Uniqueness Quantization Criteria in Cosmological Spacetimes

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Ambiguity in canonical quantization

The choice of representation for the fundamental algebra in canonical quantization is not free of ambiguity

In ordinary Quantum Mechanics, the Stone–von Neumann theorem guarantees the uniqueness of the (strongly continuous, unitary and irreducible) representation of the Weyl algebra.

In more complex situations, symmetry can help:

- In the Fock quantization of a field theory in flat spacetime $\rightsquigarrow$ Poincaré invariance
- In LQG $\rightsquigarrow$ diffeomorphism invariance [LOST; Fleishchack]

A result [Cortez, Mena Marugán, Olmedo, Velhinho]

Consider a Klein-Gordon (KG) field Φ with time-dependent mass in a compact Riemaniann manifold Σ ($\dim \Sigma \leq 3$).

The Fock representations with

1. a vacuum state invariant under the spatial symmetries of the KG equation [a complex structure (CS) invariant under those symmetries]
2. unitarily implementable field dynamics

form a unitary equivalence class.

A representative of these class is the massless representation, defined by the complex structure J_0:

$$J_0 \begin{pmatrix} \Phi \\ \pi_\Phi \end{pmatrix} = \begin{pmatrix} 0 & -(-{}^0h\,{}^0\Delta)^{-1/2} \\ -(-{}^0h\,{}^0\Delta)^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \pi_\Phi \end{pmatrix}$$

${}^0h \rightsquigarrow$ determinant of the metric ${}^0h_{ab}$ on Σ,
${}^0\Delta \rightsquigarrow$ Laplace-Beltrami operator

Applications

The result has many interesting applications
in non-stationary scenarios:

- Test KG field Φ in a FLRW universe:

$$\ddot{\Phi} + 2\frac{\dot{a}}{a}\dot{\Phi} - {}^{0}\Delta\Phi + m^2a^2\Phi = 0$$

($a \rightsquigarrow$ scale factor) With a suitable scaling, $u = a\Phi$
(the 'Mukhanov scaling'),

$$\ddot{u} + {}^{0}\Delta u + \left(m^2a^2 - \frac{\ddot{a}}{a}\right)u = 0$$

- Reduction of the Gowdy models
- Tensor perturbations and Mukhanov-Sasaki variable
in a perturbed FLRW.

Scalings of the field

[Cortez, Mena Marugán, Olmedo, Velhinho]

Another source of ambiguity: the choice of fundamental variables.

In particular, consider the following
time-dependent, linear canonical transformation:

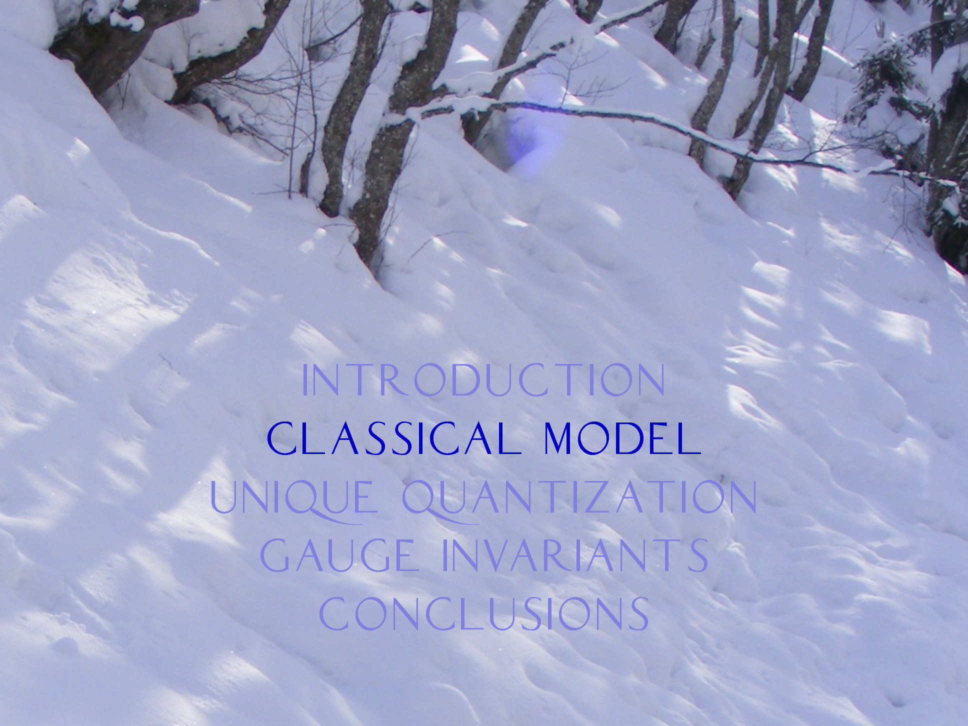
$$\Phi \mapsto F(t)\Phi,$$
$$\pi_\Phi \mapsto [F(t)]^{-1}\pi_\Phi + G(t)\sqrt{{}^0h}\,\Phi$$

Obviously, this transformation changes the field dynamics.
Furthermore,
it prevents the unitary implementation of the new dynamics.

(If $\dim \Sigma = 1$, a shift of the momentum is admissible.
But we do not get anything new $\rightsquigarrow$ unitary equivalence.)

These results hold in the presence of
corrections subdominant in the ultraviolet.

As an example of particular physical interest, we consider
the case of scalar perturbations around a FLRW universe
whose spatial sections are either three-tori (flat case $\rightsquigarrow k = 0$)
or three-spheres ($k = +1$).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FLRW model

- Metric $\rightsquigarrow$ ADM decomposition:

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dx^a\, dt + h_{ab} dx^a\, dx^b;$$

$$h_{ab}, \rightsquigarrow \text{ 3-metric, } N \rightsquigarrow \text{ lapse, } N_a \rightsquigarrow \text{ shift.}$$

- Matter content: scalar field Φ with mass $\tilde{m} = m/\sigma$.

In the FLRW model:

$$h_{ab}(t,x) = \sigma^2 e^{2\alpha(t)}\ {}^0h_{ab}(x),$$

$$N(t,x) = \sigma\ N_0(t),$$

$$N_a(t,x) = 0,$$

$$\Phi(t,x) = (l_0^{3/2}\sigma)^{-1}\ \varphi(t),$$

N_0, α, φ $\rightsquigarrow$ homogeneous variables,
${}^0h_{ab}$ $\rightsquigarrow$ unperturbed metric (in T^3 or S^3),
$\sigma^2 = 4\pi G/(3l_0^2)$, $l_0^3 = \int \sqrt{{}^0h}\, d^3x$.

Perturbed FLRW model

- Metric $\rightsquigarrow$ ADM decomposition:

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dx^a\, dt + h_{ab} dx^a\, dx^b;$$

$$h_{ab}, \rightsquigarrow \text{ 3-metric, } N \rightsquigarrow \text{ lapse, } N_a \rightsquigarrow \text{ shift.}$$

- Matter content: scalar field Φ with mass $\tilde{m} = m/\sigma$.

In the FLRW model with perturbations:

$$h_{ab}(t,x) = \sigma^2 e^{2\alpha(t)}\left[{}^0h_{ab}(x) + \epsilon_{ab}(t,x)\right],$$
$$N(t,x) = \sigma\left[N_0(t) + \delta N_0(t,x)\right],$$
$$N_a(t,x) = \delta N_a(t,x),$$
$$\Phi(t,x) = (l_0^{3/2}\sigma)^{-1}\left[\varphi(t) + \delta\varphi(t,x)\right],$$

N_0, α, φ $\rightsquigarrow$ homogeneous variables,
${}^0h_{ab}$ $\rightsquigarrow$ unperturbed metric (in T^3 or S^3),
$\sigma^2 = 4\pi G/(3l_0^2)$, $l_0^3 = \int \sqrt{{}^0h}\, d^3x$.

Eigenbasis of the Laplace-Beltrami operator

To expand the inhomogeneities, we use the orthogonal basis of the eigenfunctions of the Laplace-Beltrami (LB) operator of the reference static metric:

$$ {}^0\Delta Q^{nl} = -\omega_n^2 Q^{nl}, \quad n = 1, 2, \ldots $$

$l = 1, \ldots, \mathfrak{g}_n$ accounts for the degeneracy

(implicit in most formulas)

T^3: plane waves (sines and cosines).

S^3: (real) hyperspherical harmonics.

For vector and tensor quantities, we introduce

$$ P_a^n = \frac{1}{\omega_n^2}(Q^n)_{|a} $$

$$ P_{ab}^n = \frac{1}{3}\,{}^0h_{ab}Q^n + \frac{1}{\omega_n^2}(Q^n)_{|ab} $$

$|$ $\rightsquigarrow$ covariant derivarives w.r.t. ${}^0h_{ab}$.

We only consider harmonics that can be obtained in this way.

Expansion of the inhomogeneities

We expand the inhomogeneities in this way:

$$\epsilon_{ab}(t,x) = 2\sum_n \left[a_n(t)Q^n(x)^0h_{ab} + 3b_n(t)P^n_{ab}(x)\right],$$
$$\delta N_0(t,x) = \sum_n N_0(t)g_n(t)Q^n(x),$$
$$\delta N_a(t,x) = \sigma^2 e^{\alpha(t)}\sum_n k_n(t)P^n_a(x),$$
$$\delta\varphi(t,x) = \sum_n f_n(t)Q^n(x);$$

a_n, b_n, f_n, g_n and k_n parametrize the inhomogeneities.

We treat them as perturbative coefficients,
truncating the action at quadratic order in them.

Perturbative Hamiltonian

Naturally, the Hamiltonian is a linear combination of constraints:

$$H = N_0 \left(H_{|0} + \sum H^n_{|2} \right) + \sum N_0 g_n H^n_{|1} + \sum k_n H^n_{-1}.$$

- $H_{|0} + \sum H^n_{|2} \rightsquigarrow$ (perturbed) Hamiltonian constraint
- $H^n_{|1} \rightsquigarrow$ linear Hamiltonian constraints
- $H^n_{-1} \rightsquigarrow$ momentum constraints

We fix the local gauge freedom classically,
removing the constraints $H^n_{|1}$ and H^n_{-1}
and parametrizing the inhomogeneities with $\delta\varphi$ and its momentum.

Gauge fixing

Gauge of constant curvature	**Longitudinal gauge**
$\boxed{a_n = 0 = b_n}$	$\boxed{b_n = 0 = \pi_{a_n} - \pi_\alpha a_n - 3\pi_\varphi f_n =: C_n}$

Well-posedness:

$\det\begin{pmatrix} \{a_n, H^n_{-1}\} & \{b_n, H^n_{-1}\} \\ \{a_n, H^n_{\|1}\} & \{b_n, H^n_{\|1}\} \end{pmatrix} \propto \pi_\alpha$	$\det\begin{pmatrix} \{b_n, H^n_{-1}\} & \{C_n, H^n_{-1}\} \\ \{b_n, H^n_{\|1}\} & \{C_n, H^n_{\|1}\} \end{pmatrix} > 0$

Secondary constraints:

$\{a_n, H\} = 0 = \{b_n, H\}$	$\{b_n, H\} = 0 = \{C_n, H\}$

These constraints fix the value of the Lagrange multipliers

	$k_n = 0 \; (N_a = 0)$

Reduced symplectic structure

$\dot{a}_n \pi_{a_n}, \dot{b}_n \pi_{b_n} = 0$	$\dot{a}_n \pi_{a_n} \neq 0$
$\alpha, \pi_\alpha, \varphi, \pi_\varphi, f_n, \pi_{f_n}$ canonical	$\alpha, \pi_\alpha, \varphi, \pi_\varphi, f_n, \pi_{f_n}$ not canonical
	$\alpha \mapsto \alpha + \frac{1}{2}\sum a_n^2, \quad \pi_\alpha \mapsto \pi_\alpha$
	$\varphi \mapsto \varphi + 3\sum a_n f_n, \quad \pi_\varphi \mapsto \pi_\varphi$
	$f_n \mapsto f_n, \quad \pi_{f_n} \mapsto \pi_{f_n} - 3\pi_\varphi a_n$

Second-order Hamiltonian constraint

After the reduction,

$$H^n_{|2} = \frac{1}{2} e^{-\alpha} \left(E^n_{\pi\pi} \pi^2_{f_n} + 2 E^n_{f\pi} f_n \pi_{f_n} + E^n_{ff} f^2_n \right).$$

For instance, in the gauge $a_n = 0 = b_n$,

$$E_{\pi\pi} = e^{-2\alpha} \left(1 + \frac{3k}{\omega_n^2 - 3k} \frac{\pi_\varphi^2}{\pi_\alpha^2} \right)$$

$$E_{f\pi} = -3 \frac{\pi_\varphi^2}{\pi_\alpha} + \frac{3k}{\omega_n^2 - 3k} \frac{\pi_\varphi}{\pi_\alpha^2} \left(e^{6\alpha} m^2 \varphi - 3 \pi_\alpha \pi_\varphi \right)$$

$$E_{ff} = e^{2\alpha} \left[\omega_n^2 + m^2 e^{2\alpha} + (\text{background function}) + k O(\omega_n^{-2}) \right]$$

Is there a canonical transformation

linear in the inhomogeneous sector

that leads this Hamiltonian to the KG form?

Canonical transformation

If we restrict to local transformations, we cannot remove the terms subdominant in ω_n. But still we can scale the field and choose its momentum in the following form:

Gauge of constant curvature

$$\bar{\alpha} = \alpha - \frac{1}{2}\left(3\frac{\pi_\varphi^2}{\pi_\alpha^2} - 1\right)\sum f_n^2$$

$$\pi_{\bar{\alpha}} = \pi_\alpha - \sum\left[f_n \pi_{f_n} - \left(3\frac{\pi_\varphi^2}{\pi_\alpha} + \pi_\alpha\right) f_n^2\right]$$

$$\bar{\varphi} = \varphi + 3\frac{\pi_\varphi}{\pi_\alpha}\sum_n f_n^2$$

Longitudinal gauge

$$\bar{\alpha} = \alpha + \frac{1}{2}\sum f_n^2$$

$$\pi_{\bar{\alpha}} = \pi_\alpha - \sum\left(f_n \pi_{f_n} - \pi_\alpha f_n^2\right)$$

$$\bar{\varphi} = \varphi$$

Both gauges

$$\pi_{\bar{\varphi}} = \pi_\varphi$$

$$\bar{f}_n = e^\alpha f_n$$

Gauge of constant curvature

$$\pi_{\bar{f}_n} = e^{-\alpha}\left[\pi_{f_n} - \left(3\frac{\pi_\varphi^2}{\pi_\alpha} + \pi_\alpha\right) f_n\right]$$

Longitudinal gauge

$$\pi_{\bar{f}_n} = e^{-\alpha}\left(\pi_{f_n} - \pi_\alpha f_n\right)$$

so as to arrive at a KG Hamiltonian with subdominant corrections.
(If $k = 0$, the corrections vanish in the $a_n = 0 = b_n$ gauge.
This case will be discussed later.)

Dynamical equations

Using conformal time η, defined by $e^{\alpha} d\eta = N_0 dt$.

- Canonically conjugate momentum of $\bar{f}_{nl}$:

$$\pi_{\bar{f}_{nl}} = [1 + p_n(\eta)]\dot{\bar{f}}_{nl} + q_n(\eta)\bar{f}_{nl},$$

with p_n, $q_n \sim O(\omega_n^{-2})$.

- Equation of motion of $\bar{f}_{nl}$:

$$\ddot{\bar{f}}_{nl} + r_n(\eta)\dot{\bar{f}}_{nl} + [\omega_n^2 + s(\eta) + s_n(\eta)]\bar{f}_{nl} = 0,$$

where r_n, $s_n = (\omega_n^{-2})$.

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with $p_n,\ q_n \sim O(\omega_n^{-2})$.

- Equation of motion of $\bar{f}_{nl}$:

$$\ddot{\bar{f}}_{nl} + r_n(\eta)\dot{\bar{f}}_{nl} + [\omega_n^2 + s(\eta) + s_n(\eta)]\bar{f}_{nl} = 0,$$

where $r_n,\ s_n = (\omega_n^{-2})$.

Creation-like variables

We introduce the creation- and annihilation-like variables adapted to the massless representation, characterized by J_0:

$$\begin{pmatrix} a_{\bar{f}_{nl}} \\ a^*_{\bar{f}_{nl}} \end{pmatrix} = \frac{1}{\sqrt{2\omega_n}} \begin{pmatrix} \omega_n & i \\ \omega_n & -i \end{pmatrix} \begin{pmatrix} \bar{f}_{nl} \\ \bar{\pi}_{\bar{f}_{nl}} \end{pmatrix}.$$

The action of the complex structure J_0 is then

$$J_0 \begin{pmatrix} a_{\bar{f}_{nl}} \\ a^*_{\bar{f}_{nl}} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a_{\bar{f}_{nl}} \\ a^*_{\bar{f}_{nl}} \end{pmatrix}$$

By construction, J_0 is invariant under the symmetries of the LB operator.

Besides, it allows a unitary implementation of the dynamics.

Dynamics

Classical time evolution does not mix modes.

Therefore, it is represented by a block-diagonal matrix $\mathcal{U}$:

$$\begin{pmatrix} a_{\bar{f}_{nl}}(\eta) \\ a^*_{\bar{f}_{nl}}(\eta) \end{pmatrix} = \mathcal{U}_n(\eta,\eta_0) \begin{pmatrix} a_{\bar{f}_{nl}}(\eta_0) \\ a^*_{\bar{f}_{nl}}(\eta_0) \end{pmatrix}.$$

Each block has the form

$$\mathcal{U}_n(\eta,\eta_0) = \begin{pmatrix} \alpha_n(\eta,\eta_0) & \beta_n(\eta,\eta_0) \\ \beta^*_n(\eta,\eta_0) & \alpha^*_n(\eta,\eta_0) \end{pmatrix},$$

with $|\alpha_n|^2 - |\beta_n|^2 = 1$.

Is there a unitary operator U implementing this transformation in the quantum theory, $\hat{a}_{\bar{f}_{nl}}(\eta) = U(\eta,\eta_0)\hat{a}_{\bar{f}_{nl}}(\eta_0)U^{-1}(\eta,\eta_0)$?

$\mathcal{U}$ is unitarily implementable

$\Leftrightarrow$ $J_0\mathcal{U}J_0 + \mathcal{U}$ is a Hilbert-Schmidt operator.

$\Leftrightarrow$ $\displaystyle\sum_{n,l}|\beta_n|^2 = \sum_n \mathfrak{g}_n|\beta_n(\eta,\eta_0)|^2 < \infty, \quad \forall\eta$

It depends only on the asymptotic behaviour of β_n!

Asymptotical analysis

We write $\bar{f}_{nl}$ in the following way:

$$\bar{f}_{nl}(\eta) = A_{nl} e^{\omega_n \Theta_n(\eta)} + A^*_{nl} e^{\omega_n \Theta^*_n(\eta)},$$

$$\text{where } \Theta_n(\eta) = -i(\eta - \eta_0) + \int_{\eta_0}^{\eta} \frac{W_n(\bar{\eta})}{\omega_n} d\bar{\eta}.$$

W_n satisfies the Ricatti equation

$$\dot{W}_n = i\omega_n r_n - s_n + (2i\omega_n - r_n) W_n - W_n^2$$

$$\text{with the initial condition } W_n(\eta_0) = 0.$$

Its asymptotic behaviour is found to be

$$W_n(\eta) = \frac{1}{2i\omega_n} \left[s(\eta) - s(\eta_0) e^{2i\omega_n(\eta - \eta_0)} - e^{2i\omega_n \eta} \int_{\eta_0}^{\eta} \frac{\dot{s}(\tilde{\eta})}{e^{2i\omega_n \tilde{\eta}}} d\tilde{\eta} \right] + O(\omega_n^{-2})$$

Unitarity of the evolution

Now we can write the asymptotic behaviour
of the Bogoliubov coefficients α_n and β_n.

We find that $\begin{cases} \alpha_n = e^{i\omega_n(\eta-\eta_0)} + O(\omega_n^{-2}), \\ \beta_n \sim O(\omega_n^{-2}), \end{cases}$

Consequently, $\sum_n \mathfrak{g}_n |\beta_n|^2 < \infty$, since

$$\mathfrak{g}_n \begin{cases} = \omega_n^2 + 1, \text{ in } S^3, \\ \lesssim \omega_n^2, \text{ in } T^3. \end{cases}$$

The dynamics is indeed unitarily implementable.

But are there other inequivalent representations
with these properties?

Invariant complex structures in S^3

Isometries of $S^3 \rightsquigarrow$ SO(4) rotations.

We are interested only in SO(4)-invariant complex structures.

Each eigenspace of the LB operator carries an irreducible representation of SO(4) $\rightsquigarrow$ an invariant complex structure cannot mix different eigenspaces.

And recall that $\{\bar{f}_{nl}\}_l$ and $\{\pi_{\bar{f}_{nl}}\}_l$ transform in the same way.

By Schur's lemma,

an SO(4)-invariant CS J must be block diagonal:

$$J\begin{pmatrix}\bar{f}_{nl}\\ \pi_{\bar{f}_{nl}}\end{pmatrix} = \begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}\begin{pmatrix}\bar{f}_{nl}\\ \pi_{\bar{f}_{nl}}\end{pmatrix},$$

with $a_n,\, b_n,\, c_n,\, d_n \in \mathbb{R}$.

Invariant complex structures in T^3

[Castelló Gomar, Cortez, Martín-de Blas, Mena Marugán, Velhinho]

In T^3, the symmetry group is $U(1) \times U(1) \times U(1)$.
$\rightsquigarrow$ Abelian compact group
$\rightsquigarrow$ one-dimensional complex irreducible representations.

However, the real irreducible reps. are two-dimensional.

We can apply Schur's lemma in the complex basis to decompose any invariant complex structure in 2×2 blocks.

The requirement that $J(\Omega\cdot, \cdot)$ be positive definite implies that J transforms in the same way a mode and its complex conjugate.

This allows us to pass to the real basis, as in S^3.

Relation between representations

Any invariant complex structure J must be related to J_0 by a block-diagonal symplectomorphism $\mathcal{K}$, whose blocks have the form

$$\mathcal{K}_n = \begin{pmatrix} \kappa_n & \lambda_n \\ \lambda_n^* & \kappa_n^* \end{pmatrix},$$

with $|\kappa_n|^2 - |\lambda_n|^2 = 1$ (so $|\kappa_n| > 1$).

Thus, $J = \mathcal{K} J_0 \mathcal{K}^{-1}$.

$\mathcal{U}$ is unitarily implementable with respect w.r.t. $J = \mathcal{K} J_0 \mathcal{K}^{-1}$

$\Leftrightarrow$ $\mathcal{K}^{-1}\mathcal{U}\mathcal{K}$ is unitarily implementable w.r.t. J_0.

$\Leftrightarrow$ The sequence $\{\sqrt{\mathfrak{g}_n}\beta_n^J\}$ is square summable, where β_n^J are the 'β coefficients' of $\mathcal{K}^{-1}\mathcal{U}\mathcal{K}$.

Let us assume that this is the case.

Equivalence of representations

$$\sqrt{\mathfrak{g}_n}\beta_n^J = \sqrt{\mathfrak{g}_n}\big[(\kappa_n^*)^2\beta_n - \lambda_n^2\beta_n^* + 2i\kappa_n^*\lambda_n\mathfrak{I}(\alpha_n)\big].$$

We can substract square-summable contributions until we arrive at

$$\sum_n^M \mathfrak{g}_n \left|\frac{\lambda_n}{\kappa_n^*}\right|^2 \sin^2\left[\omega_n(\eta-\eta_0) + \int_{\eta_0}^{\eta}\frac{s(\bar\eta)}{2\omega_n}d\bar\eta\right] < \infty$$

for arbitrarily large M.

Using Luzin's theorem to integrate the above function over a suitable set, and since the integral of the sine is bounded from below, we conclude that

$$\sum_n \mathfrak{g}_n \left|\lambda_n/\kappa_n^*\right|^2 < \infty.$$

This in turn implies that $\sum_n \mathfrak{g}_n|\lambda_n|^2 < \infty$, as $1-|\lambda_n/\kappa_n|^2 = 1/|\kappa_n|^2$.

Therefore, the two representations are unitarily equivalent.

Rescaling of the variables

What if we had choosen other variables?

For example, we could perform a time-dependent linear canonical transformation using background functions:

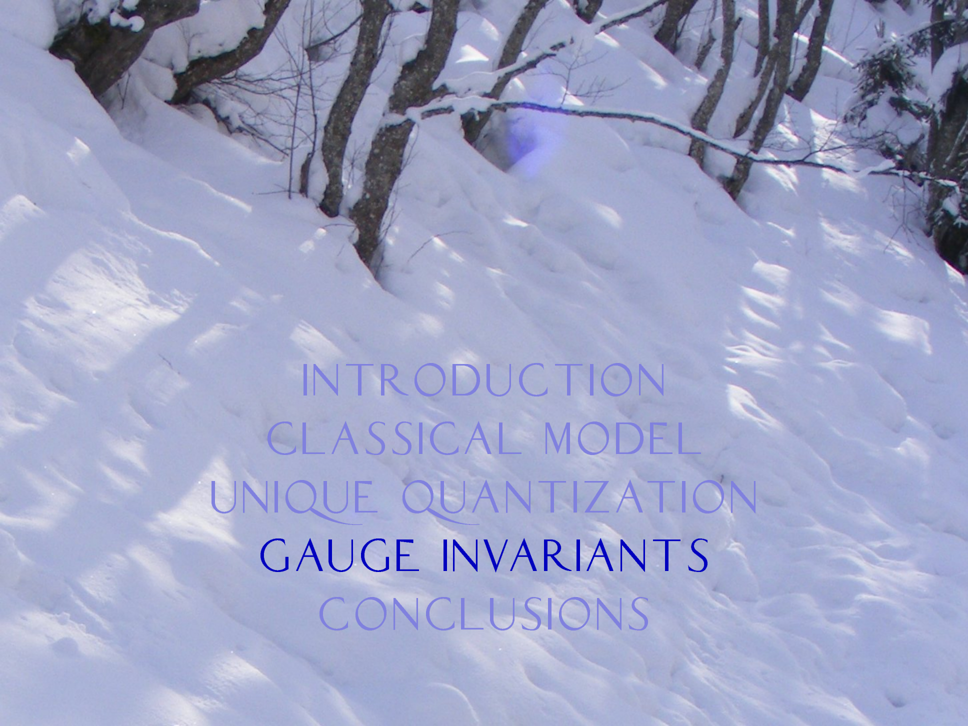
$$\breve{f}_n = F(\eta)\bar{f}_n,$$
$$\pi_{\breve{f}_n} = \pi_{\bar{f}_n}/F(\eta) + G(\eta)\bar{f}_n.$$

(we can fix $F(\eta_0) = 1$ and $G(\eta_0) = 0$).

Naturally, such a transformation changes the dynamics, with new Bogoliubov coefficients that depend on F and G.

Is there any invariant complex structure $J = \mathcal{K}J_0\mathcal{K}^{-1}$ that implements unitarily the new dynamics?

If that were the case, the sequence $\{\sqrt{\mathfrak{g}_n}\breve{\beta}_n^J\}$ would be square-summable. But a careful analysis of its asymptotic behaviour shows that then F and G must be constant!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Gauge transformations

Consider infinitesimal coordinate transformations that do not change the background metric, $x^\mu \mapsto x^\mu + \xi^\mu$, where ξ^μ is a perturbation that can be parametrized as

$$\xi_0 = \sigma^2 N_0 \sum_n T_n Q^n,$$
$$\xi_a = \sigma^2 e^\alpha \sum_n L_n P_a^n,$$

The inhomogeneities would change in this way:

$$a_n \mapsto a_n + e^{-\alpha}\left(\dot{\alpha} T_n + \tfrac{1}{3} L_n\right),$$
$$b_n \mapsto b_n - \tfrac{1}{3} e^{-\alpha} L_n,$$
$$f_n \mapsto f_n + e^{-\alpha} \dot{\varphi} T_n,$$
$$g_n \mapsto g_n + e^{-\alpha} \dot{T}_n,$$
$$k_n \mapsto k_n - N_0 e^{-\alpha}\left(\omega_n^2 T_n + \dot{L}_n - \dot{\alpha} L_n\right).$$

It is clear that these coefficients can be combined to give gauge-invariant quantities.

A gauge-invariant canonical pair

In particular, consider the gauge-invariant combination

$$\Psi_n = \frac{1}{\sqrt{\omega_n^2 - 3k}} \frac{e^{-\alpha}}{\pi_\varphi} [\pi_\varphi \pi_{f_n} + (e^{6\alpha} m^2 \varphi - 3\pi_\alpha \pi_\varphi) f_n - 3\pi_\varphi^2 a_n]$$

Motivation:

- $\{\Psi_n, \dot{\Psi}_n\} = 1$
- Ψ satisfies a KG equation with time-dependent mass.

In the reduced system,

$$\Psi_n = \frac{1}{\sqrt{\omega_n^2 - 3k}} \left(\bar{\pi}_{\bar{f}_n} + \chi \bar{f}_n\right),$$
$$\dot{\Psi}_n = \frac{\chi}{\sqrt{\omega_n^2 - 3k}} (\bar{\pi}_{\bar{f}_n} + \chi \bar{f}_n) - \sqrt{\omega_n^2 - 3k}\, \bar{f}_n,$$

where χ is a background function which depends on the gauge.

Equivalence of the representations

We can assign a new preferred CS to these variables.
Again, it will be related to J_0
by a block diagonal symplectomorphism:

$$\begin{pmatrix} a_{\Psi_n} \\ a^*_{\Psi_n} \end{pmatrix} = \frac{1}{\sqrt{2\omega_n}} \begin{pmatrix} \omega_n & i \\ \omega_n & -i \end{pmatrix} \begin{pmatrix} \Psi_n \\ \dot{\Psi}_n \end{pmatrix} = \begin{pmatrix} \kappa_n & \lambda_n \\ \lambda^*_n & \kappa^*_n \end{pmatrix} \begin{pmatrix} a_{\bar{f}_n} \\ a^*_{\bar{f}_n} \end{pmatrix}$$

The coefficients λ_n are given by

$$\lambda_n = i \frac{\chi^2 + 3k}{2\omega_n \sqrt{\omega_n^2 - 3k}}$$

Since $\sum_n \mathfrak{g}_n |\lambda_n|^2 < \infty$, the two quantizations are unitarily related.

Mukhanov-Sasaki variable

Let us define the Mukanov-Sasaki variable:

$$v_n = e^{\alpha}\left(f_n + \frac{\pi_\varphi}{\pi_\alpha}(a_n + b_n)\right),$$

In the flat case, $\{v_n, \dot{v}_n\} = 1$ and, moreover, v_n the MS equation:

$$\ddot{v}_n - \left(\omega_n^2 - \frac{\ddot{z}}{z}\right) v_n = 0,$$

with $z = -e^{\alpha}\pi_\varphi/\pi_\alpha$.

In the gauge $a_n = 0 = b_n$, $v_n = \bar{f}_n$.

In the longitudinal gauge, they do not coincide,
but the corresponding quantizations are unitarily equivalent.

A footnote

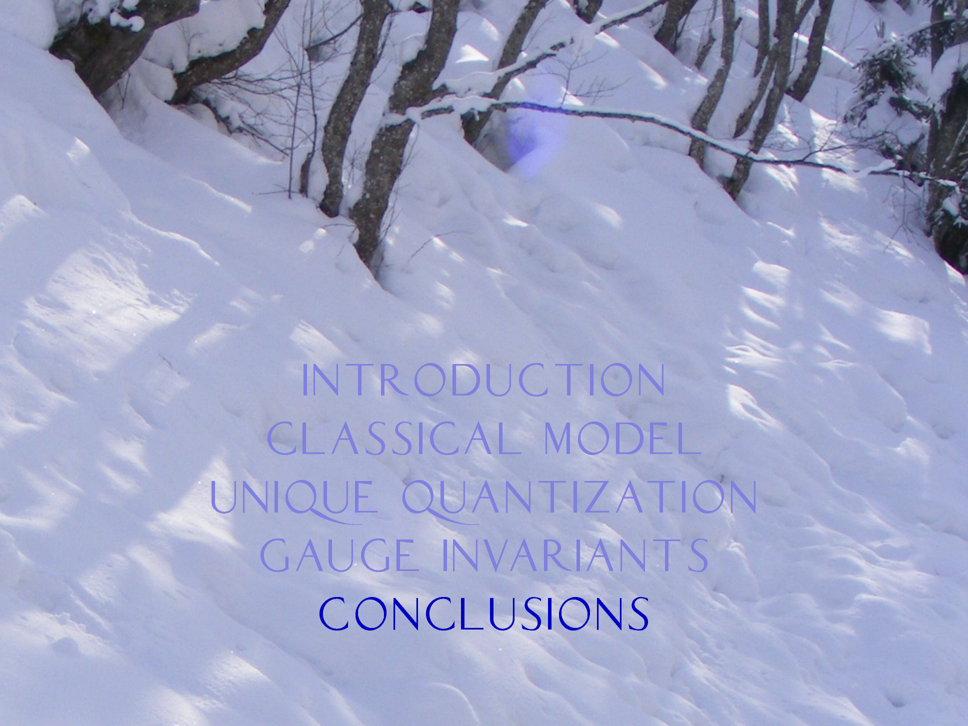
[Cortez, Fonseca, Martín-de Blas, Mena Marugán]

There is a recent further result:

Consider a KG field with time-dependent mass,
and mode- and time-dependent linear canonical transformations
compatible with the symmetries of the LB operator.

Every transformation of this kind that does not change
the KG form of the equation of motion
(but possibly changes the time-dependent mass),
must be unitarily implementable.

In the cases considered here, there are subdominant corrections,
but we have seen that they are irrelevant.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- The presence of subdominant corrections
  does not spoil the uniqueness result for a KG field
  with time-dependent mass.

- Together with the symmetry, the unitarity of the dynamics
  imposes a strong constraint in quantum field theory
  in non-stationary spacetimes.
  It selects not only a preferred equivalence class
  of representations,
  but also a privileged scaling of the field and its momentum.

- The scaling of the field is necessary to have unitary dynamics.