Black hole entropy and self-dual loop quantum gravity

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Tux 2
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Based on:
MG, K. Noui, 1312.1696 [gr-qc]
MG, K. Noui, to appear
E. Frodden, MG, K. Noui, A. Perez, 1212.4060 [gr-qc]

Surprising interplay between various questions in LQG and spin foams:

- ⋆ Interpretation of the Barbero–Immirzi parameter γ.
- ⋆ Lorentz symmetry and choice of gauge group.
- ⋆ Imposition of simplicity constraints and canonical Hamiltonian constraint.
- ⋆ Derivation of black hole entropy and fixation of $\gamma = \gamma_0$.

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What is black hole entropy in quantum gravity?

- ⋆ What are we computing (in LQG)?
  Entanglement entropy? Statistical entropy?
- ⋆ Fixing γ in the entropy calculation is a priori meaningless:
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Results described here:

- ⋆ Entropy of a BTZ black hole from the Turaev–Viro model.
- ⋆ Entropy of 4d black holes from the self-dual theory, and evidences for radiation.
- ⋆ Unified framework for 3d and 4d (and higher dimensions)!

1. BTZ black hole entropy
   - ⋆ Classical properties
   - ⋆ The Turaev–Viro model
   - ⋆ BTZ black hole entropy

2. 4d black holes in LQG
   - ⋆ Chern–Simons description
   - ⋆ Analytic continuation to $\gamma = \mathrm{i}$
   - ⋆ Radiation and temperature

3. Conclusion and outlook

Why does 3d gravity look simple?

⋆ In 3d, the Weyl tensor is identically vanishing:

→ $R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$,

→ all solutions are spaces of constant curvature, locally Minkowski, dS or AdS, depending on $\text{sign}(\Lambda)$.

⋆ No local degrees of freedom, no gravitational waves, no gravitons.

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- ⋆ If the manifold has a non-trivial topology, the geometry might be locally trivial but not globally so.
- ⋆ The case $\Lambda < 0$ admits black hole solutions (Bañados, Teitelboim, Zanelli, 1992).

The BTZ black hole

- Solution of 3d gravity with $\Lambda = -1/\ell_c^2 < 0$.
- Euclidean metric (by Wick rotation of the Lorentzian solution):

$$\mathrm{d}s^2 = N^2 \mathrm{d}\tau^2 + N^{-2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\phi + N^\phi \mathrm{d}\tau\right)^2,$$

with

$$N = \left(-8GM + \frac{r^2}{\ell_c^2} - \frac{16G^2J^2}{r^2}\right)^{1/2}, \qquad N^\phi = -\frac{4GJ}{r^2},$$

$$r_\pm^2 = 4GM\ell_c^2\left(1 \pm \sqrt{1 + \left(\frac{J}{M\ell_c}\right)^2}\right).$$

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- ⋆ First example of AdS/CFT (Brown, Henneaux, 1986).
- ⋆ Global geometry of the Euclidean solution is a solid torus (Bañados, Henneaux, Teitelboim, Zanelli, 1992 | Carlip, Teitelboim, 1995).

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- No spin foam description so far, because the gauge groups involved are non-compact:

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Group theoretical data

- ⋆ Choose an integer $\mathrm{k} \geq 1$, define $q = \exp\left(\frac{i\pi}{\mathrm{k}+2}\right)$, and

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \sin\left(\frac{\pi}{\mathrm{k}+2} n\right) \sin^{-1}\left(\frac{\pi}{\mathrm{k}+2}\right).$$

- ⋆ Consider the algebra $\mathrm{U}_q(\mathfrak{su}(2))$, and its unitary irreducible representations

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{\mathrm{k}}{2}, \qquad \text{with} \qquad \dim(\mathcal{H}_j) = 2j + 1.$$

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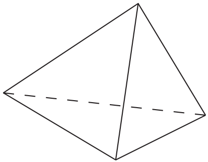

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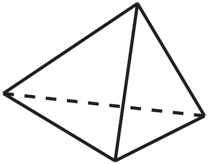

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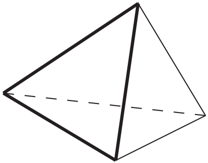

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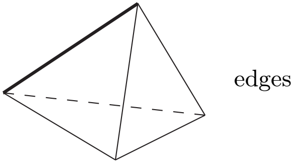

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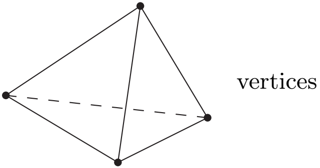

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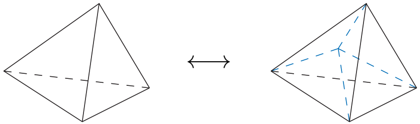
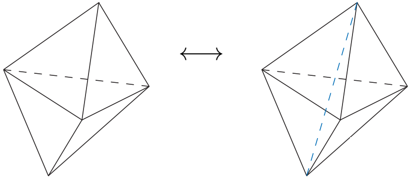
- ⋆ How is this a topological invariant?
- ⋆ How is this 3d quantum gravity?

Topological invariance

- ⋆ Triangulations of d-dimensional PL-homeomorphic manifolds can be related by elementary moves (Alexander 1930, Pachner 1978):
  - in $d = 3$ (i.e. on tetrahedra)



1–4 move



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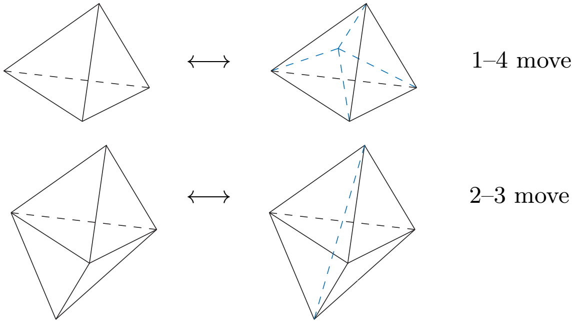

- The amplitudes assigned to the simplices satisfy algebraic relations reflecting these geometrical moves.

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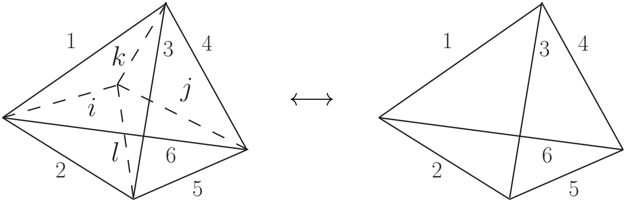

is encoded in the algebraic relation

$$\sum_{i,j,k,l} \omega_i^2 \omega_j^2 \omega_k^2 \omega_l^2 \begin{vmatrix} 1 & 2 & 3 \\ l & k & i \end{vmatrix}_q \begin{vmatrix} 3 & 4 & 5 \\ j & l & k \end{vmatrix}_q \begin{vmatrix} 1 & 4 & 6 \\ j & i & k \end{vmatrix}_q \begin{vmatrix} 2 & 5 & 6 \\ j & i & l \end{vmatrix}_q$$

$$= \quad \omega^2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \end{vmatrix}_q .$$

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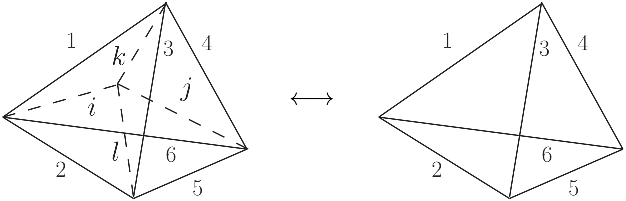

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$$= \quad \omega^2 \left| \begin{matrix} 1 & 2 & 3 \\ 5 & 4 & 6 \end{matrix} \right|_q .$$

- This explains why $\mathcal{Z}(M) = \sum_j \mathcal{Z}(M, \Delta)$ is triangulation-independent.

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In the limit

$$\Lambda \longrightarrow 0, \qquad \mathrm{k} \longrightarrow \infty, \qquad q \longrightarrow 1, \qquad \mathrm{U}_q(\mathfrak{su}(2)) \longrightarrow \mathfrak{su}(2),$$

the Turaev–Viro model becomes the Ponzano–Regge model, which can be obtained from

$$S[e,\omega] = \int_M e \wedge F[\omega].$$

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- ⋆ The asymptotics (large j) of $|6j|_q$ reproduces the Regge action with a cosmological constant (Mizoguchi, Tada, 9110057).
- ⋆ Link with the path integral of Chern–Simons theory: $\mathcal{Z}(M)=|\mathcal{Z}_{\mathrm{WRT}}(M)|^2$ (Turaev, Virelizier, 1006.3501).

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 BTZ black hole entropy

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State sum observable (Barrett, García-Islas, Martins, 0411281).

- ⋆ Choose a graph Γ consisting of a subset of n edges of Δ, and then define

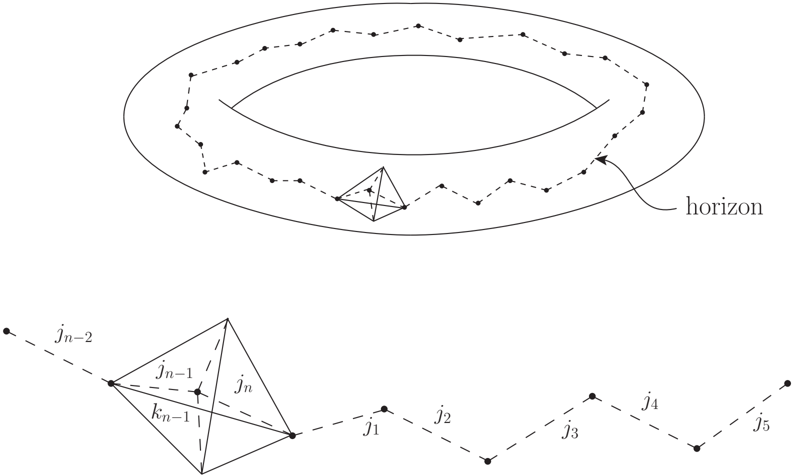
$$\mathcal{Z}_n(M,\Gamma)=\sum_{j|_\Gamma}\mathcal{Z}(M,\Delta).$$

- ⋆ This object is triangulation-independent away from Γ, and

$$\sum_{j_\Gamma}\mathcal{Z}_n(M,\Gamma)=\mathcal{Z}(M).$$

- ⋆ These observables are related to the Witten–Reshetikhin–Turaev observables in Chern–Simons theory via a Fourier transform whose kernel is a Verlinde coefficients.

Triangulation of $\mathbb{D}^2 \times \mathbb{S}^1$



Computation of the observable partition function

 BTZ black hole entropy

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- ⋆ Use recursively the partial Pachner 4–1 move

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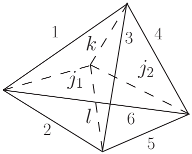

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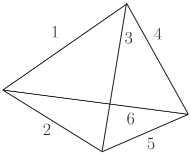

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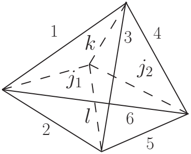

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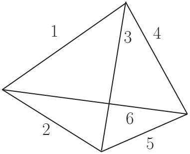

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- Upshot:

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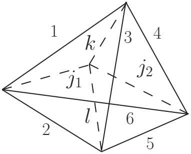

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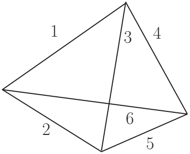

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- ⋆ This is precisely the dimension N of the Chern–Simons Hilbert space

$$\mathrm{Inv}_{\mathrm{U}_q(\mathfrak{su}(2))}(j_1 \otimes \cdots \otimes j_n), \qquad \text{with dimension} \quad \xrightarrow{\mathrm{k}\to\infty} \quad \prod_{e=1}^{n} (2j_e + 1)$$

(Blau, Thompson, 9305010 | Kaul, Majumdar, 9801080).

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- ? What does this have to do with 4d black holes?

1. BTZ black hole entropy
   - ⋆ Classical properties
   - ⋆ The Turaev–Viro model
   - ⋆ BTZ black hole entropy

2. 4d black holes in LQG
   - ⋆ Chern–Simons description
   - ⋆ Analytic continuation to $\gamma = \mathrm{i}$
   - ⋆ Radiation and temperature

3. Conclusion and outlook

4d black holes are also described by Chern–Simons theory, but for another reason.

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- ⋆ The horizon is treated as a boundary. In terms of the Ashtekar–Barbero connection, one gets (Engle, Noui, Perez, 0905.3168)

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SU(2) Chern–Simons theory

- ⋆ Extensively studied (Witten and 3d gravity ...). Path integral quantization gives relation with quantum groups, invariants, and quantization of the level.
- ⋆ Physical Hilbert space (n-punctured sphere) $\mathcal{H}(j_1,\dots,j_n)=\mathrm{Inv}_{\mathrm{U}_q(\mathfrak{su}(2))}(j_1\otimes\cdots\otimes j_n)$ has finite dimension given by the Verlinde formula

$$N(j_1,\dots,j_n)=\frac{2}{\mathrm{k}+2}\sum_{d=0}^{\mathrm{k}+1}\sin\left(\frac{\pi}{\mathrm{k}+2}d\right)^{2-n}\prod_{e=1}^{n}\sin\left(\frac{\pi}{\mathrm{k}+2}dd_e\right).$$

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⋆ Counting the number of states compatible with a given macroscopic area A by

$$\mathcal{N}(A)=\sum_{p=1}^{\infty}\sum_{j_1,\dots,j_n}N(j_1,\dots,j_n)\,\delta\left(A-8\pi\gamma\ell_{\mathrm{Pl}}^2\sum_{e=1}^{n}\sqrt{j_e(j_e+1)}\right).$$

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- ⋆ In the CMC gauge, fixing $\gamma = \gamma_0 \in \mathbb{R}$ is not possible, and only $\gamma = \mathrm{i}$ leads to the correct entropy (Bodendorfer, Stottmeister, Thurn, 1203.6525).

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- ⋆ Same as what we have obtained for the Turaev–Viro BTZ after continuing from $\Lambda > 0$ to $\Lambda < 0$.
- ⋆ In the limit $j_e \to \infty$ and $\ell_{\mathrm{Pl}} \to 0$ with $\ell_{\mathrm{Pl}} j_e$ finite,

$$S = \log \mathcal{Z} = \frac{A}{4\ell_{\mathrm{Pl}}^2} + \circ\big(\log(A)\big).$$

- ⋆ Punctures can be indistinguishable.
- ⋆ The case $\gamma = \mathrm{i}$ is also supported by the KMS interpretation of states (Pranzetti, 1305.6714).
- ⋆ In the CMC gauge, fixing $\gamma = \gamma_0 \in \mathbb{R}$ is not possible, and only $\gamma = \mathrm{i}$ leads to the correct entropy (Bodendorfer, Stottmeister, Thurn, 1203.6525).
- ⋆ Agreement with the large spin effective action for one 4-simplex (Bodendorfer, Neiman, 1303.4752).

Possible alternative (if one forgets about the level)

- ⋆ Expand the $\mathrm{SL}(2,\mathbb{C})$ representations $|(\rho,k)\rangle$ in terms of the $\mathrm{SU}(1,1)$ subgroup. The labels $(\rho,k)=\big(\mathrm{i}(j+1),j\big)$ are self-dual i.e. satisfy

$$\vec{K}-\mathrm{i}\vec{L}=0,$$

  and elements of the continuous series ($j=\mathrm{i}s-1/2$) verify the area reality condition

$$\widehat{\mathrm{Area}}\triangleright\psi=8\pi\ell_{\mathrm{Pl}}^2\sqrt{s^2+1/4}\;\psi\in\mathbb{R}.$$

- ⋆ Changing $j\to\mathrm{i}s-1/2$ in the Chern–Simons dimension leads to a similar expression for $\mathcal{Z}$.

Rewriting of the analytically-continued partition function

⋆ When the level is large, we have

$$\mathcal{Z} \approx \frac{2\sinh^2\pi}{\lambda}\prod_{e=1}^{n}\mathcal{Z}_e,$$

with

$$\mathcal{Z}_e = \sum_{m=0}^{2j_e}\exp\big(-\beta E_e(m)\big) = \sum_{m=0}^{2j_e} g_e \exp\big(-\beta a m\big),$$

where

$$\beta = \frac{2\pi}{a} \qquad \text{inverse Unruh temperature with acceleration } a,$$

$$E_e(m) = am + E_e(0) \qquad \text{discrete ``energy spectrum'' with } E_e(0) = -aj_e,$$

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- ⋆ Matches the near-horizon energy of (Frodden, Gosh, Perez, 1110.4055) seen by a stationary observer:

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- ⋆ One can introduce canonical and grand canonical partition functions for describing this system (Gosh, Noui, Perez, 1309.4563) → #punctures $\sim \sqrt{A}$.

Self-dual decomposition of a state

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- ⋆ Consider the (anti) self-dual generators $\vec{J}^{\pm} = (\vec{L} \pm \mathrm{i}\vec{K})/2$, and the helicity (or unitary spinor) basis $|\mu,\nu\rangle$ of $\mathrm{SL}(2,\mathbb{C})$, where L_3 and K_3 are diagonal:

$$L_3|\mu,\nu\rangle = \mu|\mu,\nu\rangle, \qquad K_3|\mu,\nu\rangle = \nu|\mu,\nu\rangle, \qquad \text{with} \qquad \mu \in \mathbb{Z}/2,\ \nu \in \mathbb{R},$$

$$J_3^{\pm}|\mu,\nu\rangle = m^{\pm}|\mu,\nu\rangle, \qquad \text{with} \qquad m^{\pm} = \frac{1}{2}(\mu \pm \mathrm{i}\nu),$$

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- ⋆ According (e.g.) to the dynamics of EPRL/FK spin foams, a state is labelled

$$|0(j)\rangle_{\mathrm{SF}}:=|(\gamma(j+1),j),j,j\rangle.$$

Self-dual decomposition of a state

- The overlap coefficients simplify dramatically, e.g. for EPRL and j integer,

$$\langle\mu,\nu|0(j)\rangle^2_{\mathrm{SF}}=\delta_{\mu,j}\frac{\sinh(\pi\rho)}{\cosh(\pi\rho)+\cosh(\pi\nu)}\frac{2j+1}{2\rho}\prod_{n=1}^{j}\frac{(n-1/2)^2+(\rho+\nu)^2/4}{n^2+\rho^2}.$$

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- ⋆ One can then introduce $\hat{\rho}=|0(j)\rangle\langle 0(j)|_{\mathrm{SF}}$, and compute

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- Analogous to the derivation of Bianchi (1204.5122), except with the self-dual decomposition instead of a thermometer.
- Generic behavior, independent of the spin foam model.

3d theory

- ⋆ Entropy of Euclidean BTZ from the Turaev–Viro model (Euclidean, $\Lambda > 0$).
- ⋆ No Barbero–Immirzi parameter needed in the length spectrum.

- ? Relationship with spin foam model for $q \in \mathbb{R}$ (unbounded spins)?
- ? Link with isolated horizon treatment?
- ? How to describe the Lorentzian black hole?
- ? Link with other counting methods, in particular CFT?
- ? Relation between boundary triangulation invariance, modular transformations, and log corrections?

4d theory

- ⋆ Making sense out of $\gamma = \mathrm{i}$ in the BH context.
- ⋆ New paradigm: indistinguishable punctures, large spins (effective action), radiation, ...
- ⋆ Consistent with 3d calculation and works for higher dimensional black holes.

- ? Can we make this more rigorous and extend it to other LQG/SF computations?