Configuration Spaces: Reduction vs. Quantization

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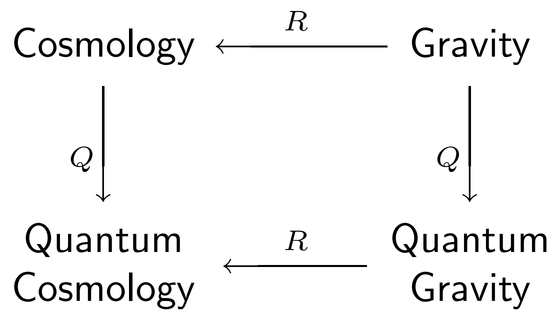
Institut für Mathematik

Tux, February 2014

1 Introduction



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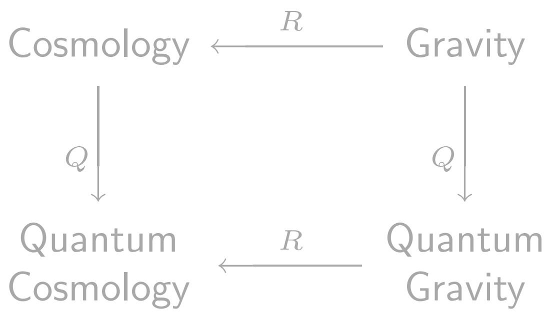
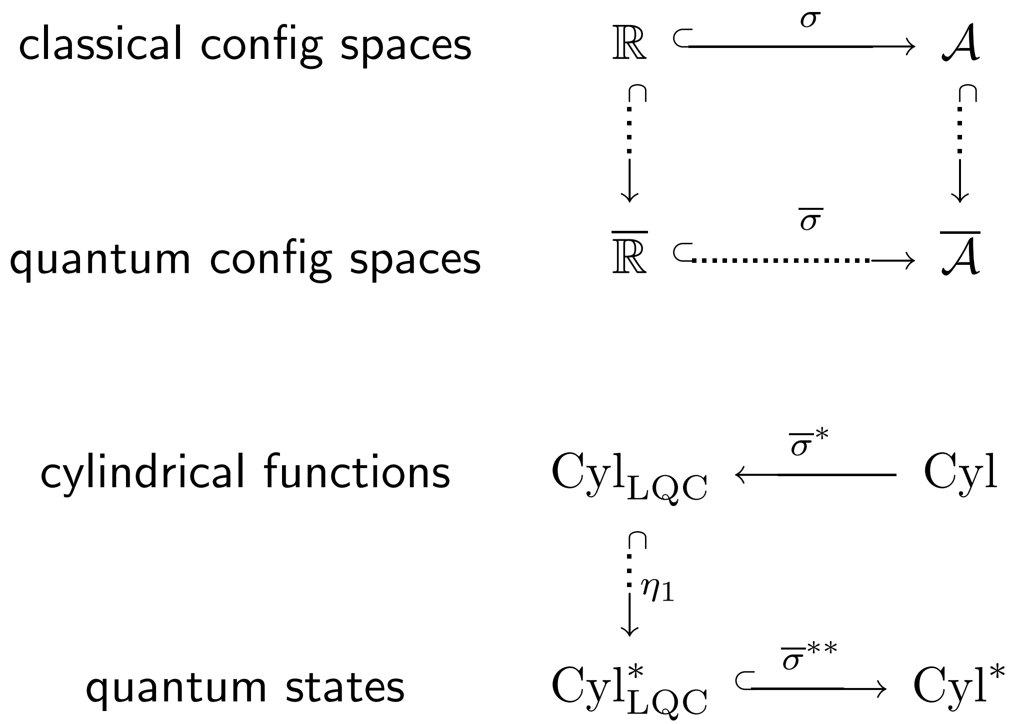
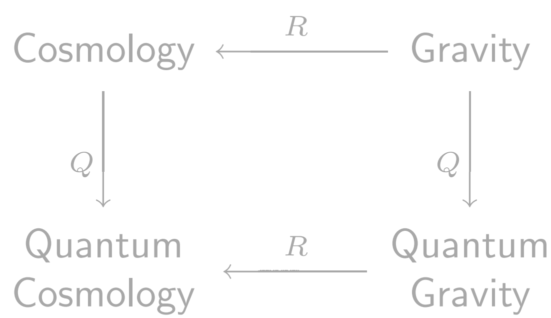




1 Introduction

Given: Embedding σ of classical configuration spaces



classical config spaces $\quad \mathbb{R} \overset{\sigma}{\hookrightarrow} \mathcal{A}$

quantum config spaces $\quad \overline{\mathbb{R}} \overset{\overline{\sigma}}{\hookrightarrow} \overline{\mathcal{A}}$

cylindrical functions $\quad \mathrm{Cyl}_{\mathrm{LQC}} \overset{\overline{\sigma}^*}{\longleftarrow} \mathrm{Cyl}$

η_1

quantum states $\quad \mathrm{Cyl}^*_{\mathrm{LQC}} \overset{\overline{\sigma}^{**}}{\hookrightarrow} \mathrm{Cyl}^*$

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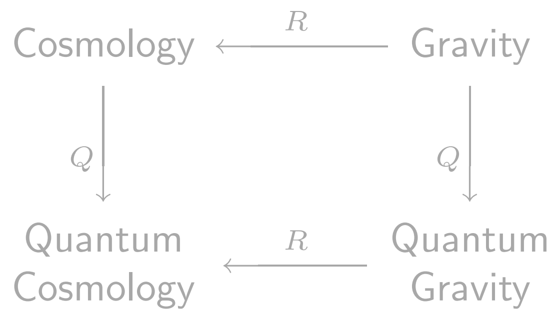
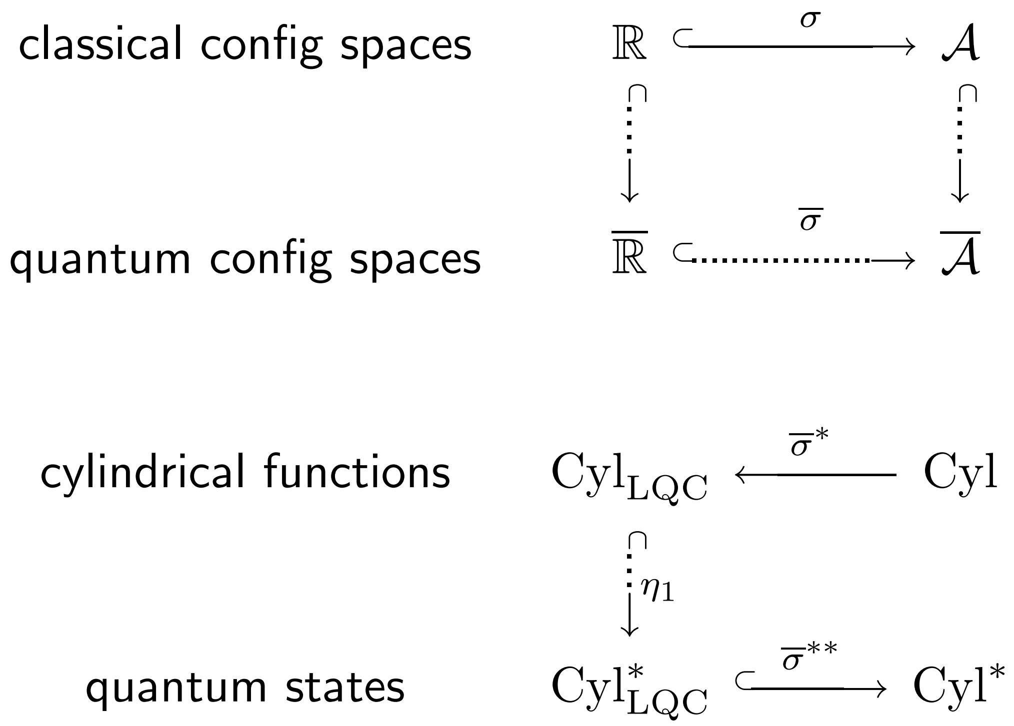
Task 1: Extend σ to $\overline{\sigma}$





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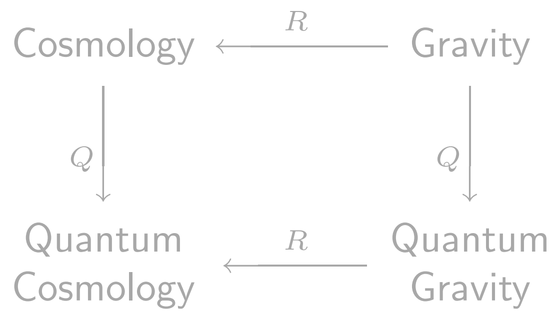

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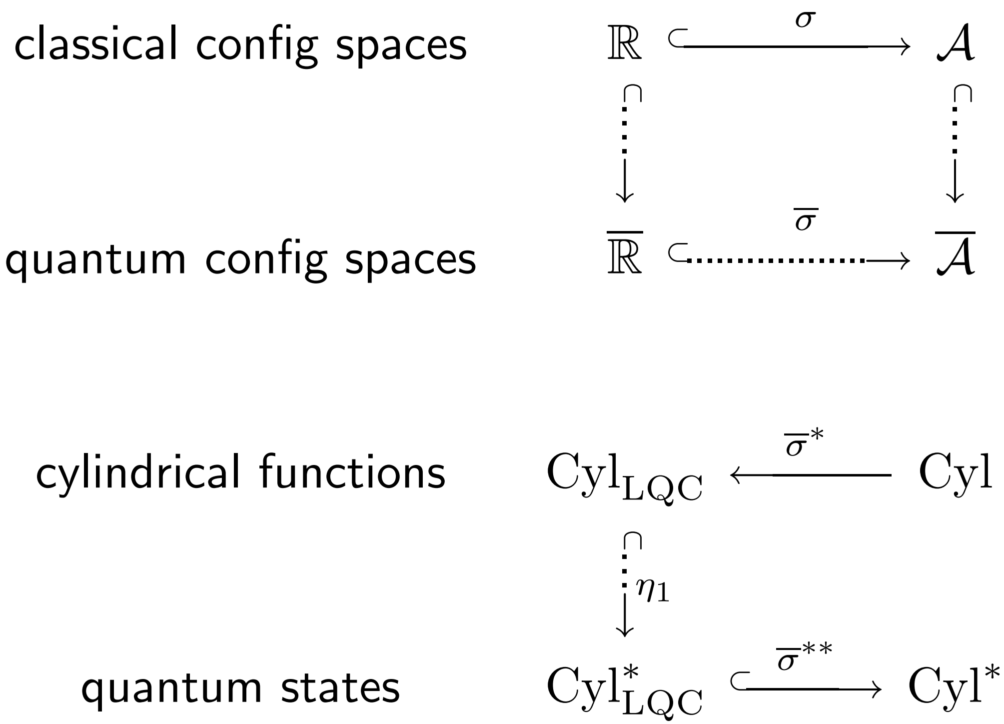
Task 1: Extend σ to $\overline{\sigma}$

Task 2: Choose Gelfand triple in LQC



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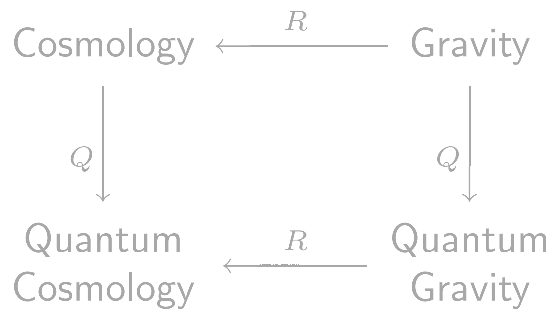
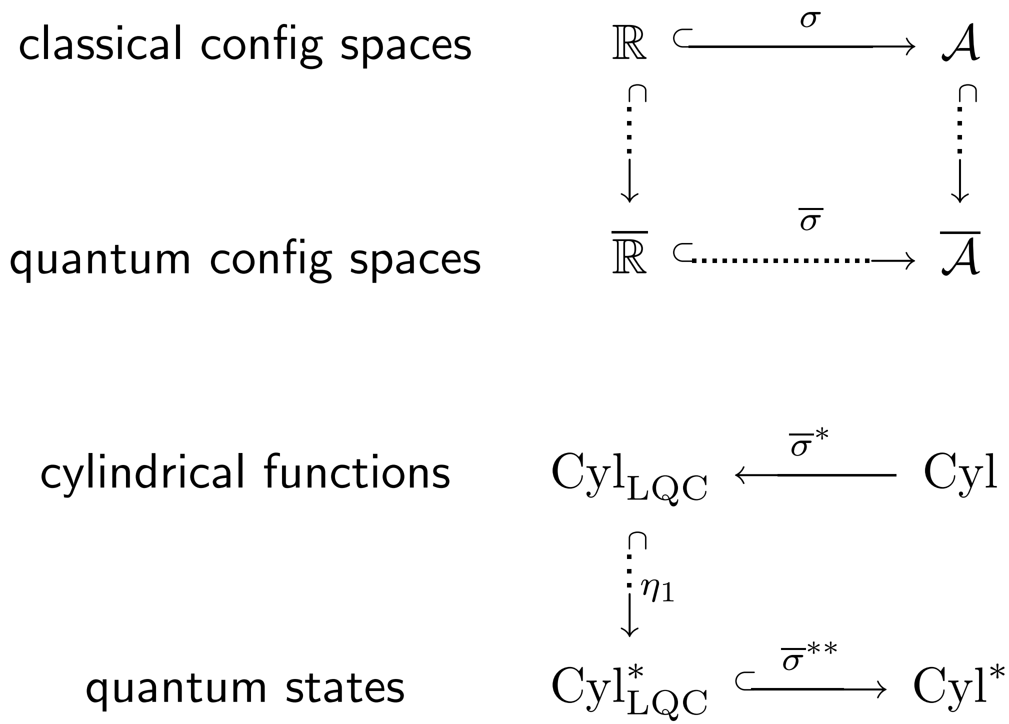


Given: Embedding σ of classical configuration spaces

Task 1: Extend σ to $\overline{\sigma}$

Task 2: Choose Gelfand triple in LQC

LQG
$\mathcal{A}$
$(h_\gamma)^i_j$
Cyl
$\overline{\mathrm{Cyl}}$
$\overline{\mathcal{A}}$
μ_{AL}
Cyl^*



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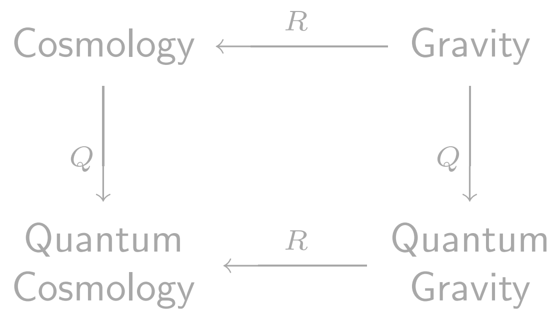

LQG		Structure	Definition
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μ_{AL}			
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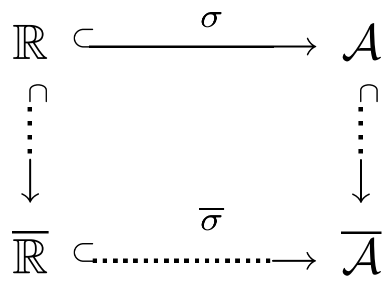

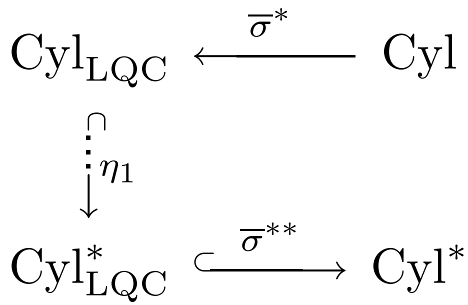

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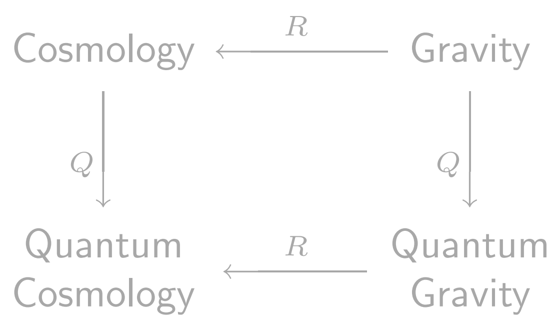

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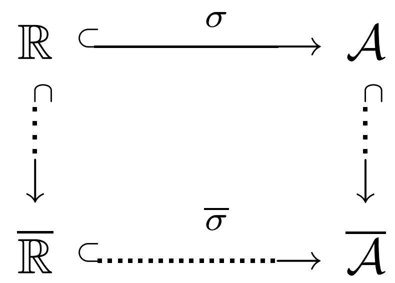

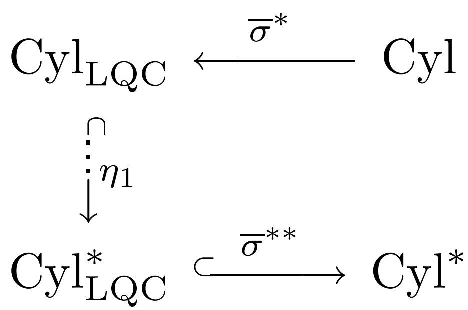

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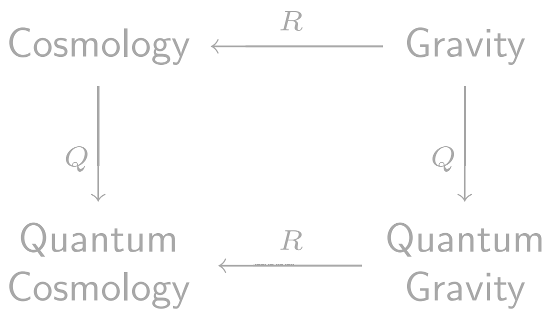

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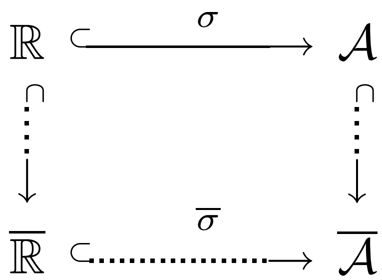

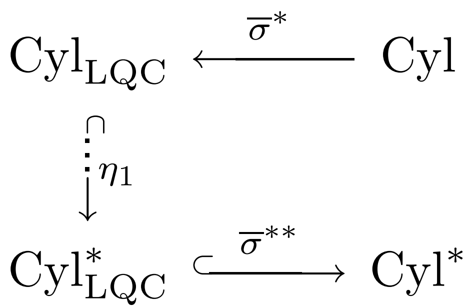

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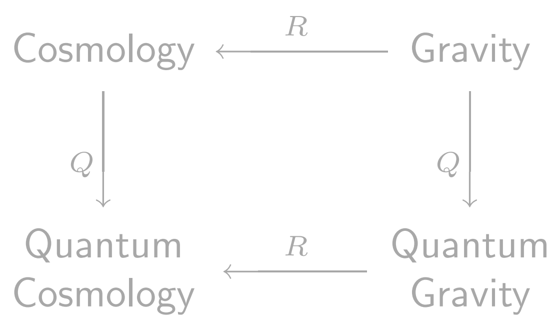

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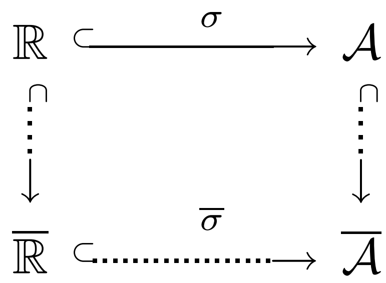

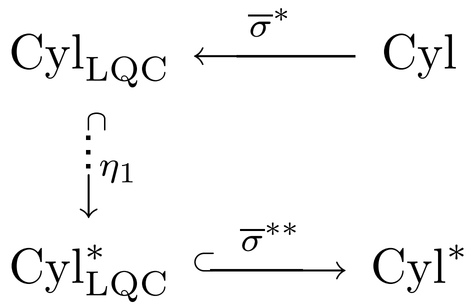

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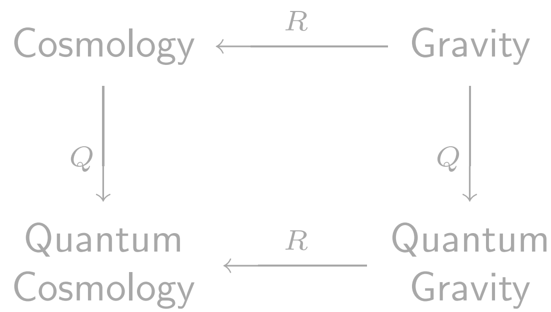

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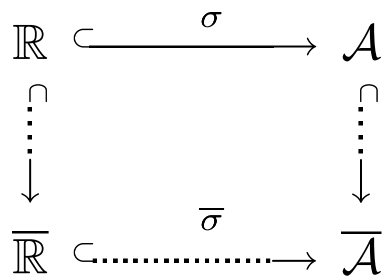

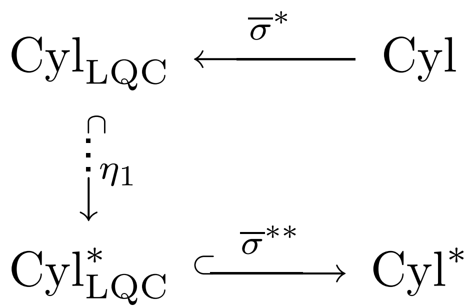

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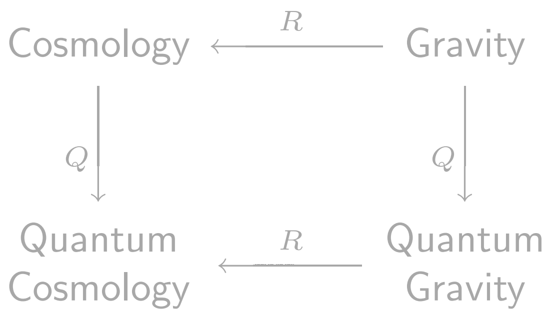

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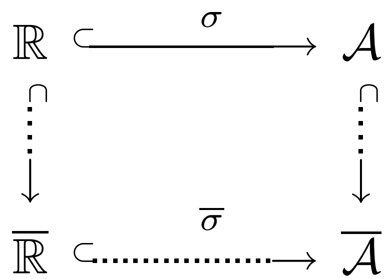

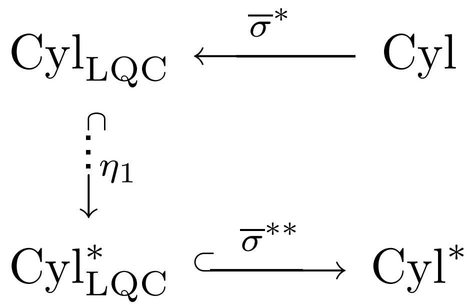

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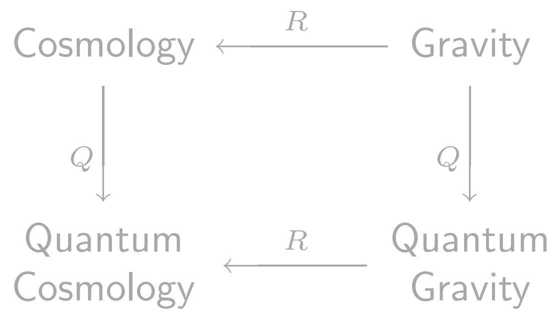

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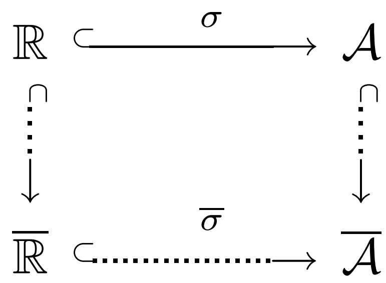

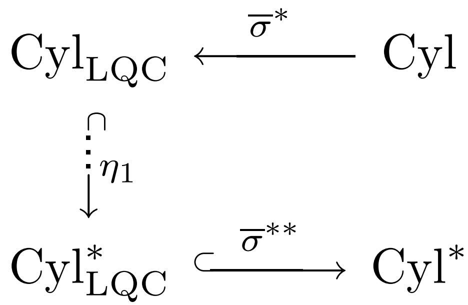

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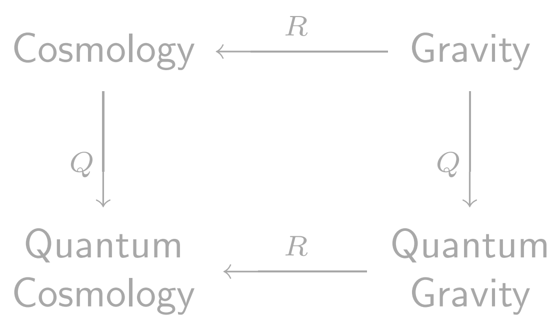

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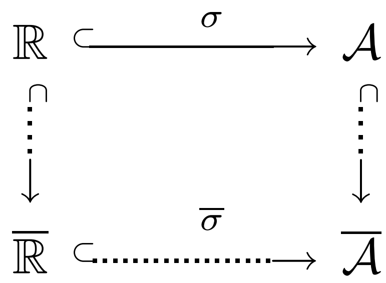

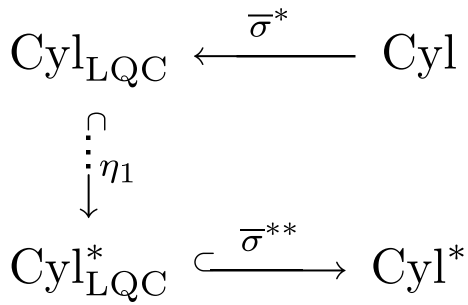

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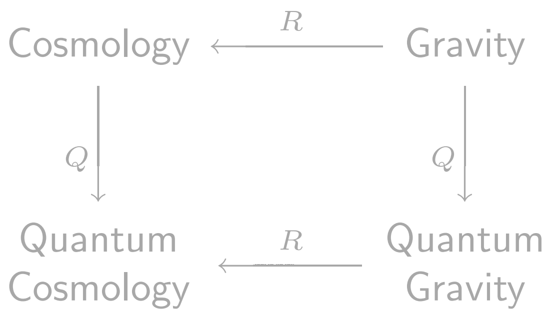

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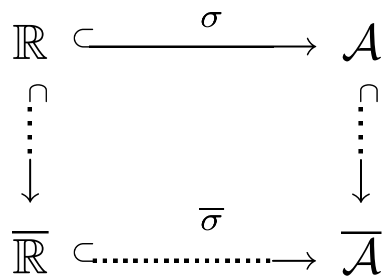

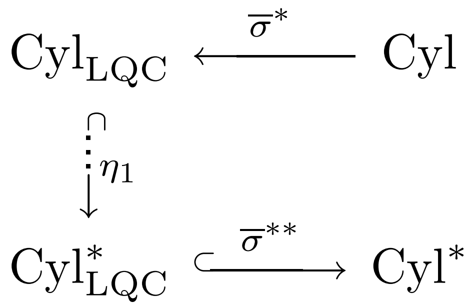

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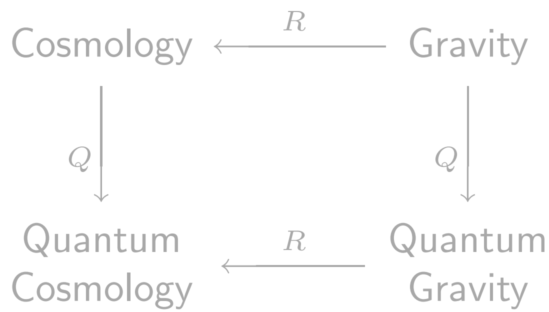

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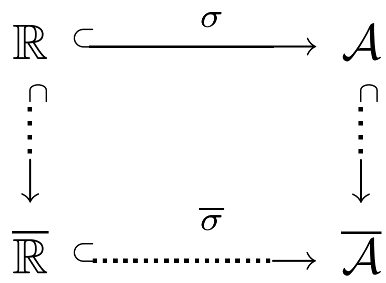
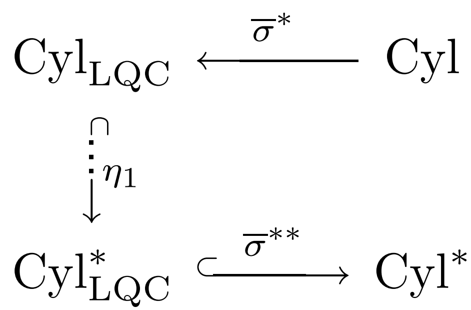
μ_{AL}		measure	*repr theory*
Cyl^*			





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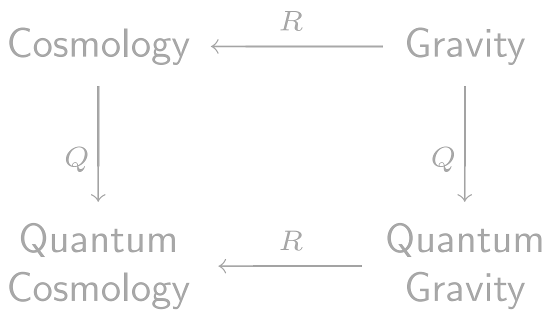

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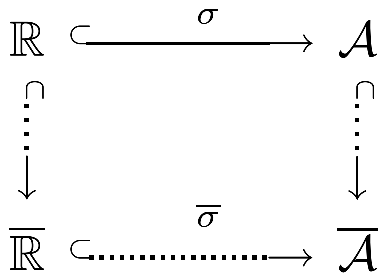
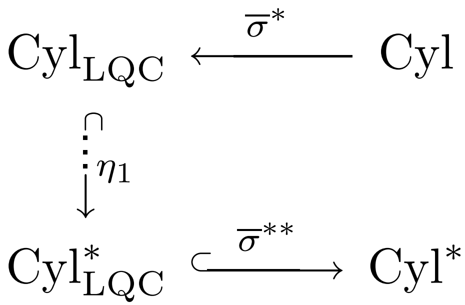
μ_{AL}		measure	*repr theory*
Cyl^*	$\mathfrak{B}^*$	state space	$\mathfrak{B}^*$





2 Quantization of Configuration Spaces

$\mathbf{S}$	$\mathcal{A}$
a	$(h_\gamma)^i_j$
$\mathfrak{B}$	Cyl
$\mathfrak{A}$	$\overline{\mathrm{Cyl}}$
$\overline{\mathbf{S}}$	$\overline{\mathcal{A}}$

2 Quantization of Configuration Spaces

Definition: **Natural Mapping** $\iota : \mathbf{S} \longrightarrow \operatorname{spec}\mathfrak{A}$

$$\begin{aligned} \iota(s) : \quad \mathfrak{A} &\longrightarrow \mathbb{C} \\ a &\longmapsto a(s) \end{aligned}$$

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Lemma: $\widetilde{a} \circ \iota = a$

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Proof:

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Lemma: $\widetilde{a} \circ \iota = a$

Proof:

Gelfand transform

$$\widetilde{a} : \quad \begin{array}{ccc} \operatorname{spec} \mathfrak{A} & \longrightarrow & \mathbb{C} \\ \chi & \longmapsto & \chi(a) \end{array}$$

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Gelfand-Naimark theorem

$$\begin{array}{rccc} \sim: & \mathfrak{A} & \longrightarrow & C_0(\operatorname{spec}\mathfrak{A}) \\ & a & \longmapsto & \widetilde{a} \end{array} \qquad \text{isometric } *\text{-isomorphism}$$

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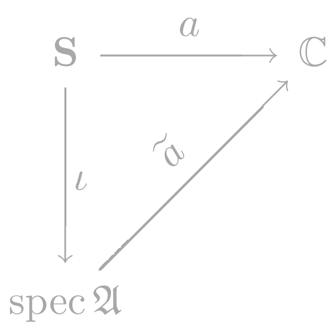

Proof: $[\widetilde{a} \circ \iota](s) \equiv \widetilde{a}(\iota(s)) = [\iota(s)](a) = a(s)$

Gelfand transform

$$\begin{array}{rccc} \widetilde{a}: & \operatorname{spec}\mathfrak{A} & \longrightarrow & \mathbb{C} \\ & \chi & \longmapsto & \chi(a) \end{array}$$

Gelfand-Naimark theorem

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isometric $*$-isomorphism

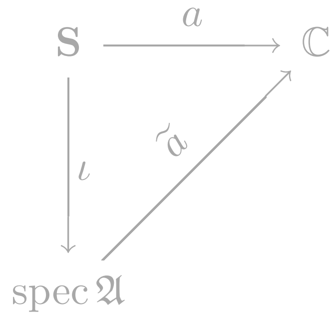
2 Quantization of Configuration Spaces

Definition: **Natural Mapping** $\iota : \mathbf{S} \longrightarrow \operatorname{spec} \mathfrak{A}$

$$\begin{array}{rccc} \iota(s) : & \mathfrak{A} & \longrightarrow & \mathbb{C} \\ & a & \longmapsto & a(s) \end{array}$$

Proposition: 1. $\iota(\mathbf{S})$ dense in $\operatorname{spec} \mathfrak{A}$

$\mathbf{S}$	$\mathcal{A}$
a	$(h_\gamma)^i_j$
$\mathfrak{B}$	Cyl
$\mathfrak{A}$	$\overline{\mathrm{Cyl}}$
$\overline{\mathbf{S}}$	$\overline{\mathcal{A}}$



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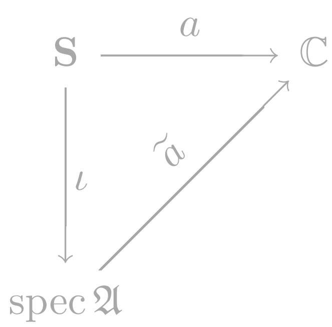
Proposition:
1. $\iota(\mathbf{S})$ dense in $\operatorname{spec} \mathfrak{A}$
2. a) ι and $\mathfrak{A}$ separate same points
   b) ι injective $\Longleftrightarrow$ $\mathfrak{A}$ separates points in $\mathbf{S}$



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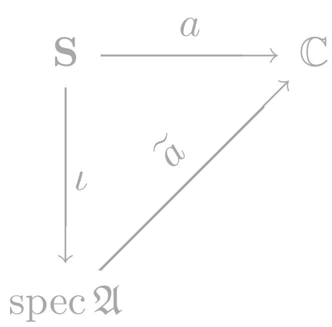

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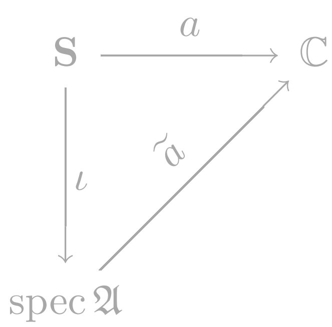

Proof: 1. $\chi \in \operatorname{spec} \mathfrak{A} \setminus \overline{\iota(\mathbf{S})}$

$\Longrightarrow$ $\exists \phi \in C_0(\operatorname{spec} \mathfrak{A})$ with $\phi(\chi) \neq 0$, but $\phi \equiv 0$ on $\iota(\mathbf{S})$

$\Longrightarrow$ $\phi = \widetilde{a} \Longrightarrow a = \widetilde{a} \circ \iota = \phi \circ \iota = 0 \Longrightarrow \phi \equiv 0$

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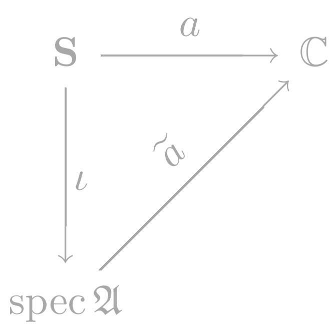

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2. $\iota(s_1) = \iota(s_2) \Longleftrightarrow a(s_1) \equiv \iota(s_1)(a) = \iota(s_2)(a) \equiv a(s_2) \quad \forall a$

2 Quantization of Configuration Spaces

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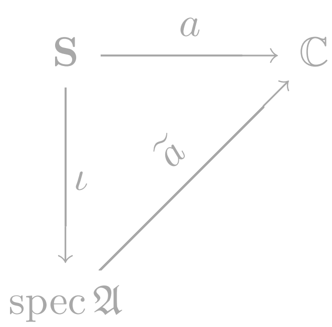

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3. ι continuous $\Longleftrightarrow a \equiv \widetilde{a} \circ \iota : \mathbf{S} \longrightarrow \mathbb{C}$ continuous $\forall a$ **qed**

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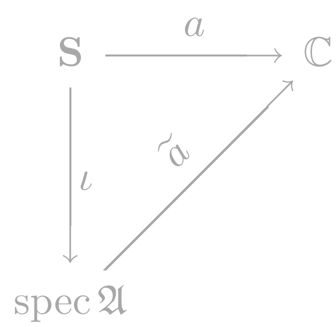

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Definition: **Quantum Configuration Space** $\overline{\mathbf{S}} := \operatorname{spec} \mathfrak{A}$

2 Quantization of Configuration Spaces

$\mathbf{S}$	$\mathcal{A}$
a	$(h_\gamma)^i_j$
$\mathfrak{B}$	Cyl
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1. $\iota(\mathbf{S})$ dense in $\operatorname{spec}\mathfrak{A}$
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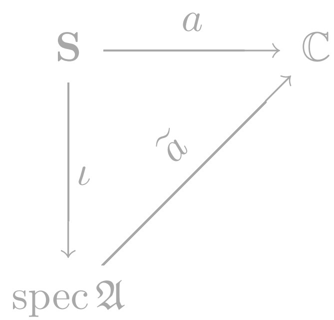
3. ι continuous $\Longleftrightarrow$ $\mathfrak{B} \subseteq C(\mathbf{S})$



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LQG	Standard LQC		Structure	Definition
$\mathcal{A}$		$\mathbf{S}$	set	*config space*
$(h_\gamma)^i_j$			function on $\mathbf{S}$	*choice*
Cyl		$\mathfrak{B}$	subset of $C_b(\mathbf{S})$	$\{b\}$
$\overline{\mathrm{Cyl}}$		$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$		$\overline{\mathbf{S}}$	locally compact	$\operatorname{spec} \mathfrak{A}$
μ_{AL}			measure	*repr theory*
Cyl^*		$\mathfrak{B}^*$	state space	$\mathfrak{B}^*$

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$\mathbb{R}$	$\mathcal{A}$	$\mathbf{S}$
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$\mathrm{Cyl}_{\mathrm{LQC}}$	Cyl	$\mathfrak{B}$
$C_{\mathrm{AP}}(\mathbb{R})$	$\overline{\mathrm{Cyl}}$	$\mathfrak{A}$
$\mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathcal{A}}$	$\overline{\mathbf{S}}$

Question: Existence, uniqueness, continuity of $\overline{\sigma}$?

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad\sigma\quad} & \mathcal{A} \\ \downarrow{\scriptstyle \iota_{\mathbb{R}}} & & {\scriptstyle \iota_{\mathcal{A}}}\downarrow \\ \overline{\mathbb{R}} & \overset{\overline{\sigma}}{\cdots\cdots\longrightarrow} & \overline{\mathcal{A}} \end{array}$$

$\mathbb{R}$	$\mathcal{A}$	$\mathbf{S}$
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$\mathrm{Cyl}_{\mathrm{LQC}}$	Cyl	$\mathfrak{B}$
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$\mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathcal{A}}$	$\overline{\mathbf{S}}$

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Proposition: continuous $\overline{\sigma}$ exists $\Longrightarrow$ *(unital case)*

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$$\begin{array}{ccccc} \mathbf{S}_1 & \xrightarrow{\sigma} & \mathbf{S}_2 & \xrightarrow{a_2} & \mathbb{C} \\ \downarrow{\scriptstyle \iota_1} & & {\scriptstyle \iota_2}\downarrow & \nearrow{\scriptstyle \widetilde{a}_2} & \\ \overline{\mathbf{S}}_1 & \overset{\overline{\sigma}}{\cdots\cdots\rightarrow} & \overline{\mathbf{S}}_2 & & \end{array}$$

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Proposition: continuous $\overline{\sigma}$ exists $\Longrightarrow$ *(unital case)*

Proof: $a_2 \in \mathfrak{B}_2$

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$\mathrm{Cyl}_{\mathrm{LQC}}$	Cyl	$\mathfrak{B}$
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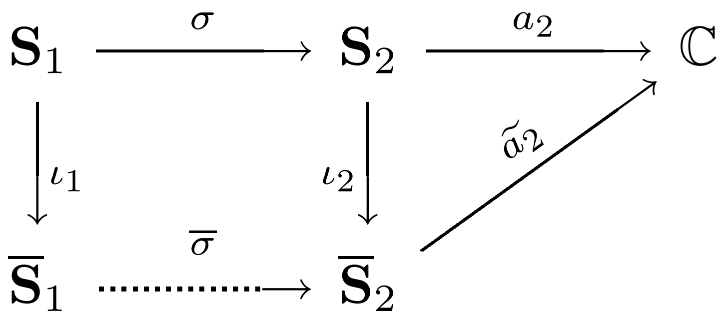

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Proof: $a_2 \in \mathfrak{B}_2 \quad \Longrightarrow \quad \widetilde{a}_2 \circ \overline{\sigma} : \overline{\mathbf{S}}_1 \longrightarrow \mathbb{C}$ continuous

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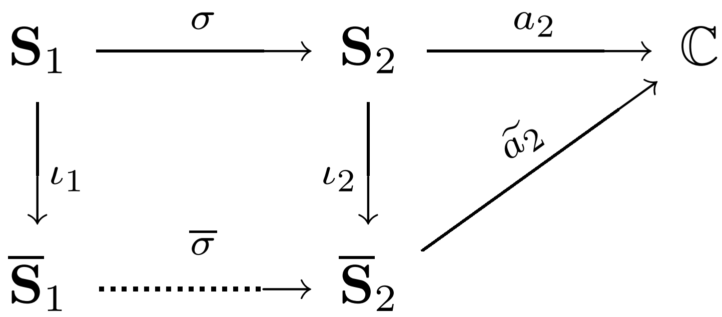
Question: Existence, uniqueness, continuity of $\overline{\sigma}$?



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$\Longrightarrow \quad \widetilde{a}_2 \circ \overline{\sigma} = \widetilde{a}_1$ (Gelfand-Naimark)

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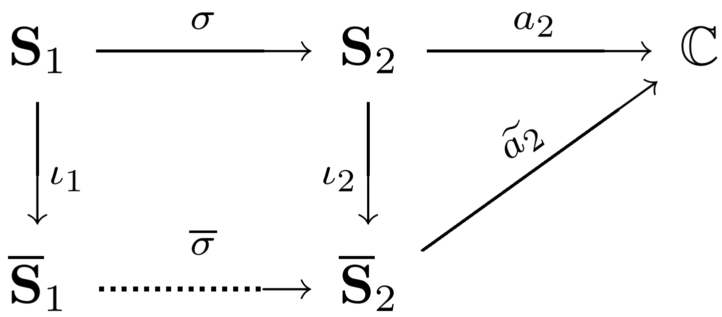

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$\Longrightarrow \quad \sigma^* a_2 \;\equiv\; a_2 \circ \sigma$

$\mathbb{R}$	$\mathcal{A}$	$\mathbf{S}$
$e^{il\bullet}$	$(h_\gamma)^i_j$	a
$\mathrm{Cyl}_{\mathrm{LQC}}$	Cyl	$\mathfrak{B}$
$C_{\mathrm{AP}}(\mathbb{R})$	$\overline{\mathrm{Cyl}}$	$\mathfrak{A}$
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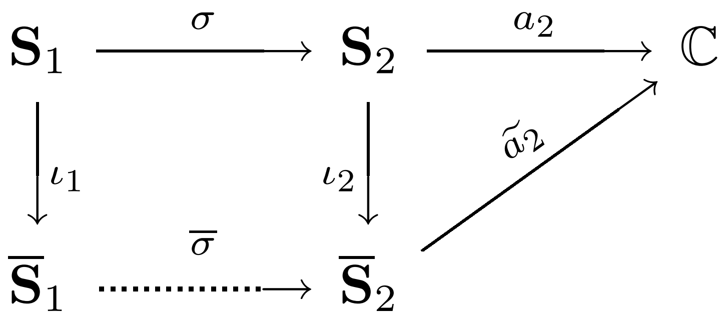
Question: Existence, uniqueness, continuity of $\overline{\sigma}$?



Proposition: continuous $\overline{\sigma}$ exists $\Longrightarrow$ *(unital case)*

Proof: $a_2 \in \mathfrak{B}_2 \Longrightarrow \widetilde{a}_2 \circ \overline{\sigma} : \overline{\mathbf{S}}_1 \longrightarrow \mathbb{C}$ continuous

$\Longrightarrow \widetilde{a}_2 \circ \overline{\sigma} = \widetilde{a}_1$ (Gelfand-Naimark)

$\Longrightarrow \sigma^* a_2 \equiv a_2 \circ \sigma = \widetilde{a}_2 \circ \iota_2 \circ \sigma$

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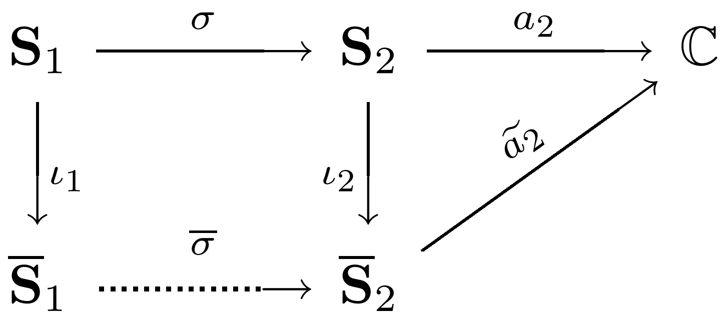

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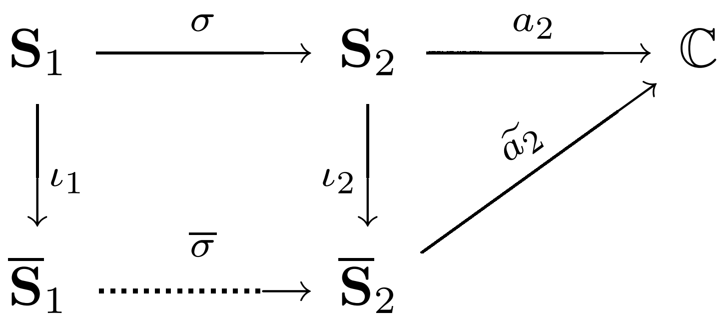

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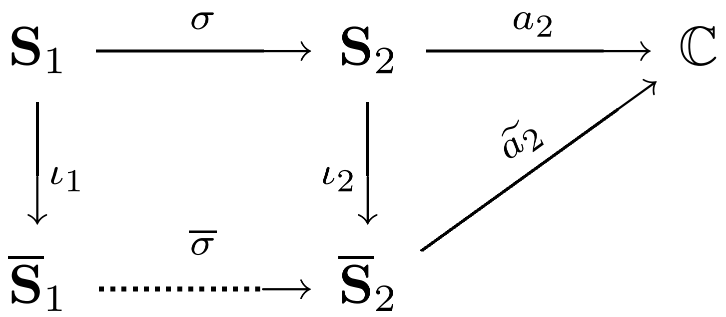

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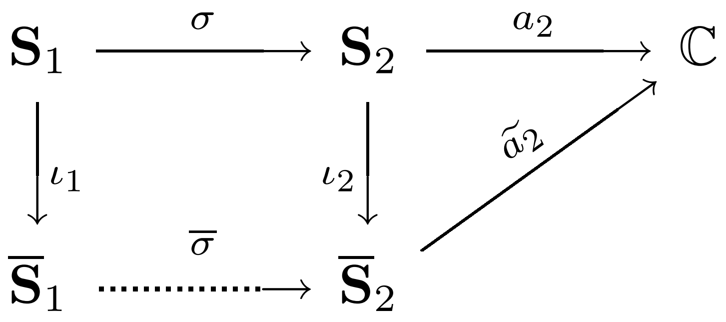

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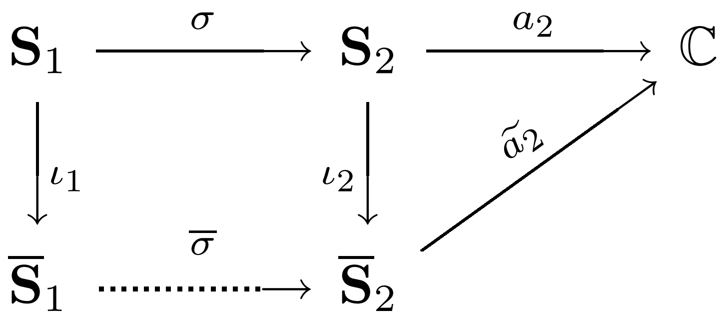

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Definition: **Restriction Algebra**

$$\sigma^*\mathfrak{B}_2 := \{\sigma^* b_2 \mid b_2 \in \mathfrak{B}_2\} \equiv \{b_2 \circ \sigma \mid b_2 \in \mathfrak{B}_2\} \subseteq \ell^\infty(\mathbf{S}_1)$$

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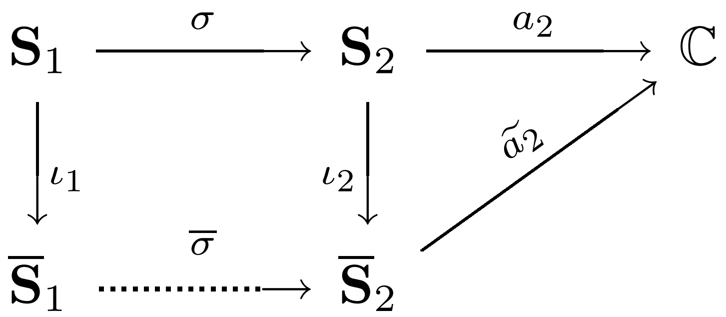

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Question: $\mathrm{Cyl}|_{\mathbb{R}} \equiv \sigma^*\mathrm{Cyl} \subseteq \overline{\mathrm{Cyl}_{\mathrm{LQC}}} \equiv C_{\mathrm{AP}}(\mathbb{R})$?

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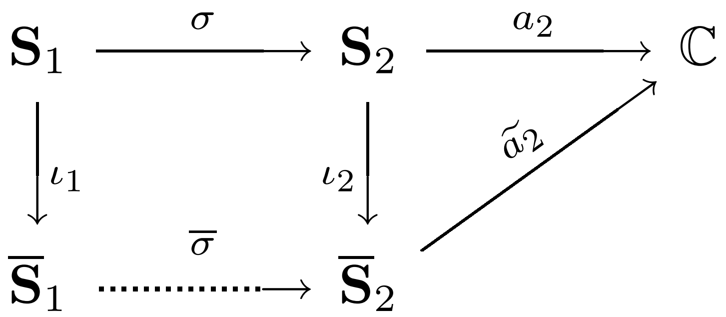

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Question: Is any parallel transport almost periodic in c?

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3 Embeddability Criterion

Question: For which γ is $c \longmapsto h_{cA_*}(\gamma)$ almost periodic?

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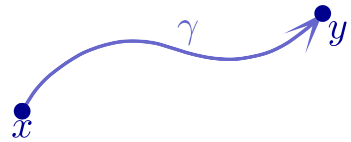
- Parallel Transport



$$g(t) \;:=\; h_{cA_*}\big(\gamma|_{[0,t]}\big)$$

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$$\begin{aligned} \dot g(t) &= -c\,A_*\big(\dot\gamma(t)\big)\,g(t) \\ g(0) &= \mathbf{1} \end{aligned}$$

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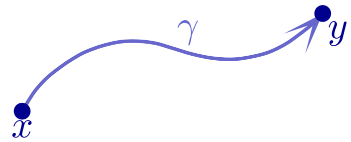

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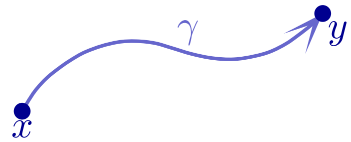

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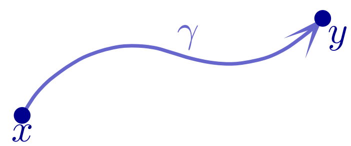

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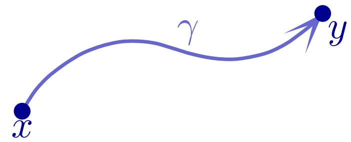

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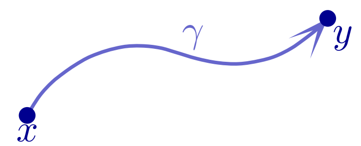

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- Differential Equation

$$\begin{pmatrix} \dot{a} & \dot{b} \\ -\dot{\overline{b}} & \dot{\overline{a}} \end{pmatrix} \quad = \quad \mathrm{i}c \begin{pmatrix} n & m \\ \overline{m} & -n \end{pmatrix} \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

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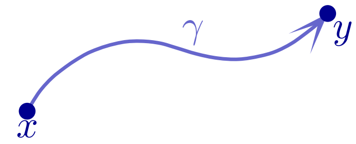

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$$\begin{aligned} \dot{a} \quad &= \quad \mathrm{i}c(na - m\overline{b}) \qquad & a(0) \quad &= \quad 1 \\ \dot{b} \quad &= \quad \mathrm{i}c(nb + m\overline{a}) \qquad & b(0) \quad &= \quad 0 \end{aligned}$$

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- Second-order Equation

$$\ddot{a} = \mathrm{i}c(\dot{n} - Mn)a - c^2 a + M\dot{a} \qquad \text{with} \quad M := \frac{\dot{m}}{m}$$

- Examples

	Straight Line	
$\dot{m}$	0	
$n,\ \dot{n}$	0	
Equations		
Initial Values $(m(0) = -\mathrm{i})$		
Solution		

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$$\ddot{a} = \mathrm{i}c(\dot{n} - Mn)a - c^2 a + M\dot{a} \qquad \text{with} \quad M := \frac{\dot{m}}{m}$$

- Examples

	Straight Line	Planar Circle
$\dot{m}$	0	$2\mathrm{i}m$
n, $\dot{n}$	0	0
Equations	$\ddot{a} + c^2 a = 0$ $\ddot{b} + c^2 b = 0$	$\ddot{a} + c^2 a = 2\mathrm{i}m\dot{a}$ $\ddot{b} + c^2 b = 2\mathrm{i}m\dot{b}$
Initial Values $(m(0) = -\mathrm{i})$	$b(0) = 0$ $\dot{b}(0) = c$	$b(0) = 0$ $\dot{b}(0) = c$
Solution	$b(t) = \sin ct$ Good!	$b(t) = \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{1+\frac{1}{c^2}}} \sin \sqrt{1 + \frac{1}{c^2}}\; ct$

3 Embeddability Criterion

Question: For which γ is $c \longmapsto h_{cA_*}(\gamma)$ almost periodic?

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Theorem: $\mathbb{R} \hookrightarrow \mathcal{A}$ can**not** be continuously extended to $\mathbb{R}_{\mathrm{Bohr}} \hookrightarrow \overline{\mathcal{A}}$.

$\mathbb{R}$	$\mathcal{A}$	$\mathbf{S}$
$\mathrm{e}^{il\bullet}$	$(h_\gamma)^i_j$	a
$\mathrm{Cyl}_{\mathrm{LQC}}$	Cyl	$\mathfrak{B}$
$C_{\mathrm{AP}}(\mathbb{R})$	$\overline{\mathrm{Cyl}}$	$\mathfrak{A}$
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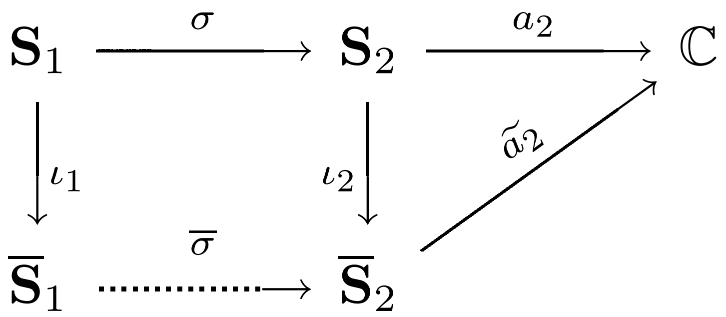

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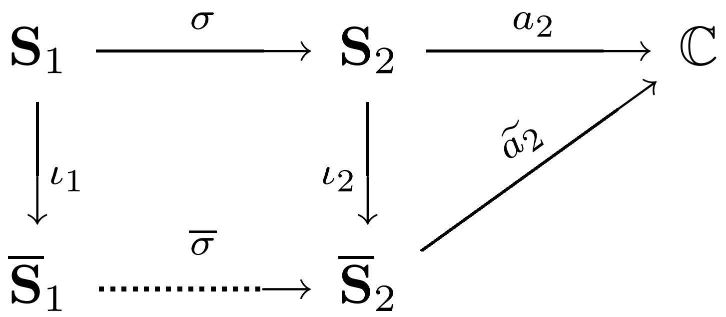

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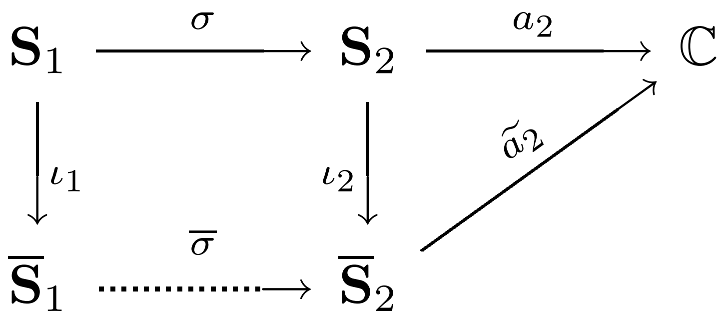

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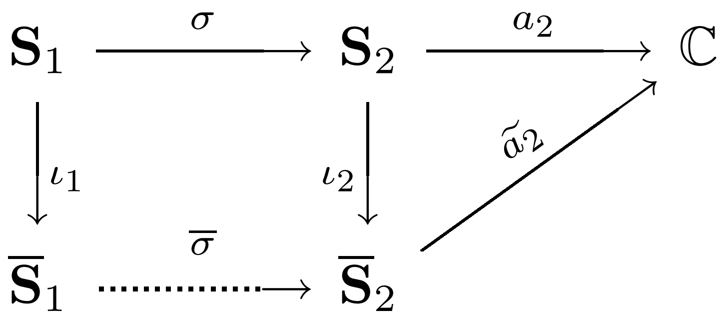

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($\iota_1(\mathbf{S}_1)$ dense, $\overline{\mathbf{S}}_1$ compact)

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Corollary: continuous $\overline{\sigma}$ exists $\Longrightarrow$ $\overline{\sigma}$ unique

3 Embeddability Criterion

Sufficient Condition

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Theorem: injective continuous $\overline{\sigma}$ exists $\iff$ $\mathfrak{A}_1 = C^*(\sigma^* \mathfrak{B}_2)$

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Theorem: injective continuous $\overline{\sigma}$ exists $\iff$ $\mathfrak{A}_1 = C^*(\sigma^*\mathfrak{B}_2)$

No topology required – neither on $\mathbf{S}_1$ nor on $\mathbf{S}_2$!

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Lesson: **Defining**

$$\mathrm{Cyl}_{\mathrm{LQC}} \quad := \quad \sigma^*\mathrm{Cyl}$$

is the **only** way to get a continuous embedding of LQC into LQG.

$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$ \ $\mathbb{R} \hookrightarrow \overline{\mathbb{R}}$	same as for LQG	piecewise linear	in fixed geodesic	incommensurable	
piecewise analytic	+	−	−	−	Ashtekar/Lewandowski
piecewise smooth	+	−	−	−	Baez/Sawin, CF
piecewise C^k	+	−	−	−	CF
piecewise linear	+	+	+	−	Zapata, Engle
in fixed graph	+	(−)	(−)	(−)	Giesel/Thiemann
in fixed PL graph	+	(∘)	(∘)	(−)	Giesel/Thiemann
barycentric subdivision	+	(∘)	(∘)	(−)	Aastrup/Grimstrup
	CF	Bojowald	Bojowald, Engle	Thiemann, CF	

+ . . . cont. inj. $\overline{\sigma}$

∘ . . . cont. non-inj. $\overline{\sigma}$

− . . . no cont. $\overline{\sigma}$

4 Configuration Space for Embeddable LQC

CF 2010, 2014

Task: Determine $\mathrm{Cyl}_{\mathrm{LQC}} := \sigma^* \mathrm{Cyl}$ for homogeneous isotropic case

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set of solutions of

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Hanusch, CF 2013

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Proof:
- Straight line $\Longrightarrow$ solution with period $L(\gamma)$ $\Longrightarrow$ $C_{\mathrm{AP}}(\mathbb{R})$
- Circle $\Longrightarrow$ asymptotically periodic solution $\Longrightarrow$ $C_0(\mathbb{R})$

$c_1 < c_2$ separated by any circle

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Proof: Estimates$^\infty$

Theorem: C^*-algebra of cylindrical functions of homogeneous isotropic LQC:

$$C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$$

4 Configuration Space for Embeddable LQC

CF 2010, 2014

Task: Determine $\mathrm{Cyl}_{\mathrm{LQC}} := \sigma^* \mathrm{Cyl}$ for homogeneous isotropic case

set of solutions of

$$\begin{aligned} \dot{a} &= \mathrm{i}c(na - m\bar{b}) & a(0) &= 1 \\ \dot{b} &= \mathrm{i}c(nb + m\bar{a}) & b(0) &= 0 \end{aligned}$$

Proposition: $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R}) \subseteq \overline{\sigma^* \mathrm{Cyl}}$

Hanusch, CF 2013

Proof:
- Straight line $\Longrightarrow$ solution with period $L(\gamma)$ $\Longrightarrow$ $C_{\mathrm{AP}}(\mathbb{R})$
- Circle $\Longrightarrow$ asymptotically periodic solution $\Longrightarrow$ $C_0(\mathbb{R})$

$c_1 < c_2$ separated by any circle

Proposition: $\sigma^* \mathrm{Cyl} \subseteq C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

Proof: Estimates$^\infty$

Theorem: C^*-algebra of cylindrical functions of homogeneous isotropic LQC:

$$C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$$

Task: Determine spectrum of $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

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		Loop Quantum Cosmology	

Task: Determine spectrum of $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

	General Situation	Loop Quantum Cosmology	
X	locally compact	$\mathbb{R}$	
$\mathfrak{A}$		$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	

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X	locally compact	$\mathbb{R}$	
$\mathfrak{A}_0$	C^*-algebra $\subseteq C_0(X)$	$C_0(\mathbb{R})$	
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		$C_0(\mathbb{R}) \cap C_{\mathrm{AP}}(\mathbb{R}) = \mathbf{0}$	
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Observation: By

$$\begin{array}{rrcl} \tau(x): & a & \longmapsto & a(x) \\ \tau(\chi_1): & a_0 + a_1 & \longmapsto & \chi_1(a_1) \end{array}$$

we get well-defined $\tau \; : \; X \sqcup \operatorname{spec} \mathfrak{A}_1 \longrightarrow \operatorname{spec} \mathfrak{A}:$

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$$\begin{aligned} [\tau(\chi_1)]\big((a_0 + a_1)(b_0 + b_1)\big) &= [\tau(\chi_1)]\big((a_0 b_0 + a_0 b_1 + a_1 b_0) + a_1 b_1\big) \\ &= \chi_1(a_1 b_1) = \chi_1(a_1)\,\chi_1(b_1) \\ &= [\tau(\chi_1)](a_0 + a_1)\,[\tau(\chi_1)](b_0 + b_1). \end{aligned}$$

Task: Determine spectrum of $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

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Proposition: $\operatorname{spec}\mathfrak{A} \;\cong\; X \sqcup \operatorname{spec}\mathfrak{A}_1$

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$\operatorname{spec} \mathfrak{A}$	$X \sqcup \operatorname{spec} \mathfrak{A}_1$		
ι	canonical embedding		

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$\operatorname{spec}\mathfrak{A}$	$X \sqcup \operatorname{spec}\mathfrak{A}_1$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	
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ι	canonical embedding		$\mathbb{R} \hookrightarrow \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	
open I	$V \sqcup \varnothing$	open V		
open II	$K^c \sqcup \operatorname{spec}\mathfrak{A}_1$	compact K		
open III	$a_1^{-1}(U) \sqcup \widetilde{a}_1^{-1}(U)$	$a_1 \in \mathfrak{A}_1$		

Proposition: $\operatorname{spec}\mathfrak{A} \;\cong\; X \sqcup \operatorname{spec}\mathfrak{A}_1$

Task: Determine spectrum of $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

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$\mathfrak{A}$	C^*-algebra (sic!) $\mathfrak{A}_0 + \mathfrak{A}_1$	$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	$C_0(X) + \mathbb{C}\cdot\mathbf{1}$
$\operatorname{spec}\mathfrak{A}$	$X \sqcup \operatorname{spec}\mathfrak{A}_1$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	
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$\operatorname{spec} \mathfrak{A}$	$X \sqcup \operatorname{spec} \mathfrak{A}_1$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	$X \sqcup \{\infty\}$
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open I open II open III	$V \sqcup \varnothing$ — open V $K^c \sqcup \operatorname{spec} \mathfrak{A}_1$ — compact K $a_1^{-1}(U) \sqcup \widetilde{a}_1^{-1}(U)$ — $a_1 \in \mathfrak{A}_1$		$V \sqcup \varnothing$ $K^c \sqcup \{\infty\}$ $\mathbb{R} \sqcup \{\infty\}, \quad \varnothing \sqcup \varnothing$

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Task: Determine spectrum of $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

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	$\mathfrak{A}_0 \cap \mathfrak{A}_1 = \mathbf{0}$ $\mathfrak{A}_0 \mathfrak{A}_1 \subseteq \mathfrak{A}_0$	$C_0(\mathbb{R}) \cap C_{\mathrm{AP}}(\mathbb{R}) = \mathbf{0}$ $C_0(\mathbb{R}) C_{\mathrm{AP}}(\mathbb{R}) \subseteq C_0(\mathbb{R})$	$C_0(X) \cap \mathbb{C} \cdot \mathbf{1} = \mathbf{0}$ $C_0(X) \cdot \mathbb{C} \subseteq C_0(X)$
$\mathfrak{A}$	C^*-algebra (sic!) $\mathfrak{A}_0 + \mathfrak{A}_1$	$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	$C_0(X) + \mathbb{C} \cdot \mathbf{1}$
$\operatorname{spec} \mathfrak{A}$	$X \sqcup \operatorname{spec} \mathfrak{A}_1$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	$X \sqcup \{\infty\}$
ι	canonical embedding	$\mathbb{R} \hookrightarrow \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	$\mathbb{R} \hookrightarrow \mathbb{R} \sqcup \{\infty\}$
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Task: Find measures on $\overline{\mathbb{R}} = \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

5 Measures

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Problem: No continuous group structure on $\overline{\mathbb{R}} \implies$ no Haar measure

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Idea: Determine measures separately on $\mathbb{R}$ and $\mathbb{R}_{\mathrm{Bohr}}$

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Hanusch 2013

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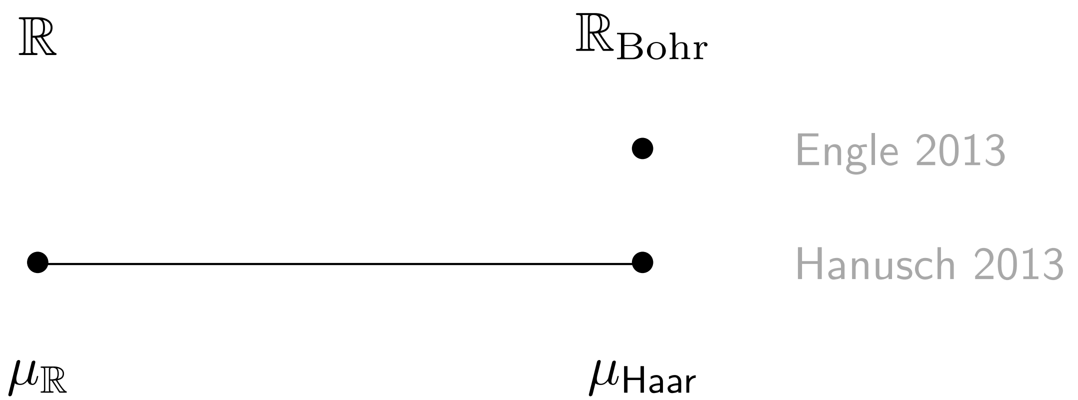

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Hanusch 2013

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Engle 2013

Hanusch 2013

Strategy: $\overline{\mathbb{R}} \cong \varprojlim_E \big((\mathrm{im}\ \varrho) \sqcup U(1)^{\#E}\big)$ Hanusch 2013

- $E \subseteq \mathbb{R}$ independent over $\mathbb{Z}$
- $\varrho : 1 + U(1) \longrightarrow \dot{\mathbb{R}}$ homeo with $\varrho(0) = \infty$

$\mu_{\mathbb{R}} = \varrho_* \mu_{U(1)}$

6 Conclusions

LQG	Embeddable LQC		Structure	Definition
$\mathcal{A}$	$\mathbb{R}$	$\mathbf{S}$	set	*config space*
$(h_\gamma)^i_j$			function on $\mathbf{S}$	*choice*
Cyl		$\mathfrak{B}$	subset of $C_b(\mathbf{S})$	$\{b\}$
$\overline{\mathrm{Cyl}}$		$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$		$\overline{\mathbf{S}}$	locally compact	$\operatorname{spec}\mathfrak{A}$
μ_{AL}			measure	*repr theory*
Cyl^*		$\mathfrak{B}^*$	state space	$\mathfrak{B}^*$

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LQG	Embeddable LQC		Structure	Definition
$\mathcal{A}$	$\mathbb{R}$	S	set	*config space*
$(h_\gamma)^i_j$	$h_\gamma(\bullet A_*)^i_j$		function on $\mathbf{S}$	*choice*
Cyl	$\sigma^*\mathrm{Cyl}$	$\mathfrak{B}$	subset of $C_b(\mathbf{S})$	$\{b\}$
$\overline{\mathrm{Cyl}}$		$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$		$\overline{\mathrm{S}}$	locally compact	$\operatorname{spec}\mathfrak{A}$
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$\overline{\mathrm{Cyl}}$	$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$		$\overline{\mathsf{S}}$	locally compact	$\operatorname{spec}\mathfrak{A}$
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$\overline{\mathrm{Cyl}}$	$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathsf{S}}$	locally compact	$\mathrm{spec}\,\mathfrak{A}$
μ_{AL}			measure	*repr theory*
Cyl^*		$\mathfrak{B}^*$	state space	$\mathfrak{B}^*$

6 Conclusions

Coming Next: • Other symmetries $\Longrightarrow$ other invariant connections

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$(h_\gamma)^i_j$	$h_\gamma(\bullet A_*)^i_j$		function on S	*choice*
Cyl	$\sigma^*\mathrm{Cyl}$	$\mathfrak{B}$	subset of $C_b(\mathrm{S})$	$\{b\}$
$\overline{\mathrm{Cyl}}$	$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$	$\mathfrak{A}$	C^*-algebra	$C^*(\mathfrak{B})$
$\overline{\mathcal{A}}$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathrm{S}}$	locally compact	$\operatorname{spec} \mathfrak{A}$
μ_{AL}			measure	*repr theory*
Cyl^*		$\mathfrak{B}^*$	state space	$\mathfrak{B}^*$

6 Conclusions

Coming Next:
- Other symmetries $\Longrightarrow$ other invariant connections
- Lifting of group actions

LQG	Embeddable LQC		Structure	Definition
$\mathcal{A}$	$\mathbb{R}$	$\mathbf{S}$	set	*config space*
$(h_\gamma)^i_j$	$h_\gamma(\bullet A_*)^i_j$		function on $\mathbf{S}$	*choice*
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6 Conclusions

Coming Next:

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- Symmetric quantum configuration space

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6 Conclusions

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Open Issues:
- Still further symmetries

6 Conclusions

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Open Issues:

- Still further symmetries
- Measures

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Open Issues:
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- Measures
- Stone-von Neumann theorem

6 Conclusions

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Open Issues:

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- Measures
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6 Conclusions

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