Higher Spin Gravity in 3-Dimensions and Unitarity

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Outline

1. Motivation
2. Gravity in 3 Dimensions
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Motivation

- Black hole evaporation and unitarity
- Role of geometry in quantum gravity
- Experience of infalling observers?
- Microscopic description of black hole entropy?
- Want one theory in which semiclassical and quantum regime can both be understood
Gravity in 3D with $\Lambda \leq 0$

$$I = \frac{1}{16\pi G_N} \int \sqrt{-g} (R + 2)$$

- One dimensionless coupling $G_N$ (in units of curvature radius)
- Vacuum solution: Anti de-Sitter space (AdS)
- Admits black holes with finite horizon size [BTZ92]
- (Boundary) Gravitons
First Order Formalism

- Dreibein $e^a_\mu$ and spin connection $\omega^{ab}_\mu$ independent variables
- 3D trick: use invariant antisymmetric rank 3 symbol $\epsilon^{abc}$ to construct $\omega^a_\mu$
- Action becomes a difference of two $\mathfrak{sl}_2$ Chern-Simons theories [AT86, Wit88]

$$I = \frac{k}{4\pi} \left( \int \text{tr} \left[ A \wedge dA + \frac{2}{3} A^3 \right] - \int \text{tr} \left[ \bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A}^3 \right] \right)$$

where $A = \frac{1}{2}(\omega + e)$, $\bar{A} = \frac{1}{2}(\omega - e)$, $k = \frac{1}{4G_N}$

- $g_{\mu\nu} = \frac{1}{2} \text{tr} \left[ (A - \bar{A})_\mu (A - \bar{A})_\nu \right]$
First Order Formalism: Gauge Symmetries

Gauge symmetries

\[ \delta_\varepsilon A = d\varepsilon + [A, \varepsilon] \]
\[ \delta_{\bar{\varepsilon}} \bar{A} = d\bar{\varepsilon} + [\bar{A}, \bar{\varepsilon}] \]

- \( \varepsilon - \bar{\varepsilon} \) generate local translations (diffeomorphisms)
- \( \varepsilon + \bar{\varepsilon} \) generate local Lorentz transformations

\[ \delta_{\varepsilon-\bar{\varepsilon}} e = d(\varepsilon - \bar{\varepsilon}) + [\omega, \varepsilon - \bar{\varepsilon}] \]
\[ \delta_{\varepsilon+\bar{\varepsilon}} e = [e, \varepsilon + \bar{\varepsilon}] \]
\[ \delta_{\varepsilon-\bar{\varepsilon}} \omega = [e, \varepsilon - \bar{\varepsilon}] \]
\[ \delta_{\varepsilon+\bar{\varepsilon}} \omega = d(\varepsilon + \bar{\varepsilon}) + [\omega, \varepsilon + \bar{\varepsilon}] \]
Imposing appropriate boundary conditions → gravity in asymptotically AdS$_3$

$$ds^2 = d\rho^2 + (e^{2\rho} \eta_{\mu\nu} + O(1))
\ dx^\mu
dx^{\nu}$$

- Canonical Analysis [BH86], originally in 2nd order formalism
- Later repeated in 1st order formalism, same result, but simpler to understand
Connection and Boundary Conditions

- Partially gauge fix $A = b^{-1}ab$, $\tilde{A} = bab^{-1}$ with $b = e^{\rho L_0}$
- Split connection into background, state-dependent fluctuations, and state-independent (subleading in $\rho$) fluctuations $a = \hat{a}^{(0)}(t, \phi) + a^{(0)}(t, \phi) + a^{(1)}(\rho, t, \phi)$
- It is necessary [convenient] for $\hat{a}^{(0)} [a^{(0)}]$ to satisfy the asymptotic equations of motion $F = 0 = \overline{F}$

### Boundary Conditions for Asymptotically AdS$_3$

\[
\begin{align*}
\hat{a}^{(0)} &= L_0 d\rho + L_1 dx^+ \\
\hat{a}^{(0)} &= -L_0 d\rho - L_{-1} dx^- \\
a^{(0)} &= \mathcal{L}(x^+) L_{-1} dx^+ \\
\overline{a}^{(0)} &= \overline{\mathcal{L}}(x^-) L_1 dx^- \\
a^{(1)} &= \mathcal{O}(e^{-\rho}) \\
\overline{a}^{(1)} &= \mathcal{O}(e^{-\rho})
\end{align*}
\]
Canonical Analysis of CS Theories I: Hamiltonian

- Convenient to use a $2 + 1$ decomposition

\[ I_{CS} [A] = \frac{k}{4\pi} \int_{\mathbb{R}} dt \int_{\mathcal{D}} d^2x \epsilon^{ij} \kappa_{ab} \left( \ddot{A}_i^a A_j^b + A_0^a F_{ij}^b \right) \]

- Canonical momenta $\pi^\mu_a$ generate primary constraints $\varphi^\mu_a$

\[ \varphi^0_a := \pi^0_a \approx 0 \quad \varphi^i_a := \pi^i_a - \frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} A_j^b \approx 0 \]

- Total Hamiltonian density $\mathcal{H}_T = -\frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} A_0^a F_{ij}^b + u^a_\mu \varphi^\mu_a$

- Conservation of the primary constraints $\dot{\varphi}^\mu_a = \{ \varphi^\mu_a, \mathcal{H}_T \} \approx 0$ leads to secondary constraints

\[ \mathcal{K}_a := -\frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} F_{ij}^b \approx 0 \quad D_i A_0^a - u_i^a \approx 0 \]
Canonical Analysis of CS Theories II: Charges

- Let $\overline{K}_a = K_a - D_i \varphi_i^a$. Then total Hamiltonian density expressed as sum of constraints

$$\mathcal{H}_T = A_0^a \overline{K}_a + u_0^a \varphi_0^a$$

- $\varphi_0^a$, $\overline{K}_a$ are first class, $\varphi_i^a$ are second class
- Construct gauge generators via Castellani’s algorithm.

$$\overline{G} [\epsilon] = \int_D d^2x \left( D_0 \epsilon^a \pi_0^a + \epsilon^a \overline{K}_a \right)$$

- Demanding functional differentiability determines the charges

$$\delta \mathcal{G} [\epsilon] = \delta \overline{G} [\epsilon] + \delta Q [\epsilon]$$

$$\delta Q [\epsilon] = \frac{k}{2\pi} \oint_{\partial D} d\phi \text{tr} \left( \epsilon \delta A_\phi \right)$$
Motivation
Gravity in 3 Dimensions
Higher Spins

Asymptotic Symmetry Algebra

- 2 copies of Witt Algebra

\[ \{ \mathcal{L}(\theta), \mathcal{L}(\theta') \} = \delta(\theta-\theta')\mathcal{L}'(\theta') - 2\delta'(\theta-\theta')\mathcal{L}(\theta') - \frac{k}{4\pi} \delta^{(3)}(\theta-\theta') \]

- Quantizing $\rightarrow$ 2 copies of Virasoro Algebra

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{k}{2}(n^3 - n)\delta_{n+m,0} \]

- $c_L = c_R = 6k$

- Boundary gravitons are descendents of the vacuum created by $L_{-n_1} \cdots L_{-n_m} |0\rangle$ with $n_i > 1$
Dual CFT and Unitary Models

Requirements for Unitary CFT (partial list)

- Central charge $c > 0$
- Modular invariant partition function

- $c = 1/2$ Ising Model
- $c = 7/10$ Tricritical Ising Model
- Only examples, no unitary semiclassical models ($c \gg 1$) [CGH+12]

Need an alternative theory with more allowed values of $c$
Generalization to Higher Spins

- Enlarge $\mathfrak{sl}_2$ to $\mathfrak{sl}_N$ (or other gauge group containing $\mathfrak{sl}_2$)
- Choice of embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_N$ determines other field content
- Spins of other field content given by weight under gravitational $\mathfrak{sl}_2$ action
- Number of embeddings grows exponentially with $N$
- Typical choice: Principal embedding, integer spins $2 \ldots N$

\[ g_{\mu\nu} = \frac{1}{2} \text{tr} \left( (A - \bar{A})_\mu (A - \bar{A})_\nu \right) \]
\[ \phi_{\mu\nu\rho} = \frac{1}{3!} \text{tr} \left( (A - \bar{A})_\mu (A - \bar{A})_\nu (A - \bar{A})_\rho \right) \]
\[ \vdots \]
Boundary conditions

\[
\hat{a}^{(0)} = L_0 d\rho + L_1 dx^+
\]

\[
a^{(0)} = \left(\mathcal{L}(x^+) L_{-1} + \sum_{n=2}^{N-1} \mathcal{W}_n(x^+) \mathcal{W}_n^{-1}\right) dx^+
\]

\[
a^{(1)} = \mathcal{O}(e^{-\rho})
\]

\[
\hat{a}^{(0)} = -L_0 d\rho - L_{-1} dx^-
\]

\[
a^{(0)} = \left(\overline{\mathcal{L}}(x^-) L_1 + \sum_{n=2}^{N-1} \overline{\mathcal{W}}_n(x^-) \mathcal{W}_n^+(x^-)\right) dx^-
\]

\[
\overline{a}^{(1)} = \mathcal{O}(e^{-\rho})
\]

ASA: two copies of \(\mathcal{W}_n\) algebra with central charges

\[
c_L = c_R = 6k
\]
Possible unitary representations again (partially) classified by [CGH+12]

- $c = 4/5$ Potts Model
- $c = 6/7$ Tricritical Potts Model
- $c = 2\frac{N-1}{N+2}, N \in \{5,6,7,8\}$ Parafermions

Again, no semi-classical limit $c \gg 1$ allowed. What about non-Principal embeddings?
No-Go Theorem

- No-Go Theorem for Unitary representations in the limit $k \to \infty$ [CHLJ12]
- All non-principal embeddings include a singlet, which leads to a Kac-Moody algebra as part of the ASA

$$[J_n, J_m] = \kappa n \delta_{n+m,0} + \cdots$$

- Unitarity requires $\kappa \geq 0$
- Unitary also requires central charge $c \geq 0$ for Virasoro algebra (from $\mathfrak{sl}_2$)

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}$$

- In the limit $|c| \to \infty$, $\text{sign}(c) = -\text{sign}(\kappa)$. 
No-Go Guides The Way

- Work at finite (but possibly large) central charge $c$
- Next-to-principal ($W_N^{(2)}$) higher spin gravity
- For a given value of $N$, allows a discrete spectrum of unitary representations
- Central charge $c$ ranges from $O(1)$ to $O(N/4)$
- Asymptotic Symmetry Algebra is Feigin-Semikhatov algebra
Example: Polyakov-Bershadsky Algebra $\mathcal{W}_3^{(2)}$

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \]
\[ [J_n, J_m] = \kappa n\delta_{n+m,0} \]
\[ [J_n, L_m] = nJ_{n+m} \quad [J_n, G_m^\pm] = \pm G_{n+m}^\pm \]
\[ [G_n^+, G_m^-] = \frac{\lambda}{2}(n^2 - \frac{1}{4})\delta_{n+m,0} + \ldots \]
\[ [L_n, G_m^\pm] = \left(\frac{n}{2} - m\right) G_{n+m}^\pm \]

- Level $\kappa = \frac{2k+3}{3}$
- Central charge $c = 25 - \frac{24}{k+3} - 6(k + 3)$
- $G^\pm$ central term $\lambda = (k + 1)(2k + 3)$
Example: Polyakov-Bershadsky Algebra $\mathcal{W}_3^{(2)}$

- Like $\mathcal{N} = 2$ Superconformal Algebra, but with commuting $G^\pm$
- Leads to negative norm descendents of the vacuum if $G^\pm$ generate physical states
- $G^\pm$ must be null for unitary theory $\Rightarrow \lambda = 0$
- No other restrictions on unitarity

**Unitary Representations of $\mathcal{W}_3^{(2)}$**

- $c = 0$ trivial theory
- $c = 1$ theory of $\hat{u}(1)$ current algebra
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Feigin-Semikhatov Algebra $\mathcal{W}_{N}^{(2)}$

\[
\begin{align*}
[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[J_n, J_m] &= \kappa n\delta_{n+m,0} \\
[J_n, L_m] &= nJ_{n+m} \quad [J_n, G_m^\pm] = \pm G_{n+m}^\pm \\
[G_n^+, G_m^-] &= \lambda f(n)\delta_{n+m,0} + \cdots \\
[L_n, G_m^\pm] &= \left(n\left(\frac{N}{2} - 1\right) - m\right) G_{n+m}^\pm
\end{align*}
\]

- $G^\pm$ central term

\[
\lambda = \prod_{m=1}^{N-1} \left(m(N + k - 1) - 1\right)
\]

Unitarity requires $\lambda = 0$
Unitary Representations of $W_{100}^{(2)}$

Allowed values of central charge $c$ for $N = 100$
Unitary Representations of $W_N^{(2)}$

- Demanding unitarity $\rightarrow$ Newton’s constant automatically quantized
- Critical values $\alpha = \frac{\hat{N}}{N - \hat{N} - 1}$ where $\hat{N} \in \mathbb{N}$, $\hat{N} \leq \frac{N-1}{2}$
- Allowed values of central charge (let $m = N - 2\hat{N} - 1$)
  \[
  c(\hat{N}, m) - 1 = (\hat{N} - 1) \left( 1 - \frac{\hat{N}(\hat{N} + 1)}{(m + \hat{N})(m + \hat{N} + 1)} \right)
  \]
- Exactly values of central charge $W_{\hat{N}}$ minimal models, up to shift by 1 due to $\hat{u}(1)$ current algebra
- Small $\alpha$ quantum regime with $c \sim \mathcal{O}(1)$
- Intermediate $\alpha$ semiclassical regime with $c \sim \mathcal{O}\left(\frac{N}{4}\right)$
- $\alpha \sim \mathcal{O}(1)$ dual quantum regime with $c \sim \mathcal{O}(1)$
Open Issues

- Existence of modular invariant partition function
- \( N \to \infty \) limit?
- Other non-principal embeddings
- Gauge invariance vs. Geometry

Thank You
Motivation

Gravity in 3 Dimensions

Higher Spins

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