   for all local fields of the theory, where $U(a) = e^{ia^\mu P_\mu}$.

2. If (and only if) $T^{\mu\nu}$ is symmetric, then also $J^\nu_{\kappa\lambda} = x_\kappa T^\nu_\lambda - x_\lambda T^\nu_\kappa$ is conserved ($\partial_\nu J^\nu_{\kappa\lambda} = T_{\kappa\lambda} - T_{\lambda\kappa}$), and

   $$M_{\kappa\lambda} = \int_{x^0=t} J^0_{\kappa\lambda}(x)\; d^sx$$

   is independent of t.

[1] This name comes from elasticity theory. The German "energy-momentum tensor" would be more appropriate.

3. If (and only if) $T^{\mu\nu}$ is also traceless, $T = T^\kappa_\kappa = 0$, then also $J^\nu = x_\kappa T^{\kappa\nu}$ conserved ($\partial_\nu J^\nu = T^\kappa_\kappa$), and
$$D = \int_{x^0=t} J^0(x)\, d^sx$$
is independent of t. In this case, also $J^\nu_\mu = 2x_\mu x_\kappa T^{\kappa\nu} - x^2 T^\nu_\mu$ is conserved ($\partial_\nu J^\nu_\mu = 2x^\kappa(T_{\kappa\mu} - T_{\mu\kappa}) + x_\mu T^\kappa_\kappa$), and
$$K_\mu = \int_{x^0=t} J^0_\mu(x)\, d^sx$$
is independent of t.

4. Since (1) holds also for $\phi = T^{\mu\nu}$, one can compute the commutators:
$$\begin{aligned} i[P_\mu, P_\nu] &= 0, \\ i[P_\mu, M_{\kappa\lambda}] &= \eta_{\mu\lambda} P_\kappa - \eta_{\mu\kappa} P_\lambda, \\ i[P_\mu, D] &= -P_\mu, \\ i[P_\mu, K_\nu] &= -2\eta_{\mu\nu} D + 2M_{\mu\nu}. \end{aligned}$$

5. The components M_{i0} can be recognized as the angular momentum (integral over $\vec{x}\times$ momentum density). Thus, in a relativistic theory, $M_{\kappa\lambda}$ should generate the Lorentz transformations: $U(\Lambda)\phi_{\dots\mu\dots}(x)U(\Lambda)^* = (\cdots(\Lambda^{-1})^{\mu'}_\mu\cdots)\phi_{\dots\mu'\dots}(\Lambda x)$,
$$i[M_{\kappa\lambda}, \phi_{\dots\mu\dots}] = (x_\kappa\partial_\lambda - x_\lambda\partial_\kappa)\phi_{\dots\mu\dots} + \sum(\eta_{\kappa\mu}\phi_{\dots\lambda\dots} - \eta_{\lambda\mu}\phi_{\dots\kappa\dots}), \qquad (2)$$
where the fields may be vector or tensor fields. From this, one can also compute the commutators with $M_{\kappa\lambda}$, using (2):
$$\begin{aligned} i[M_{\kappa\lambda}, P_\mu] &= \eta_{\kappa\mu} P_\lambda - \eta_{\lambda\mu} P_\kappa, \\ i[M_{\kappa\lambda}, D] &= 0, \\ i[M_{\kappa\lambda}, M_{\mu\nu}] &= \eta_{\kappa\mu} M_{\lambda\nu} \pm \dots, \\ i[M_{\kappa\lambda}, K_\mu] &= \eta_{\kappa\mu} K_\lambda - \eta_{\lambda\mu} K_\kappa. \end{aligned}$$

6. A QFT is called conformal (CFT), if also the commutation relations
$$i[D, K_\mu] = -K_\mu,$$
$$i[K_\mu, K_\nu] = 0,$$
hold, which turn them into the Lie algebra of the conformal group $G = SO(2,n)_0$ (n = dimension of spacetime), and if D acts on the fields as the generator of the scale transformations
$$x \to \lambda x,$$
and K_μ as the generator of the special conformal transformations
$$x \to \frac{x - x^2\, b}{1 - 2x\cdot b + b^2\, x^2}.$$
We assume that the generators integrate to a (projective) unitary representation U.

7. How do fields transform under D and K_μ? Consider the point $x = 0$ which is fixed under Lorentz transformations, scaling and special transformation. Thus, these generators act on the space spanned by "operators" $\phi(0)$. One may diagonalize the commuting adjoint actions of D and M: $i[D, \phi(0)] = d\phi(0)$, $i[M_{\mu\nu}, \phi_A(0)] = \pi(m_{\mu\nu})^B_A \phi_B(0)$, and then, by using (1) and the commutator between P and D, find
$$i[D, \phi(x)] = (x^\kappa \partial_\kappa + d)\phi(x). \tag{3}$$
d is called the scaling dimension of the field ϕ: in integrated form, the law reads
$$e^{itD}\phi(x)e^{-itD} = \lambda^d \phi(\lambda x) \qquad (\lambda = e^t).$$
The commutator between D and K implies that K lowers the eigenvalue d by one unit. Since (by reasons to become clear later) d must not be negative, there must be fields such that $i[K, \phi(0)] = 0$. These are called "quasiprimary fields". Again, using (1) and the conformal Lie algebra, one finally gets
$$i[K_\mu, \phi(x)_{\dots\nu\dots}] = \big(2x_\mu(x\partial) + x^2\partial_\mu + 2dx_\mu\big)\phi(x)_{\dots\nu\dots} + 2\sum(\eta_{\mu\nu}x^\kappa - \delta^\kappa_\mu x_\nu)\phi_{\dots\kappa\dots}. \tag{4}$$
Thus, for quasiprimary fields all infinitesimal transformation laws are fixed by the nature of the field as a Lorentz tensor (two quantum numbers $j_1, j_2 \in \frac{1}{2}\mathbb{N}_0$ specifying the matrix representation $i[M_{\mu\nu}, \cdot] = \pi(m_{\mu\nu})$ of the Lorentz Lie algebra on $\phi(0)$), and by the scaling dimension d. The derivative of a quasiprimary field is no longer quasiprimary, because the commutator with K_μ will also contain the primitive field.

8. A representation is a Hilbert space where the fields and symmetries act, satisfying all the above commutation relations. A repn has "positive energy" if the generator P^0 is positive. A vacuum repn is a PER with a unique U-invariant vector Ω. The transformation laws then uniquely determine the two-point functions $(\phi(x)\Omega, \phi(y)\Omega)$ (and strongly constrain higher correlation functions). These turn out to be distributions. Therefore also the fields cannot be operator valued functions, but must be operator valued (OPV) distributions.

   By integrating the two-point distribution with test functions $\overline{f(x)}f(y)$, one obtains the norm square $\|\phi(f)\Omega\|^2$. In order that this is positive for all f, the scaling dimension must satisfy a certain unitarity bound $d \geq d_{\min}(j_1, j_2)$. In particular, the SET has dimension $d = 4$ and $j_1 = j_2 = 1$. The massless Klein-Gordon field has $d = 1$, $j_1 = j_2 = 0$, thee free electromagnetic field $d = 2$, $(j_1, j_2) = (0, 1) \oplus (1, 0)$.

We now have the essential features of conformal QFT: a Hilbert space $\mathcal{H}$; fields as OPV distributions $f \mapsto \phi(f) = \phi(\overline{f})^*$ over spacetime, among them a distinguished field: the SET; an automorphic transformation law $\alpha_g : \phi(f) \mapsto \phi(f_g)$ of the fields under the conformal group G acting geometrically on its arguments x; a unitary projective representation U of G on $\mathcal{H}$ whose generators are integrals over (components of) the SET, and which implements the conformal symmetry: $\alpha_g(\cdot) = U(g) \cdot U(g)^*$. In addition, one has to require causality[2]: Fields at spacelike separation must be commuting operators, in order to exclude acausal effects caused by a quantum measurement.

[2]Most commonly, the term "locality" is used instead.

The question is: what other fields may there be present in a CFT? What are their quantum numbers (d, j_1, j_2), what are their algebraic relations, what are the vacuum correlation functions $(\Omega, \phi_1(x_1) \dots \phi_n(x_n)\Omega)$?

Can one classify such theories?
In four-dimensional (4D) spacetime, this is an ambitious, and rather hopeless program.

2 CFT in two dimensions

Session 1.2

In two dimensions (2D), enormous progress has been made in the last three decades.

The physical interest in 2D CFT is manifold. Some of the models (cf Sect. 5) have well-known realizations as scaling limits of critical Stat Mech systems (cf Sect. 4). Current algebras are used to describe the quantum Hall effect. In String Theory, CFT are understood as describing the "internal degrees of freedom" of the two-dimensional string surface swept out in a 10-dimensional spacetime, where the specific CFT model distinguishes different "compactifications" from 10 to the "physical" 4 dimensions of our "low-energy experience".

1. The conformal symmetry simplifies drastically. The Lie algebra splits into to copies of $sl(2, \mathbb{R})$:
$$i[P, D] = -P, \qquad i[K, D] = K, \qquad i[P, K] = -2D.$$
Here, $P = P_\pm = \frac{1}{2}(P_0 \pm P_1)$ are the translations in lightlike directions, $D = D_\pm = \frac{1}{2}(D \pm M_{01})$ are the independent scale transformations of the lightcone coordinates $x^\pm = t \pm x$ (such that $D_+ - D_-$ is a Lorentz transformation), and $K = K_\pm = \frac{1}{2}(K_0 \mp K_1)$ are the generators for the transformation
$$x \mapsto \frac{x}{1 - bx}.$$
The conformal group $G = G_+ \times G_-$ consists of the fractional linear transformations
$$x \mapsto \frac{ax + b}{cx + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$
Under the Cayley transformation $z := \frac{1+ix}{1-ix} \in S^1$, this becomes
$$z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \qquad \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1, 1).$$
This group is called the Möbius group. Positivity of P^0 in every Lorentz frame is equivalent to positivity of P_+ and P_-, and this in turn is equivalent to positivity of K_+ and K_-, because the one-parameter subgroups generated by P and K are inner conjugate (by the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$) in the Möbius group.

2. Now consider the SET. In 2D, its scaling dimension is $d = 2$. Being symmetric and traceless, it has only two independent components $T^{00} = T^{11}$ and $T^{01} = T^{10}$. Being conserved, one finds that $T^\pm = \frac{1}{2}(T^{00} \mp T^{01})$ satisfy

$$(\partial_0 \mp \partial_1)T^\pm = 0,$$

hence $T^\pm$ actually depend only on the lightcone coordinates $x^\pm$. Such fields are called "chiral", $\phi^+(x^+)$ are called "leftmovers", $\phi^-(x^-)$ are called "rightmovers". For both copies $T \equiv T^\pm$, we have (suppressing the sign) $P = \int T(x)dx$, $D = \int xT(x)dx$, $K = \int x^2 T(x)dx$, transforming chiral fields according to

$$i[P, \phi(x)] = \partial\phi(x), \quad i[D, \phi(x)] = (x\partial + h)\phi(x), \quad i[D, \phi(x)] = (x^2\partial + 2hx)\phi(x).$$

The parameter h is the chiral scaling dimension, $h = 2$ for the SET.
Because two fields at distance $x = x_1 - x_2$ must commute whenever $x^2 = x^+x^- < 0$, chiral fields ϕ^+ must commute with ϕ^- unconditionally, and $\phi^+(x_1^+)$ must commute with $\phi^+(x_2^+)$ whenever $x_1^+ \neq x_2^+$. Therefore, the commutator can only be a sum of derivatives of δ-functions times other fields.
Making an ansatz for the commutator $[T^+, T^+]$, the coefficient fields turn out to be determined by the commutation relations with the generators [14], except for the term δ''' (again suppressing the sign):

$$[T(x), T(y)] = i\,(T(x) + T(y))\,\delta'(x-y) - \frac{ic}{24\pi}\,\delta'''(x-y)\cdot 1. \tag{5}$$

Under the Cayley transformation, $\widehat{T}(z) := -(dx/dz)^2 T(x(z))$, $\widehat{T}$ is an OPV distribution on the circle without the point -1. It can be shown that it extends periodically to the entire circle. One may then take its Fourier decomposition:

$$\widehat{T}(z) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} L_n z^{-2-n},$$

and rewrite the commutation relation as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \qquad L_n^* = L_{-n}. \tag{6}$$

Re-expressed on the real line,

$$L_n = \frac{1}{2}\int (1-ix)^{1-n}(1+ix)^{1+n}T(x)dx.$$

This is the Virasoro algebra. Physically, it expresses causal commutativity of the chiral SET along with its physical role of being the generator of conformal transformations. Mathematically, it is a central extension of the Lie algebra of diffeomorphisms $\mathrm{Diff}(S^1)$ ($\delta_n(z) = iz^{n+1}$).)

The algebra is universal, only the coefficient c, called "central charge" is model-specific.

In particular,

$$L_0 = \frac{1}{2}(P + K), \quad L_{\pm 1} = \frac{1}{2}(P - K) \pm iD.$$

L_0 is the generator of the rotations of the circle, $z \mapsto e^{i\alpha}z$. Positivity of P, and hence of K, implies positivity of L_0. In fact, both are equivalent because L_0 is inner conjugate (by the dilations) to $\frac{1}{2}(e^tP + e^{-t}K)$ for any t.

3. A chiral CFT is a QFT with chiral fields = OPV distributions on the real line, with chiral quasiprimary fields satisfying causal commutativity at different points and transforming under Möbius transformations as displayed above, among them the SET $T(x)$ with $h = 2$ as a density for the Möbius generators P, D, K. The transformation law can be equivalently written as

$$i[L_m, \phi(x)] = D^h_m\phi(x) \equiv \frac{1}{2}\Big(\frac{1+ix}{1-ix}\Big)^m \cdot \big((1+x^2)\partial + (x+im)h\big)\phi(x) \qquad (7)$$

for $m = 0, \pm 1$. A field satisfying the same commutation relations for all $m \in \mathbb{Z}$, is called primary field. Equivalently, a primary field satisfies

$$i[T(f), \phi(x)] = f(x)\,\phi'(x) + h\,f'(x)\,\phi(x) \qquad (8)$$

(whereas for quasiprimary fields there may be further terms involving fields of lower dimension.) (8) integrates to the adjoint action of $U(\gamma) = e^{iT(f)}$ on primary fields:

$$U(\gamma)\phi(x)U(\gamma)^* = \Big(\frac{d\gamma(x)}{dx}\Big)^h \cdot \phi(\gamma(x)),$$

where $\gamma = \gamma_1$ is obtained from f by integrating $\partial_t\gamma_t(x) = f(\gamma_t(x))$, $\gamma_0(x) = x$ to a one-parameter group of diffeomorphisms γ_t. The map $\gamma \mapsto U(\gamma)$ is then a projective unitary repn of Diff(S^1).

Notice that because of the central charge, the SET is quasiprimary but NOT primary; in fact, the central term $-\frac{ic}{24\pi}f'''(x)$ in the infinitesimal transformation law (= commutator $i[T(f), T(x)]$) integrates to

$$U(\gamma)T(x)U(\gamma)^* = \Big(\frac{d\gamma(x)}{dx}\Big)^2 \cdot T(\gamma(x)) - \frac{c}{24\pi}\cdot\frac{D\gamma}{Dx}\cdot 1,$$

where $\frac{D\gamma}{Dx} = \frac{\gamma'''}{\gamma'} - \frac{3}{2}(\frac{\gamma''}{\gamma'})^2$ is the Schwarz derivative.

Necessarily, $h \in \mathbb{N}$ for chiral fields, and the Cayley transformed fields again extend to the circle and may be described in terms of their Fourier components. The commutation relations then become

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n}$$

($m = 0, \pm 1$ for quasiprimary fields, all m for primary fields).

A vacuum representation is a representation of the field algebra with a vector Ω invariant under the Möbius group: $L_m\Omega = 0$ for $m = 0, \pm 1$. This implies $L_m\Omega = 0$ for $m > 0$ because otherwise it is an eigenvector of L_0 with eigenvalue $-m$, whereas L_0 is a positive operator. Notice that the central term in the Virasoro algebra excludes the existence of a diffeomorphism invariant vector.

4. A 2D CFT is a QFT on 2D Minkowski spacetime = Cartesian product of two chiral axes $\mathbb{R}$, with fields $\phi(t, x) = \phi(x^+, x^-)$ transforming covariantly wrt both chiral Möbius groups, with two chiral scaling dimensions (h_+, h_-) such that $d = h_+ + h_-$ is the scaling dimension, and $s = h_+ - h_-$ is the Lorentz spin (more precisely: the helicity). The fields must commute among each other if $x^+x^- < 0$ for $x = x_1 - x_2$.

3 The quantization of c and h

Session 2.1

The two-point functions of 2D conformal fields are

$$\sim \Big(\frac{-i}{x_+ - y_+ - i\varepsilon}\Big)^{2h_+} \cdot \Big(\frac{-i}{x_- - y_- - i\varepsilon}\Big)^{2h_-}$$

where "$-i\varepsilon$" stands for the distributional definition as the boundary limit from $Im(y) > Im(x)$. Under the Cayley transform, this becomes

$$\sim \frac{1}{(z_+ - w_+)^{2h_+}} \frac{1}{(z_+ - w_+)^{2h_-}},$$

where the branch cut is defined by "radial ordering" as the distributional limit taken from $|w| < |z|$. Such fields will therefore in general NOT extend periodically to $S^1 \times S^1$, but only to a covering space.

Causal commutativity requires that $h_+ - h_- \in \mathbb{N}$ (so that the complex phases from the branch cuts cancel at spacelike distance). Therefore, the relevant covering space may be viewed as $S^1 \times \mathbb{R}$, where $\mathbb{R}$ is the time direction.

The adjoint action of $L^\pm_n$ on a primary field, expressing its diffeo covariance, is as in (7)

$$[L^+_m, \phi(x_+, \cdot)] = D^{h_+}_m \phi(x_+, \cdot), \quad [L^-_m, \phi(\cdot, x_-)] = D^{h_-}_m \phi(\cdot, x_-). \tag{9}$$

Let us look at the vector-valued distribution (wrt y_+) $\phi(y_+, \cdot)\Omega$. It is the boundary value of a function of y_+ in the upper half-plane. (This follows from positivity of the energy: $P_+ > 0$ implies that $e^{iy_+P_+}$ is bounded for $Im(y_+)$.) In particular, $\phi(y_+ = i, \cdot)\Omega$ is a vector $|h\rangle$ in the Hilbert space. From the commutation relations with the Virasoro algebra, it follows that $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0$ for all $n > 0$, ie $|h\rangle$ is a ground state for the conformal Hamiltonian L_0.

This provides a "simple" classification scheme [7] for the possible values of h and c. Namely, by applying arbitrary products of L_n and using the Virasoro commutators, one can write every resulting vector as a linear combination of vectors $L_{-k_s} \cdots L_{-k_1}|h\rangle$ with $k_s \geq \cdots \geq k_1 > 0$. The linear span of these symbols is called a Verma module (carrying a representation of the Virasoro algebra), but the corresponding vectors in the QFT Hilbert space need not be linearly independent. Imposing also the reality condition $L^*_{-k} = L_k$, the inner product of any two vectors in the Verma module can be computed recursively as a polynomial in h and c. Example: $\|L_{-k}|h\rangle\|^2 = \langle h|[L_k, L_{-k}]|h\rangle = (2kh + \frac{k(k^2-1)}{12}c) \cdot \langle h|h\rangle$.

This inner product on the Verma module is obviously not positive definite in general. One directly sees from the example (taking $k = 1$ and $k \to \infty$) that both h and c must be nonnegative. The systematic evaluation of positive-definiteness, giving a classification of the admissible values of h and c [7] yields the following result. If $c < 0$, the inner product is indefinite. If $c = 0$, the only possible value is $h = 0$, and all vectors except $|h\rangle = \Omega$ are null vectors. If $c > 1$, the inner product is positive-definite for all values of h. If $c = 1$, it is semi-definite for quantized values $h \in \frac{1}{2}\mathbb{Z}^2$, and positive-definite for all other h. Finally, for $0 < c < 1$, it is semi-definite only if

$$c = 1 - \frac{6}{(n+2)(n+3)}, \qquad (n \in \mathbb{N}),$$

and if

$$h = h_{pq} = \frac{[(n+3)p-(n+2)q]^2-1}{4(n+2)(n+3)}, \qquad (p=1,\dots n+1, q=1\dots n+2).$$

For all other values it is indefinite.

> This is a remarkable result. It is a classification of possible QFTs "from nothing" except symmetry requirements. Neither can the central charge assume every positive value, nor can primary bulk fields have arbitrary scaling dimensions. Moreover, for $c < 1$, there are necessarily null vectors in the Verma module.

The presence of null vectors has important consequences [1]: If $P(L_{-k})|h\rangle$ is a null vector in the Verma module (P some polynomial), then

$$P(L_{-k})|h\rangle = 0$$

in the physical Hilbert space. Inserting this into any correlation function,

$$(\phi(y_+ = i, \cdot)\Omega, P(L_{-k})^*\phi_1\cdots\phi_n\Omega) = 0.$$

But $P(L_{-k})^*$ is a polynomial in L_{+k}, and $L_{+k}\Omega = 0$. So by using the commutation relations (9), one obtains a partial differential equation (of the Fuchsian type) for the correlation function. The solutions to these PDEs are called "conformal blocks", = certain hypergeometric functions or generalizations thereof, exhibiting branch cuts. The correlation functions are therefore products of leftmoving and rightmoving conformal blocks:

$$\sim \sum_{A_+A_-} C_{A_+A_-}B_{A_+}(\{x_+\})B_{A_-}(\{x_-\}).$$

The conformal blocks $B_{A_-}(\{x_-\})$ exhibit power-like singularities at coinciding points, reflecting the singular nature of quantum fields in general. The regular multipliers of the singular powers are again conformal blocks for a field occurring in the operator product expansion (OPE)

$$\phi_1(x)\phi_2(y) \sim \sum_{\lambda^+,\lambda^-} (x^+-y^+)^{\lambda^+}(x^- - y^-)^{\lambda^-}\cdot\phi_\lambda(y) \tag{10}$$

which in general can be proven to exist in a certain technical sense. The primary fields appearing in the OPE of two primary fields define the "fusion rules". The line of argument shows that the knowledge of the null vectors fixes the fusion rules, although the computation will be difficult.

The branch cut structure of the conformal blocks allows to compute commutators: in order to exchange the position of two fields in a correlation function, one must analytically continue the conformal blocks to the swapped configuration point, where the path of the continuation depends on the sign of the difference variable. Under this kind of analytic continuation, one solution of the PDE goes into another solution, ie, the space of conformal blocks exhibits "braiding" transformations:

$$B_A(\dots x_k, x_{k+1}\dots) = \sum_B R_{AB}(\pm)B_B(\dots x_{k+1}, x_k\dots),$$

where $\pm = \text{sign}\,(x_{k+1}-x_k)$.

Exercise ($c = \frac{1}{2}$, $h = h_{1,2} = \frac{1}{16}$): The four-point conformal blocks are $f_\pm = \prod_{i<j} x_{ij}^{-\frac{1}{8}} \cdot$ $\cdot \sqrt{\sqrt{x_{13}x_{24}} \pm \sqrt{x_{14}x_{23}}}$, where $x_{ij} = x_i - x_j - i\varepsilon$. Study the leading singular powers as $x_{12} \to 0$ or $x_{23} \to 0$, and the braiding as $x_1 \leftrightarrow x_2$ (giving a complex phase) or $x_2 \leftrightarrow x_3$ (giving a braiding matrix; tricky! [16]).

The distance of two points $x, y \in \mathbb{R}^2$ is spacelike iff $(x-y)^+$ and $(x-y)^-$ have opposite signs. Therefore, in order to have causal commutativity at spacelike distance, one must have

$$\sum_{A_+, A_-} R_{A_+B_+}(\pm) R_{A_-B_-}(\mp) C_{A_+A_-} \stackrel{!}{=} C_{B_+B_-}.$$

This condition also puts constraints on the (a priory unknown) pairing coefficients C.

Chosing a solution, amounts to specifying a model. There exist “trivial solutions”, namely essentially $C = \mathbf{1}$, exploiting unitarity of the R matrix in a suitable basis, but the classification of nontrivial solutions does not look very simple from this perspective. I will come back to fusion and braiding later in a more appropriate framework.

4 Euclidean CFT

Session 2.2

Correlation functions are boundary values of analytic functions in a certain cone in complex spacetime. The precise situation is very involved, eg, trying to analytically continue the fields themselves by “complex translations”

$$\phi(z_+, z_-) := e^{i(z_+P_+ + iz_-P_-)}\phi(0,0)e^{-i(z_+P_+ + iz_-P_-)}$$

makes sense only if $Im(z) > 0$ (so that the left operator is bounded by 1) and on vectors of positive energy (so that the right vector is bounded by some exponential of the energy).

Yet, it makes sense to study the analytically continued correlation functions. Of particular interest are the points of purely imaginary time, where the Lorentzian metric becomes the Euclidean metric. With $z_+ = t + x = iy + x$ and $z_- = t - x = iy - x$, these are the point where z_+ and $-z_-$ are each others' conjugates, one calls them z and $\bar{z}$. At the Euclidean points, left (right) chiral fields turn into (anti) holomorphic fields. (In 2D, “imaginary space” is equivalent to “imaginary time” up to some changes of sign, but not so in 4D.)

Euclidean field theory has its own place in physics. It arises in Statistical Mechanics, eg as the continuum limit of lattice models at a critical point (where the physical length given by the decay of correlations diverges relative to the lattice unit). Indeed, in these situations, the Hilbert space axiom of relativistic QFT is inappropriate, also the notion of adjoint operators looses its meaning, because it would relate fields at complex points to fields at their complex conjugate points. Instead, the statistical correlation functions possess another positivity property coming from the classical statistical measure in the critical limit, which is much easier to satisfy as the QFT Hilbert space positivity. Therefore, the Euclidean setting is much more flexible, and the previous classification restrictios do not apply. There exist models with negative c, and logarithmic quantum field theories which exhibit logarithmic correlation functions (whereas in QFT, all singularities are of power type).

Also the causality axiom no longer makes sense in Euclidean CFT. It is replaced by a condition of single-valuedness of correlation functions viewed as functions of z and $\bar{z}$, eg, $z^{-h_+}\bar{z}^{-h_-}$ is single-valued only if $h_+ - h_- \in \mathbb{Z}$. This gives the same restriction on the "spin" as in relativistic QFT, whereas the restriction on the dimension $h_+ + h_-$ is relaxed.

An ECFT model is often defined by a specification of the OPE of its holomorphic fields, ie, the asymptotic short-distance behavior of the form[3]

$$\phi_A(z)\phi_B(w) = \sum_{n\in\mathbb{Z}} (z-w)^n \cdot \phi_n(w).$$

One has $h_A + h_B = h_n - n$, hence $n \in \mathbb{Z}$ is bounded from below and fixes the dimensions of the fields ϕ_n in the expansion. The presence of negative n reflects the short-distance singularities of correlation functions. The individual contributions ϕ_n can be extracted from the series by suitable countour integrations, making use of Cauchy's theorem.

From the singular part, one can recover the commutator by taking the "radial-ordered" difference: in the limit $r \nearrow 1$, eg,

$$\frac{1}{z-rw} - \frac{1}{rz-w} = \frac{1}{z}\sum_{n\geq 0}\big(r\frac{w}{z}\big)^n + \frac{1}{w}\sum_{n\geq 0}\big(r\frac{z}{w}\big)^n \to \frac{1}{z}\sum_{n\in\mathbb{Z}}\big(\frac{z}{w}\big)^n = \frac{2\pi}{z}\cdot\delta_{2\pi}(\arg z - \arg w).$$

Going back to the expansion formula, let V be the linear span of all symbols $\phi(0)$, ϕ a conformal field (including derivatives of quasiprimary fields). Putting $w = 0$, one may read the left action of $\phi_A(z)$ on V

$$\phi_A(z)\phi_B(0) = \sum_{n\geq -n_0} z^n \cdot \phi_n(0),$$

as a Laurent polynomial in linear maps $Y^A_n : V \to V, \phi_B \mapsto \phi_n$ on the field space. With $Y(A,z) = \sum z^n Y^A_n$, the associativity of the operator product reads

$$Y(A,z)Y(B,w) = Y(Y(A,z-w)B). \tag{11}$$

In this form, the OPE is one of the axioms of a vertex operator algebra (VOA) [2, 11], supplemented with further axioms concerning the grading of V by the scaling dimensions, translation covariance via an operator $T : V \to V$, $\phi \mapsto \partial\phi$, causal commutativity, and the presence of the SET as a distinguished element of V.

ECFT on $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ lends itself to a generalization to Riemannian surfaces. Since correlation functions are singular at coinciding points, they are functions in either variable on a punctured surface, where the punctures are the positions of the other fields. Correlation functions are (in classes of models) still products of holomorphic and anti-holomorphic conformal blocks, and one may axiomatize CFT by properties of the latter. In the extreme case, one rather axiomatizes not the conformal blocks themselves, but rather the abstract vector spaces they span, depending on the moduli of the Riemannian surface and the quantum numbers of the fields "inserted" at the punctures. When two surfaces are glued along some boundary, these spaces must satisfy some consistency conditions. Axiomatizing these conditions, amounts to define "QFT as a functor from cobordisms into vector spaces" [17], cf Schweigert's lectures.

[3] The OPE of chiral fields in QFT has exactly the same form, but the meaning is different. Here, z is the analytically continued variable in $\mathbb{C}$, there, z is the Cayley transformed variable in S^1. The reason for this "form invariance" is that the Cayley transform is in fact a complex Möbius transformation.

5 Chiral models

Session 3.1

Chiral CFT models exist in abundance. Apart from the SET, one may have (primary) currents of dimension 1. In physics, "current" stands for the density of some "charge". The simplest ("canonical") commutation relation (CCR) are given by the Heisenberg algebra

$$[j(x), j(y)] = \frac{i}{2\pi}\delta'(x-y) \qquad \Leftrightarrow \qquad [j(f), j(g)] = i\sigma(f,g)$$

with the symplectic form on $\mathcal{S}(\mathbb{R})$

$$\sigma(f,g) = \frac{1}{4\pi}\int (fg' - f'g). \tag{12}$$

In Weyl form $W(f) = e^{ij(f)}$ for real f, the CCR read

$$W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)} \cdot W(f+g), \qquad W(f) \text{ unitary},$$

defining the C* algebra CCR(σ).

The non-abelian version is given by

$$[J^a(x), J^b(y)] = i\, f^{ab}_c\, J^c(x)\,\delta(x-y) + i\,\frac{\kappa}{2\pi}\cdot h^{ab}\,\delta'(x-y), \tag{13}$$

where f^{ab}_c are the structure constants of some Lie algebra $\mathfrak{g}$, and $h^{ab} = \mathrm{Tr}\,(\mathrm{ad}_{X^a}\mathrm{ad}_{X^b}) = -f^{ac}_d f^{bd}_c$ the Cartan-Killing metric. κ is an (at this point undetermined) parameter. Rewritten in terms of the Fourier modes

$$\widehat{J}^a(z) = i\frac{dx}{dz}\cdot J^a(x(z)) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},$$

the algebra turns into the Kac-Moody algebra $\widehat{\mathfrak{g}}$

$$[J^a_m, J^b_n] = f^{ab}_c J_{m+n} + h^{ab}\cdot m\delta_{m+n,0}\cdot\kappa, \qquad J^{a*}_n = J^a_{-n}, \qquad [\kappa, J^a_n] = 0. \tag{14}$$

In an irreducible representation, the central generator κ is a multiple of 1. This algebra, in turn, can be regarded as the infinitesimal version of a unitary projective representation of the loop group

$$LG = \{g: S^1 \to G \text{ smooth}\}, \qquad U(g)U(h) = e^{ic(g,h)}U(g\cdot h).$$

The cocycles $c(g,h)$ of the loop group can be classified: it turns out to be unique for semisimple Lie groups up to a factor. In the infinitesimal version, it is responsible for the central term, with κ the undetermined factor.

We ask again, whether these commutation relations can be realized on a Hilbert space. By the primary field commutation relations,

$$[L_m, J^a_n] = -nJ^a_{m+n},$$

the positive Fourier modes act like lowering operators for the conformal energy L_0, hence there must be ground states $J^a_n|\lambda\rangle = 0$ $(n > 0)$. Because the zero modes commute with L_0, the ground states carry a unitary representation π of the Lie algebra $\mathfrak{g}$ of G. Unless one admits infinite degeneracy of energy eigenstates, these representations must be finite-dimensional, thus G should be a compact group.[4]

Now, one may proceed as in the Virasoro case: the vectors $J^{a_s}_{-k_s} \cdots J^{a_1}_{-k_1}|\lambda\rangle_i$ span a Verma module, whose inner product can be computed by the commutation relations and the reality condition. Its positivity depends only on κ and the representation π of $\mathfrak{g}$ on the ground states. It turns out [12] that κ is quantized: if θ denotes the highest root of $\mathfrak{g}$, then $\kappa = k \cdot \frac{1}{2}(\theta, \theta)$ where k (called the level) is a positive integer; and for every value of the level k, only finitely many representations π are admitted, namely those whose highest weight λ satisfies the bound $(\lambda, \theta) \leq \kappa$. Eg, for $SU(2)$, the spin is limited to be $j \leq \frac{k}{2}$. Again, the inner product is only semi-definite, and the corresponding null vectors give rise to PDEs for the correlation functions among primary fields, on which the current algebra acts like infinitesimal gauge transformations:

$$i[J^a(x), \phi_A(y, \cdot)] = \pi(X^a)_{BA}\phi_B(y)\delta(x - y).$$

The nontrivial task is to construct these algebras by operators on a Hilbert space. There exist several methods:

"Field theoretic constructions"

F.0 Construction of the action on the Verma module and quotient by the null space. Bad control over the resummation of the Fourier modes to define the local fields.

F.1 The abelian case (CCR) can be directly constructed on a symmetric Fock space with creation and annihilation operators $a(k)^* = a(-k)$ for $k \in \mathbb{R}$, $[a(k), a(k')^*] = k\,\delta(k - k')$:

$$j(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, a(k)e^{-ikx} = j^*(x),$$

The ground state ($a(k)\Omega = 0$ for $k > 0$) gives rise to the 2-point function

$$(\Omega, j(x)j(y)\Omega) = \frac{1}{4\pi^2}\int_{\mathbb{R}+} k\, dk\, e^{-i(x-y)} = \frac{1}{4\pi^2}\Big(\frac{-i}{x - y - i\varepsilon}\Big)^2.$$

The SET of the current is given by

$$T(x) = \frac{1}{4\pi}{:}j(x)^2{:} \qquad (c = 1). \tag{15}$$

F.2 Kac-Frenkel construction [6] = embedding of nonabelian current algebras into an extension of a free Bose field theory (CCR). Example: $SU(2)$.
Let the abelian current be given by its Fourier modes satisfying $[j_m, j_n] = n\delta_{n+m,0}$. Extend $q = j_0$ to a pair of "quantum mechanical" zero modes $[q, p] = i$, and define

$$J^{\pm}(z) = \exp\Big(\mp\sqrt{2}\sum_{n<0}\frac{j_n}{n}z^{-n}\Big) \cdot \exp\big(\pm\sqrt{2}(q + ip\log z)\big) \cdot \exp\Big(\mp\sqrt{2}\sum_{n>0}\frac{j_n}{n}z^{-n}\Big).$$

[4]Another argument is that the inner product between vectors $J^a_{-1}|0\rangle$ is given by the Cartan metric, which therefore must be positive-definite.

Formally, this can also be written in x-space as

$$J^\pm(x) = :\exp \pm\sqrt{2}\, i \int_{-\infty}^{x} j(u)du:,$$

where $:\cdot:$ stands for the necessary regularization of the highly singular (because of the sharp cutoff) expression.
Then $J^3 = j/\sqrt{2}$ and $J^\pm = J^1 \pm iJ^2$ satisfy the $su(2)$ current algebra at level $k = 4\kappa = 1$.
For other Lie algebras $\mathfrak{g}$, one starts from a multi-component current with test functions taking values in the Cartan subalgebra, and replaces the coefficient $\pm\sqrt{2}$ by the root vectors of $\mathfrak{g}$.

F.3 Embedding into a free Fermi field theory (CAR). Physicists define a complex free Fermi fields satisfying $\{\psi^*(x), \psi(y)\} = 2\pi\delta(x-y)$ by creation and annihilation operators, $a(k)^* = b(-k)$ for $k \in \mathbb{R}$, $\{a(k), a(k')^*\} = \delta(k-k') = \{b(k), b(k')^*\}$ on the antisymmetric Fock space with a ground state $a(k)\Omega = b_i(k)\Omega = 0$ if $k > 0$:

$$\psi(x) = \int_{\mathbb{R}} dk\, a(k)e^{-ikx}, \quad \psi^*(x) = \int_{\mathbb{R}} dk\, b(k)e^{-ikx}$$

This implies the canonical anti-commutator $\{\psi(x), \psi^*(y)\} = 2\pi\,\delta(x-y)$ and the 2-point fn

$$(\Omega, \psi(x)\psi^*(y)\Omega) = (\Omega, \psi^*(x)\psi(y)\Omega) = \int_{\mathbb{R}+} dk\, e^{-i(x-y)} = \frac{-i}{x-y-i\varepsilon}.$$

The abelian current and the SET with $c = 1$ can be embedded by

$$j(x) = \frac{1}{2\pi}:\psi^*(x)\psi(x):, \qquad T(x) = \frac{1}{8\pi}:\partial\psi^*(x)\psi(x) - \psi^*(x)\partial\psi(x):,$$

where the Wick product is defined by subtraction of the vacuum expectation value.
The current is the generator of gauge transformations:

$$e^{ij(f)}\psi(x)e^{-ij(f)} = e^{-if(x)}\,\psi(x), \qquad e^{ij(f)}\psi^*(x)e^{-ij(f)} = e^{+if(x)}\,\psi^*(x)$$

and the SET generates diffeomorphisms. (A real Fermi field has no current, and gives a SET with $c = \frac{1}{2}$.)
The generalizations to N complex Fermi fields is obvious. Let now τ^a be a unitary N-dimensional matrix repn of some Lie algebra $\mathfrak{g}$. Then

$$J^a(x) = \frac{1}{2\pi}\sum_{ij} :\psi_i^*(x)(\tau^a)_{ij}\psi_j(x):$$

is a representation of the $\mathfrak{g}$ current algebra on the Fock space of the free Fermi fields.

F.4 The Sugawara construction. Given a current algebra, the SET is a multiple of $h_{ab}{:}j^aj^b{:}$. For $su(2)_k$, the central charge turns out to be $c_k = \frac{3k}{k+2}$ ($= 1$ for $k = 1$).

"Operator algebraic constructions":

A.1 (Analog of F.1) One fixes a state on the Weyl algebra CCR(σ) (cf (12)):

$$\omega(W(f)) = e^{-\frac{1}{2}\|f\|_+^2}, \qquad \|f\|_+^2 = \frac{1}{2\pi}\int_{\mathbb{R}^+} k\,dk|\hat f(k)|^2 = \|j(f)\Omega\|^2,$$

then the GNS construction gives $\pi_\omega(W(f)) = e^{ij(f)}$.

A.2 (Generalizes F.2) The symplectic form σ extends (by the same formula (12)) to $\widetilde{\mathcal{S}}(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} : g' \in \mathcal{S}(\mathbb{R})\}$, which includes smooth "step functions". Denote by CCR($\widetilde\sigma$) the associated extended CCR algebra. We call $q(g) = -\frac{1}{2\pi}\int g' = \frac{1}{2\pi}(g(-\infty) - g(+\infty))$ the "charge" of $W(g)$.[5] If g_1' and g_2' have support in disjoint intervals, then $\sigma(g_1, g_2) = \pm q_1 q_2 \cdot \pi$, where the sign depends on the order of the supports. Thus, one has commutation relations

$$W(g_1)W(g_2) = e^{\mp q_1 q_2 \cdot \pi} \cdot W(g_2)W(g_1).$$

Therefore, the subalgebra of operators with charges in an even lattice $q \cdot \mathbb{Z}$ (ie, q^2 is an even integer) satisfies causal commutation relations. (This algebra can be viewed as a crossed product of $CCR(\sigma)$ by $\mathbb{Z}$, where the action of $n \in \mathbb{Z}$ is given by $\mathrm{Ad}_{W(ng)}$ for any g of charge q. The choice of g is irrelevant because, up to a complex phase, $W(g_2) \sim W(f)W(g_1)$, where $f = g_2 - g_1$ has zero charge.) The vacuum state of CCR(σ) extends to the crossed product algebra by composition with the conditional expectation which projects onto the charge zero operators.
To make contact with F.2, one has to establish, that in the GNS representation of this state, the properly regularized limits of $W(g)$ as $g \to \pm g_x(\cdot) = \pm 2\pi\sqrt{2}\,\theta(x - \cdot)$ exist (as operator-valued distributions) and give the expressions of F.1.

More generally, every even lattice (including the root lattices of semisimple Lie algebras) give rise to some local extension of the current algebra [3].

A.3 (Analog of F.3) Define the CAR C* algebra by the anti-commutation relations $\{B(f), B(g)\} = (\Gamma f, g)_{\mathcal{H}}$ over the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^N \oplus \mathbb{C}^N)$, $\Gamma f = \Gamma(\underline{f}_1, \underline{f}_2) = (\overline{\underline{f}_2}, \overline{\underline{f}_1})$. Then $\sqrt{2\pi}B(f) = \sum_i \psi_i(f_1^i) + \sum_i \psi_i^*(f_2^i)$ is the complex free Fermi field (which is bounded after smearing, unlike general quantum fields).
Every projection satisfying $\Gamma P \Gamma = 1 - P$ defines a quasifree state $\omega_P(B(f)B(g)) = (\Gamma f, Pg)_{\mathcal{H}}$, giving rise to inequivalent GNS representations π_P in general. The Fock representation is obtained from P_0 = projection onto the positive-frequency part of the Fourier transform.
Now, let u be a unitary operator on $\mathcal{H}$ commuting with Γ, then

$$B(f) \mapsto B(uf)$$

is an automorphism α_u of the CAR algebra. The state $\omega_P \circ \alpha_u$ will be equal to ω_P iff P commutes with u. More generally, its GNS repn will be unitarily equivalent

[5] If $f \in \mathcal{S}(\mathbb{R})$ is constant $= 1$ on the support of g, then $W(tf)W(g)W(tf)^* = e^{itq(g)}W(g)$, ie, $W(tf)$ approximates (as $f \to 1$) the unitary 1-parameter group e^{itQ} with $Q = \int j(x)dx$ the charge operator.

to π_P iff $[P,u]$ (or equivalently $uPu^* - P$) has finite Hilbert-Schmidt norm. In this case, α_u is implemented by a unitary operator in the GNS representation,

$$\pi_P(\alpha_u(\cdot)) = U\pi_P(\cdot)U^*.$$

For $P = P_0$, one can verify the Hilbert-Schmidt condition for gauge transformations $u = g \oplus \overline{g}$, where $g : S^1 \to G$ is a smooth G-valued function, ie $g \in LG$, acting on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^N \oplus \mathbb{C}^N)$ by pointwise matrix multiplication in an N-dimensional unitary representation. (For $G = U(1)$, $g(x) = e^{i\alpha(x)}$, this amounts to $\int dxdy\, |e^{i\alpha(x)} - e^{i\alpha(y)}|^2/(x-y)^2 < \infty$.) The implementing unitaries $U(g)$ then form a projective repn of LG, and the generators of its one-parameter subgroups give a representation of the current algebra on the Fock space of the free Fermi fields [15].
Also the (orientation-preserving) diffeomorphisms act unitarily on $\mathcal{H}$ by $f \mapsto uf = (\gamma')^{\frac{1}{2}} \cdot f \circ \gamma$, and again $[P_0, u]$ is Hilbert-Schmidt. The unitaries $U(\gamma)$ then form a projective repn of $\mathrm{Diff}(S^1)$, and the generators of its one-parameter subgroups give a representation of the SET on the Fock space of the free Fermi fields.

A.4 (Analog of F.4) The diffeomorphisms act by automorphisms $g \mapsto g \circ \gamma$ on the loop group. Every unitary projective representation of LG extends to the semidirect product $\mathrm{Diff}(S^1) \ltimes LG$ [15]. The generators of one-parameter subgroups of $\mathrm{Diff}(S^1)$ can be identified with SET field operators $T(f)$.

Then there is an arsenal of methods to construct new models from given models. Trivially, one may take tensor products (where central charges and levels just add). One may take fixed point subalgebras under some automorphism group of the field algebra, like the global G-symmetry of the $\mathfrak{g}$-current algebra, which preserves the Sugawara SET T, among other fields.

The coset construction consists in taking the subalgebra A^{coset} of all fields that commute with a given subalgebra $\widetilde{A} \subset A$. The SET $\widetilde{T}$ of $\widetilde{A}$ has the same CR (8) with all fields of $\widetilde{A}$, as the full SET T. Therefore, the difference $T^{\mathrm{coset}} = T - \widetilde{T}$ belongs to the commutant. In particular, it commutes with $\widetilde{T}$. It follows that T^{coset} is a SET of its own with central charge $c^{\mathrm{coset}} = c - \tilde{c}$. Eg, the SET of two complex Fermi fields has $c = 2$, whereas the Sugawara SET for the embedded $su(2)_{k=1}$ subtheory has $c = 1$. Thus, there is a coset SET with $c = 1$. Indeed, the coset theory is the "abelian" current algebra corresponding to $U(1) \subset SU(2)$, and its SET is the SET (15).

An important class of coset models is the following. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, in general the Sugawara SETs $T_{\mathfrak{h}}$ and $T_{\mathfrak{g}}$ do not coincide. Then the coset SET has central charge $c_{\mathfrak{g}} - c_{\mathfrak{h}}$. By this method, all Virasoro algebras with $c < 1$ can be obtained via the diagonal embedding of $su(2)_{k+1}$ into $su(2)_k \otimes su(2)_1$: $c_k + c_1 - c_{k+1} = 1 - \frac{6}{(k+2)(k+3)}$ [8], and all its positive-energy representations (cf Sect. 3) are contained in PERs of $su(2)_k \otimes su(2)_1$.

All these models exhibit a rich representation theory, in many cases "rational" (ie, finitely many inequivalent irreducible representations). Quantities of physical interest are: spectral quantum numbers (lowest eigenvalues of L_0 and characters like $\chi(\beta) = \mathrm{Tr}\, e^{-\beta L_0}$, possibly including generators of further symmetries), correlation functions (conformal blocks), fusion rules, subtheories and branching rules (decomposition of representations upon restriction to subtheories), ...

6 Superselection sectors in AQFT

Session 4.1

I turn to the formulation of CFT in the setting of "algebraic QFT" (AQFT) [9]. Here, we don't specify local fields, but only local vN algebras in their vacuum representation

$$A(I) \subset B(\mathcal{H}_0), \qquad (I \subset \mathbb{R} \text{ or } \subset S^1 \text{ any interval}).$$

We have done so before: The Weyl algebra construction of the abelian current algebra gave unitaries $W(f)$; then $A^{U(1)}(I) = \{W(f) : \mathrm{supp}\, f \subset I\}''$. The CAR constructions gave unitary implementers of gauge or diffeo transformations; then $A_{\widehat{\mathfrak{g}}}(I) = \{U(g) : g : S^1 \to G, \mathrm{supp}\, f \subset I\}''$ or $A_{\mathrm{SET}}(I) = \{U(\gamma) : \gamma \in \mathrm{Diff}(S^1), \mathrm{supp}\, \gamma \subset I\}''$.

Although a general proof exists only for (large classes of) models, the field-theoretic and the AQFT approach are believed to be equivalent.

The axioms of AQFT require

- Isotony: local algebras increase with O,
- Causality (= locality, cf footnote 2): Local algebras at spacelike distance (or: of disjoint intervals in the chiral case) commute,
- Covariance: there is a repn of the Poincaré (conformal, Möbius) group G by automorphisms such that $\alpha_g A(O) = A(gO)$,
- Vacuum: there is an invariant state $\omega \circ \alpha_g = \omega$, hence a unitary repn U of G in the GNS repn of ω; and the GNS vector is a ground state for the energy (in particular, U is a positive-energy representation). In the chiral case, these axioms imply Haag duality: $A(I') = A(I)'$.

If one also requires that the repn U of the Möbius group extends to a projective repn of the diffeomorphism group $\mathrm{Diff}(S^1)$ such that $U(\gamma)A(I)U(\gamma)^* = A(\gamma I)$ and localized diffeos act trivial on the complement: $\mathrm{supp}\, \gamma \subset I \Rightarrow \mathrm{Ad}_{U(\gamma)}|_{A(I')} = \mathrm{id}$, then by Haag duality, $U(\gamma) \in A(I)$. The subalgebras $A_0(I)$ generated by $U(\gamma)$ ($\mathrm{supp}\, \gamma \subset I$) define a Virasoro subtheory.

A positive-energy representation (PER) of A is a representation π on a Hilbert space equipped with a positive-energy representation U_π of the Möbius group, such that

$$U_\pi(g)\pi(A(I))U_\pi(g)^* = \pi(A(gI)).$$

(I suppress here some subtleties with the extension from $\mathbb{R}$ to S^1.)

One can show that every positive-energy representation, if restricted to the complement of any interval $I_0 \subset \mathbb{R}$, is unitarily equivalent to the (defining) vacuum representation $\pi_0(a) = a$:

$$\pi(a) = VaV^* \qquad (a \in A(I_0')).$$

By causality, $V^*\pi(a)V$ commutes with $A(J')$ whenever $a \in A(I)$ and J contains both I and I_0. Hence by Haag duality, it belongs to $A(J)$. Therefore, $a \mapsto \rho(a) = V\pi(a)V^*$ is an endomorphism of the C*-inductive limit of $A(I)$, $I \subset \mathbb{R}$, and ρ acts trivially on the subalgebra $A(I_0')$. Such endomorphisms are called "localized", or DHR endo's [4].

In other words: every PER is unitarily equivalent to a localized endomorphism: $\pi \sim \pi_0 \circ \rho$. The localization interval I_0 can be chosen arbitrarily.

If two unitarily equivalent ρ_1 and ρ_2 are localized in intervals I_1 and I_2, then by Haag duality, the unitary intertwiner $\rho_2 = \mathrm{Ad}_u \circ \rho_1$ is an element of $A(I)$, where I contains both I_1 and I_2.

Example: the local algebras $A(I)$ of the CCR algebra are generated by Weyl operators $W(f)$, supp $f \subset I$. The maps

$$W(f) \mapsto e^{i\int gf} \cdot W(f), \qquad j(x) \mapsto j(x) + g(x) \cdot 1$$

are localized endomorphisms if $g \in \mathcal{S}(\mathbb{R})$ has compact support. Any two such maps are inner equivalent by $W(\gamma)$ where $\gamma' = g_1 - g_2$, iff $\int g_1 = \int g_2$. The equivalence classes are therefore given by the "charge" $q = \int g$.

A unitary equivalence class of PERs (hence of DHR endomorphisms) is called a superselection sector of A, or a "charge".

Consider the C* category of PERs. The morphisms $t \in \mathrm{Hom}(\rho, \sigma)$ are intertwining operators satisfying $t\rho(a) = \sigma(a)t$ for all $a \in A$ (eg, the unitary "charge transporters" $u : \rho_1 \to \rho_2$ changing the localization interval), and they always belong to A by Haag duality. Direct sums and subrepresentations can be defined in terms of isometric intertwiners:

$$\rho(a) = w_1\rho_1(a)w_1^* + w_2\rho_2(a)w_2^* \qquad (w_1w_1^* + w_2w_2^* = \mathbf{1}, \quad w_i^*w_j = \delta_{ij} \cdot \mathbf{1});$$

$$\rho \prec \sigma \quad \text{iff} \quad \rho(\cdot) = w^*\sigma(\cdot)w \quad \text{with} \quad ww^* \in \mathrm{Hom}(\sigma, \sigma),\ w^*w = 1.$$

This category is in fact a tensor category, with the "tensor product" given on the objects by the composition (in particular, NOT by the tensor product, which is not even a representation): $\pi_\sigma \times \pi_\rho = \pi_{\sigma\circ\rho}$, and on the intertwiners by $t_1 \times t_2 = t_1\rho_1(t_2) = \sigma_1(t_2)t_1$.

The irreducible decomposition of a product of irreducibles defines the "fusion rules":

$$\rho_a\rho_b \sim \bigoplus_c N_{ab}^c \rho_c.$$

In particular, there is a notion of "conjugate": Irreducibles ρ and $\bar\rho$ are conjugates iff $\mathrm{id} \prec \rho\bar\rho$ iff $\mathrm{id} \prec \bar\rho\rho$. The relation of this notion of fusion rules with the one introduced earlier in terms of the OPE of two primary fields, will be made in Sect. 7.

The tensor category of superselection sectors is also unitarily braided [4, 5]. One shows that $\rho_1\rho_2 = \rho_2\rho_1$ if the two DHR endo's are localized in disjoint intervals I_i. One then defines $\varepsilon_{\rho_1,\rho_2} := \mathbf{1}$ if I_1 is to the right of I_2, and

$$\varepsilon_{\mathrm{Ad}_{u_1}\rho_1, \mathrm{Ad}_{u_2}\rho_2} := u_2\rho_2(u_1)\varepsilon_{\rho_1,\rho_2}\rho_1(u_2)^*u_1^*$$

in general. Thanks to causality, this definition is well-defined, $\varepsilon_{\rho,\sigma} \in \mathrm{Hom}(\rho\sigma, \sigma\rho)$, and satisfies the naturality axiom for a braiding. Notice that in general, $\varepsilon_{\rho_1,\rho_2} \neq \mathbf{1}$ if I_1 is to the left of I_2, because the respective endomorphisms act nontrivially on the charge transporters needed to swap the localizations.

Example: The Virasoro theory with $c = \frac{1}{2}$ has three sectors with $h = 0, \frac{1}{2}, \frac{1}{16}$. One may choose representative DHR endo's in these sectors satisfying $\tau\circ\tau = \mathrm{id}$ and $\tau\circ\sigma = \sigma$. Then $\sigma\circ\tau \sim \sigma$, and $\sigma\circ\sigma \sim \mathrm{id} \oplus \tau$ with isometric intertwiners $r \in \mathrm{Hom}(\mathrm{id}, \sigma^2)$, $t \in \mathrm{Hom}(\tau, \sigma^2)$, and $u = rr^* - tt^* \in \mathrm{Hom}(\sigma, \sigma\tau)$ unitary. The tensor category structure is given by the formulae $\tau(r) = t$, $\tau(t) = r$ and $\sigma(r) = (r + t)/\sqrt{2}$, $\sigma(t) = (r - t)u/\sqrt{2}$. The braiding is given by $\varepsilon_{\tau,\tau} = -1$, $\varepsilon_{\tau,\sigma} = \varepsilon_{\sigma,\tau} = -iu$ and $\varepsilon_{\sigma,\sigma} = \kappa_\sigma^{-1}(rr^* + itt^*)$, where $\kappa_\sigma = e^{2\pi i h_\sigma} = e^{2\pi i/16}$, on the representatives, and by naturality on all their equivalents.

7 Local extensions

Session 4.2

A QFT is described by a covariant assignment of local vN algebras $I \mapsto A(I)$. An extension is then a covariant assignment $I \mapsto B(I)$ such that $A(I) \subset B(I)$, and the covariance α_g of B extends that of A. It is also required that $B(I)$ commutes with $A(J)$ if I and J are disjoint ("relative locality"). If $B(I)$ commutes with $B(J)$, the extension is called local. In order to exclude trivial extensions like tensor products, we shall require that the relative commutant $A(I)' \cap B(I)$ is trivial; in particular, there can be no coset SET, hence A and B share the same SET (if they have one).

We have encountered several extensions before: The lattice construction is an extension of the abelian current algebra, and it is local if the lattice is even. A nonabelian current algebra is a local extension of its Sugawara SET. The real or complex free Fermi algebra is a nonlocal extension of the SET with $c = \frac{1}{2}$ resp $c = 1$.

A two-dimensional CFT is also a local extension of its pair of chiral subalgebra $A_+ \otimes A_+ \subset B_2$, cf Sect. 3 for the case that A_+ and A_- are Virasoro theories with $c < 1$.

Constructing an extension amounts to "add" more operators to $A(I)$ and specify their algebraic relations in a consistent and covariant way. AQFT offers a way to do so in a most efficient way [13]. Namely, there is a 1:1 correspondence between extensions and "Q-systems" (= C* Frobenius algebras) in the category of superselection sectors. In the rational case, there exist only finitely many Q-systems, hence only finitely many extensions. Whether the extension is local or not, can be directly read off the Q-system.

A Q-system consists of a (reducible) DHR endo θ (ie, a represention of A), an intertwiner $w \in \mathrm{Hom}(\mathrm{id}, \theta)$ and an intertwiner $x \in \mathrm{Hom}(\theta, \theta^2)$, such that the relations

$$w^*x = \theta(w^*)x = 1, \qquad xx = \theta(x)x, \qquad w^*w = x^*x = d \cdot 1 \quad (d > 1)$$

hold. These relations take care of all consistency conditions for the extension. In order to have irreducible extensions, we must require that $\mathrm{id} \prec \theta$ with multiplicity 1, hence w is uniquely fixed by θ up to a phase, and the intertwiner x contains the proper information.

Let θ be localized in I_0. Then $B(I_0)$ is the algebra $A(I_0)$ supplemented by an element v such that

$$va = \theta(a)v, \quad v^2 = xv, \quad v^* = w^*x^*v.$$

This is again a vN algebra. Shifting the localization with a unitary u, ($\tilde{\theta} = \mathrm{Ad}_u\theta, \tilde{w} = uw, \tilde{x} = u\theta(u)xu^*$) is another Q-system, defining $A(I) \subset B(I)$ for any other region, with $\tilde{v} = uv$. B is local iff u_1v commutes with u_2v for I_1 and I_2 disjoint; this in turn is equivalent to the condition $\varepsilon_{\theta,\theta}x = x$.

A more physical picture emerges, if we "split" the generator v into "charged fields": if $\rho \prec \theta$ is irreducible and $w_\rho \in \mathrm{Hom}(\rho, \theta)$, then ψ_ρ satisfies

$$\psi_\rho a = \rho(a)\psi_\rho \tag{16}$$

Moreover, (fixing representatives of the irreducible subsectors) every element of B has a unique decomposition $b = \sum_\rho a_\rho\psi_\rho$, with $\psi_{\mathrm{id}} = 1$. The map $\mu : b \mapsto a_{\rho=\mathrm{id}}$ is the (unique) conditional expectation $B \to A$, and $\omega_B = \omega_A \circ \mu$. This implies that ψ_ρ generate the charged repn from the vacuum repn: for any two vectors $\Phi_i \in \mathcal{H}_0$,

$$(\psi_\rho^*\Phi_1, a\psi_\rho^*\Phi_2) = (\Phi_1, \mu(\psi_\rho a\psi_\rho^*)\Phi_2) = (\Phi_1, \rho(a)\mu(\psi_\rho\psi_\rho^*)\Phi_2) = (\Phi_1, \pi_\rho(a)\Phi_2).$$

The product $\psi_\sigma\psi_\rho$ then generates the representation $\pi_\sigma \times \pi_\rho = \pi_{\sigma\circ\rho}$. Moreover, one has (from $v^2 = xv$ resp from $v^* = w^*x^*v$)

$$\psi_a\psi_b = \sum_c t^c_{ab} \cdot \psi_c, \qquad \psi^*_a = r^*_a\psi_{\bar{a}}$$

where $t^c_{ab} = w^*_a\theta(w^*_b)xw_c \in \mathrm{Hom}(\rho_c, \rho_a\rho_b)$, and $r_a = w^*_{\bar{a}}\theta(w^*_a)xw \in \mathrm{Hom}(\mathrm{id}, \bar{\rho}_a\rho_a)$. This multiplication algebra of charged fields (with A-valued coefficients) is the analog of the OPE of primary fields in AQFT, making the announced contact with the notion of fusion rule in Sect. 3. If B is local, then $\psi_\rho\psi_\sigma = \varepsilon_{\sigma,\rho} \cdot \psi_\sigma\psi_\rho$.

Thus, the extension described by a Q-system consists in adding a set of charged field operators ψ_ρ to A, whose algebraic relations are specified by the intertwiners w and x in the Q-system. If ρ is localized in I, then ψ_ρ belongs to $B(I)$. Indeed, by (16) it commutes with $A(J)$, I and J disjoint. Thus, the charged fields are transported to arbitrary intervals by $\psi_{\mathrm{Ad}_u\rho} = u\psi_\rho$ (because $w_\rho u^* \in \mathrm{Hom}(\mathrm{Ad}_u\rho, \theta)$).

Example (continued from Sect. 6): The Virasoro theory with $c = \frac{1}{2}$ admits only one nontrivial Q-system ($\theta = \sigma^2, w = 2^{\frac{1}{4}}r, x = 2^{-\frac{1}{4}}(r + t)$). Since $\theta \sim \mathrm{id} \oplus \tau$, there is only one charged field operator $\psi_\tau = 2^{\frac{1}{4}} \cdot t^*v$ satisfying

$$\psi_\tau a = \tau(a)\psi_\tau, \quad \psi^2_\tau = 1, \quad \psi\psi^*_\tau = \psi_\tau.$$

Now, if τ and $\mathrm{Ad}_u\tau$ are localized in disjoint intervals, we have

$$\psi_\tau\psi_{\mathrm{Ad}_u\tau} = \psi_\tau u\psi_\tau = \tau(u)\psi_\tau\psi_\tau = (\tau(u)u^*) \cdot \psi_{\mathrm{Ad}_u\tau}\psi_\tau.$$

But $\tau(u)u^*$ equals the braiding $\varepsilon_{\tau,\tau} = -1$, hence the charged field anti-commutes. In fact, ψ is a real free Fermi field.

The problem to construct two-dimensional local CFTs with given chiral subtheories now consists in finding a Q-system for $A_+ \otimes A_-$.

Example: Both A_+ and A_- are the Virasoro theory with $c = \frac{1}{2}$. There is (among others) a Q-system with $\Theta = W_1(\mathrm{id}\otimes\mathrm{id})W^*_1 + W_2(\tau\otimes\tau)W^*_2 + W_3(\sigma\otimes\sigma)W^*_3$, $W = \sqrt{2}W_1$,

$$X = \sum_{a,b,c} \Theta(W_b)W_a(t_{ab,c} \otimes t_{ab,c})W^*_c/\sqrt{2},$$

where $t_{ab,c} \in \mathrm{Hom}(\rho_c, \rho_a\rho_b)$ are given by $1 \in \mathrm{Hom}(\rho, \mathrm{id}\rho)$, $1 \in \mathrm{Hom}(\rho, \rho\mathrm{id})$, $1 \in \mathrm{Hom}(\mathrm{id}, \tau\tau)$, $1 \in \mathrm{Hom}(\sigma, \tau\sigma)$, $(u\otimes u) \in \mathrm{Hom}(\sigma, \sigma\tau)$, $\sqrt{2}(r\otimes r) \in \mathrm{Hom}(\mathrm{id}, \sigma\sigma)$, $\sqrt{2}(t\otimes t) \in \mathrm{Hom}(\tau, \sigma\sigma)$, respectively.

One obtains two nontrivial charged fields $\Psi_{\tau\tau} = W^*_2V/\sqrt{2}$ and $\Psi_{\sigma\sigma} = W^*_3V$, satisfying

$$\Psi_{\tau\tau}a = (\tau \otimes \tau)(a)\Psi_{\tau\tau}, \qquad \Psi_{\sigma\sigma}a = (\sigma \otimes \sigma)(a)\Psi_{\sigma\sigma}, \qquad (a \in A \otimes A)$$

and the algebraic relations among each other

$$\Psi_{\tau\tau}\Psi_{\tau\tau} = 1, \quad \Psi_{\tau\tau}\Psi_{\sigma\sigma} = \Psi_{\sigma\sigma}, \quad \Psi_{\sigma\sigma}\Psi_{\tau\tau} = (u\otimes u)\Psi_{\sigma\sigma}, \quad \Psi_{\sigma\sigma}\Psi_{\sigma\sigma} = ((r\otimes r)+(t\otimes t)\Psi_{\tau\tau})/\sqrt{2},$$

$$\Psi^*_{\tau\tau} = \Psi_{\tau\tau}, \qquad \Psi^*_{\sigma\sigma} = \sqrt{2}(r^* \otimes r^*)\Psi_{\sigma\sigma}, \qquad \Psi^*_{\tau\tau}\Psi_{\tau\tau} = \Psi^*_{\sigma\sigma}\Psi_{\sigma\sigma} = 1.$$

Exercises: Verify the defining properties of the Q-system. Verify the displayed relations of the charged fields from their definitions via the Q-system! Verify their consistency as an algebraic extension of $A \otimes A$. Verify that the charged fields commute among each other at spacelike distance, but not at timelike distance.

Of course, there is a general theory behind this example, including the notions of "modularity" and "α-induction" [10, 13] which cannot be treated here. Every (possibly nonlocal) chiral extension induces a local two-dimensional extension (this is not 1:1; the extension in the example is induced from the trivial chiral extension $B = A$, but also from the extension with $\theta = \sigma^2$ presented above), and every local two-dimensional extension is intermediate to one of the induced extensions (in the last example, one may just omit $\Psi_{\sigma\sigma}$). The multiplicities Z_{ab} of the irreducible subsectors $\rho_a \otimes \rho_b$ of $A \otimes A$ as subrepns of the induced representation Θ turn out to form a modular invariant matrix (the 3×3 unit matrix in the example). Since the latter can been classified independently, one has an apriori restriction on the possible representations Θ (but not all modular invariant matrices have such a realization).

References

[1] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov: Nucl.Phys. B241 (1984) 333–380.

[2] R.E. Borcherds: Proc.Natl.Acad. Sci.USA 83 (1986) 3068–3071, Invent.Math. 109 (1992) 405.

[3] D. Buchholz, G. Mack, I. Todorov: Nucl.Phys. B (Proc.Suppl.) B5 (1988) 20–56.

[4] S. Doplicher, R. Haag, J.E. Roberts: Comm.Math.Phys. 23 (1971) 199–230, Comm. Math.Phys. 35 (1974) 49–85.

[5] K. Fredenhagen, K.-H. Rehren, B. Schroer: Comm.Math.Phys. 125 (1989) 201–226.

[6] I.B. Frenkel, V.G. Kac: Invent.Math. 62 (1980) 23–66.

[7] D. Friedan, Z. Qiu, S. Shenker: Comm.Math.Phys. 107 (1986) 535–542.

[8] P. Goddard, A. Kent, D. Olive: Comm.Math.Phys. 103 (1986) 105–119.

[9] R. Haag: Local Quantum Physics, Springer 1996.

[10] Y. Kawahigashi, R. Longo, M. Müger: Comm.Math.Phys. 219 (2001) 631–669.

[11] V.G. Kac: Vertex Algebras for Beginners, AMS University Lecture Series 10, 1998.

[12] V.G. Knizhnik, A.B. Zamolodchikov: Nucl.Phys. B247 (1984) 83–103.

[13] R. Longo, K.-H. Rehren: Rev.Math.Phys. 7 (1995) 567–597, Rev.Math.Phys. 16 (2004) 909–960.

[14] M. Lüscher, G. Mack: in: Nonperturbative Quantum Field Theory, eds. G. 'tHooft et al, Plenum 1988.

[15] A. Pressley, G. Segal: Loop Groups, Clarendon 1988.

[16] K.-H. Rehren, B. Schroer: Phys.Lett. B198 (1987) 84–88.

[17] G. Segal: in: Topology, Geometry and Quantum Field Theory, London Math.Soc. LNS 308 (2004) 421–577.