The phase shift of the Ponzano-Regge formula is correct; it is an example of a Maslov index. [Esterlis et al, math-ph/1402.0786]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12V}} \cos\left(\sum_r J_r \theta_r + \frac{\pi}{4} \right)$$

Maslov index

Also the origin of the zero-point energy

Maslov index

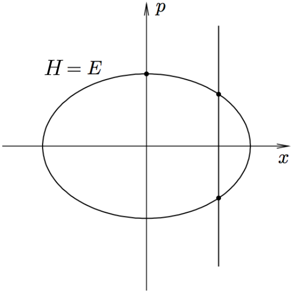
$$E_n = (n + \frac{1}{2})\hbar\omega.$$

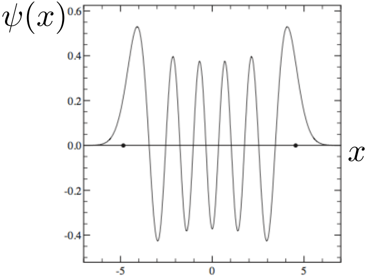
WKB ansatz, $\psi(x) = R(x) e^{\frac{i}{\hbar} S(x)}$, is really

$$\psi(x) = \sum_{\text{br } k} R_k(x) e^{\frac{i}{\hbar}(S_k(x) + \mu_k \frac{\pi}{2})} \ .$$

The Maslov index produces the correct quantum interference.

$$\psi(x) = \left\langle \begin{matrix} \hat{x} \\ x \end{matrix} \middle| \begin{matrix} \hat{H} \\ E \end{matrix} \right\rangle \sim \sum_{k=1}^{2} \frac{1}{\sqrt{|\{x, H\}|}} e^{\frac{i}{\hbar}(S_k + \mu_k \frac{\pi}{2})} \propto \frac{1}{\sqrt{|p|}} \cos\left(S + \frac{\pi}{4}\right)$$





We show: $\mu_k = \operatorname{sgn} \alpha\{\{x, H\}, H\}$, α computed by linear algebra.

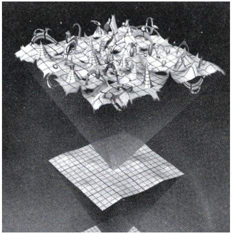
Finite regions, entanglement, and quantum gravity

Hal Haggard
in collaboration with E. Bianchi, G. Chirco, A. Riello, C. Rovelli

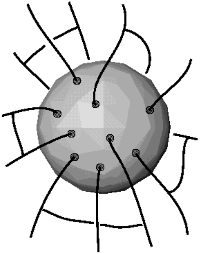
February 13th, 2014

Second EFI winter conference on quantum gravity
Tux, Austria

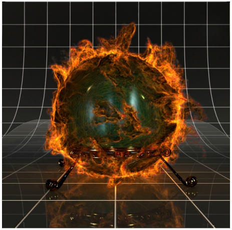
Entanglement provides a vibrant tool for quantum gravity



Entanglement correlations can distinguish vacua, ...



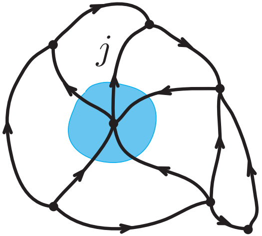
... provide a compelling calculation of black hole entropy, ...

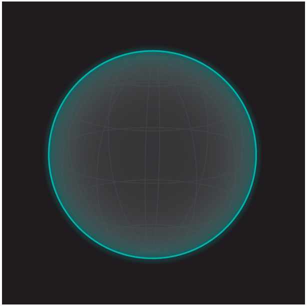


... and offer insight into spacetime thermodynamics.

What is the detailed entanglement of a finite region of spacetime?

Spin networks describe finite spatial regions $\rightsquigarrow$ their entanglement can provide consistency checks and design principles





Casini, Huerta & Myers have extensively studied the entanglement entropy of spherical causal domains.
[Casini, Huerta & Myers '11]

- New result $\rightsquigarrow$ the entanglement spectrum (or eigensystem).

Outline

Entanglement and the entanglement spectrum

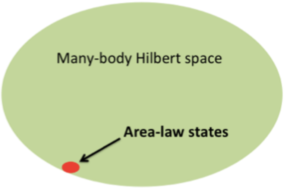
The entanglement spectrum of a sphere

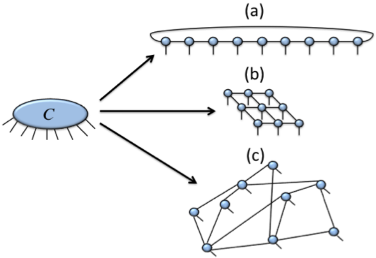
Interest for quantum gravity

Tensor networks, already valuable in condensed matter, are a natural fit for loop gravity

$$|\Psi\rangle = \sum_{i_1, i_2, \ldots, i_N} C_{i_1 i_2 \cdots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

The many body Hilbert space is too large ($\sim p^N$)





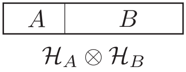
(a) Matrix Prod States (MPS)
(b) Proj Ent Pair Sts (PEPS)
(c) Tensor networks (TNs)

$\rightsquigarrow$ Area laws

- Spin networks are tensor networks

Entanglement

Pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.



Schmidt decomposition:

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}|i_A\rangle \otimes |i_B\rangle$$

with $|i_A\rangle$ and $|i_B\rangle$ orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Leads to the reduced density matrix

$$\begin{aligned}\rho_B &= \mathrm{Tr}_A|\Psi\rangle\langle\Psi| \\ &= \sum_i \lambda_i|i_B\rangle\langle i_B|,\end{aligned}$$

and the entanglement entropy

$$S_E \equiv -\mathrm{Tr}\,\rho_B \log \rho_B = -\sum_i \lambda_i \log \lambda_i.$$

The entanglement spectrum unveils a thorough description of entanglement.

Can always write

$$\rho_B = e^{-H_E}, \quad \text{i.e.} \qquad H_E \equiv -\log \rho_B,$$

the "entanglement Hamiltonian".

Already diagonalized H_E:

$$\rho_B = \sum_i e^{-\epsilon_i} |i_B\rangle\langle i_B|$$

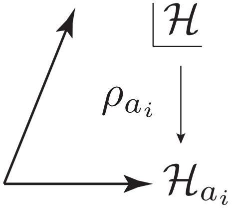
with ϵ_i $(i = 1, 2, \dots)$ the "entanglement spectrum". $(\lambda_i = e^{-\epsilon_i})$

[Li & Haldane '08]

- Study for spacetime fields.

The von Neumann entropy can quantify both (i) entanglement and (ii) our ignorance of the specific microstate.

If eigenvalues a_i of a few macroscopic observables describe state,



the density matrix ρ_{a_i} projects onto the (large) subspace $\mathcal{H}_{a_i}$.

The microcanonical von Neumann entropy is

$$S_{\mathsf{vN}} = -\sum p_i \ln p_i = \ln n,$$

with $n = \dim \mathcal{H}_{a_i}$.

Outline

Entanglement and the entanglement spectrum

The entanglement spectrum of a sphere

Interest for quantum gravity

Boundary conditions for the remainder

For simplicity I restrict to:

Scalar field $\varphi(x)$,

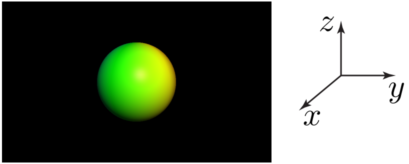
with $m = 0$

on flat $D = 3 + 1$ spacetime

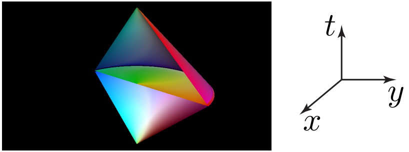
unless otherwise stated. Metric signature $(-, +, +, +)$.

- Results apply to any conformal field theory.

The finite region of interest is the Cauchy development of a 3-ball with boundary 2-sphere of radius R:

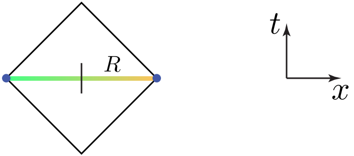


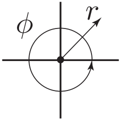
Spatial 3-ball B $\rightsquigarrow$ Cauchy development D(B).



Entangling surface the boundary 2-sphere $\partial B = S^2$.

Choose adapted coordinates that preserve S^2: similar to how polar coords fix $(0, 0)$...





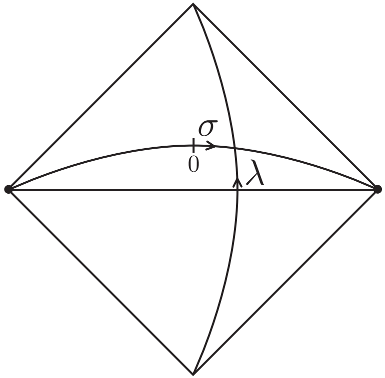
Use hyperbolas to build coordinates on the diamond

Diamond coords $(\lambda, \sigma, \theta, \phi)$:

$$t = R\frac{\mathsf{sh}\,\lambda}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

$$r = R\frac{\mathsf{sh}\,\sigma}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

with $\lambda \in (-\infty, \infty)$, $\sigma \in [0, \infty)$.



The Minkowski metric becomes

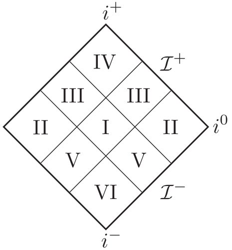
$$ds^2 = \frac{R^2}{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2],$$

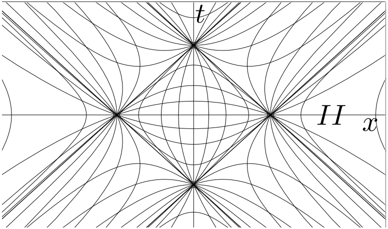
a conformal rescaling of static $\kappa = -1$ FRW.

Diamond coordinates can be extended to all of Minkowski space

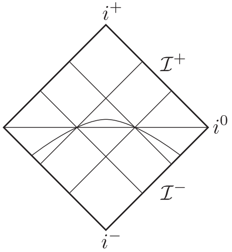
E.g. region II:

$$\left.\begin{aligned} t &= R\frac{\mathsf{sh}\,\tilde{\lambda}}{\mathsf{ch}\,\tilde{\sigma} - \mathsf{ch}\,\tilde{\lambda}}, \\ r &= R\frac{\mathsf{sh}\,\tilde{\sigma}}{\mathsf{ch}\,\tilde{\sigma} - \mathsf{ch}\,\tilde{\lambda}} \end{aligned}\right\} \quad |\tilde{\lambda}| \leq \tilde{\sigma}.$$





All hyperbolas asymp. null:

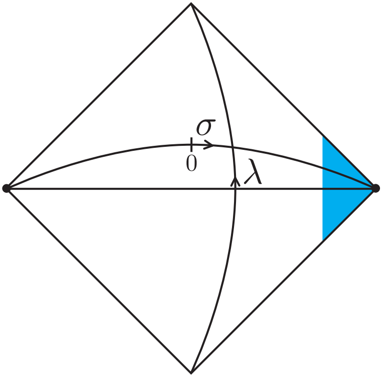


Near the L and R corners the diamond is approximately Rindler.

Large σ limit:

$$t \approx 2Re^{-\sigma}\mathsf{sh}\,\lambda = \ell\,\mathsf{sh}\,\lambda$$
$$R - r \approx 2Re^{-\sigma}\mathsf{ch}\,\lambda = \ell\,\mathsf{ch}\,\lambda$$

coord transformation to (left) Rindler wedge.



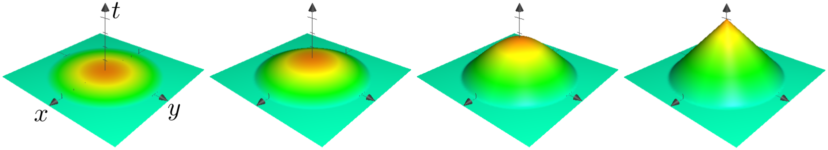
The proper distance from right corner is $\ell = 2Re^{-\sigma}$.

Entanglement (or foliation) Hamiltonian

$\xi^\mu = \left(\frac{\partial}{\partial\lambda}\right)^\mu \rightsquigarrow$ current $J^\mu = T^{\mu\nu}\xi_\nu$ with $T_{\mu\nu}$ the stress-tensor. If $T^\mu{}_\mu = 0$ the charge is conserved

$$C = \int T_{\mu\nu}\xi^\mu d\Sigma^\nu$$

and generates the spatial foliation discussed above:

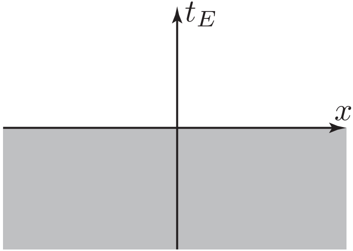


Explicitly this charge is,

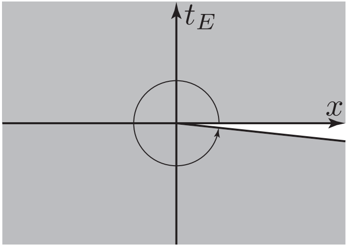
$$C_{in} = \frac{1}{2R}\int_B r^2 dr\, d\tilde{\Omega}\,\left(\frac{1}{2}(R^2 - r^2)(\dot{\varphi}^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi) + \varphi^2\right)$$

The density matrix for B is given by a Euclidean path integral

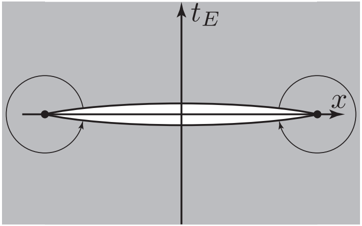
Minkowski vacuum:



Rindler density matrix:



Spherical density matrix:



$$\left.\begin{aligned} t_E &= R\frac{\sin\lambda_E}{\cos\lambda_E + \mathsf{ch}\,\sigma} \\ r &= R\frac{\mathsf{sh}\,\sigma}{\cos\lambda_E + \mathsf{ch}\,\sigma} \end{aligned}\right\}\ \text{Bipolar coords}$$

$$\boxed{\rho_B = \int \mathcal{D}\varphi\, e^{-S_E} = e^{-2\pi C_{in}}}$$

Inverted strategy: study finite region of flat spacetime using QFT on a curved background

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (\cancelto{0}{m^2} + \frac{1}{6}\mathcal{R}(x))\varphi^2].$$

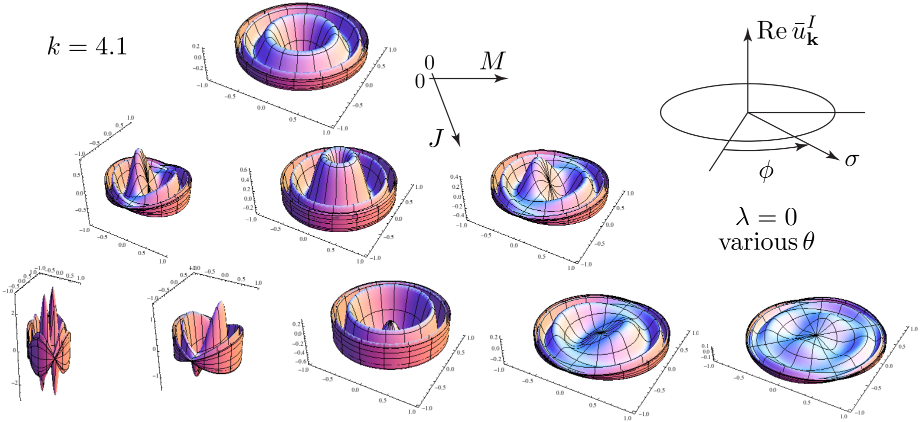
Action is conformally invariant under $\begin{cases} \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \\ \bar{\varphi} = \Omega(x)^{-1}\varphi \end{cases}$

With $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the EOM transform as ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi.$$

Find $\bar{\varphi}$ by finding φ.

Sphere modes



Sphere modes:

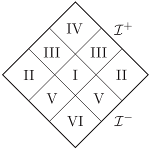
$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^{-}_{kJ}(\sigma)\,Y^M_J(\theta,\phi)\,e^{-ik\lambda}$$

Mode decomposition defines vacuum $|0\rangle_S$ and excitations above it

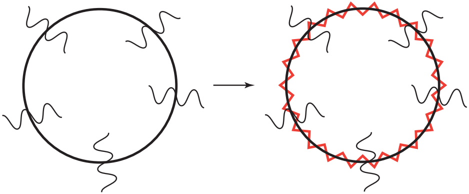
Minkowski space:
$\varphi(x) = \int d^{D-1}k[a_{\mathbf{k}} u_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(x)]$ with vacuum $a_{\mathbf{k}}|0\rangle_M = 0$.

Sphere: $\varphi(x) = \int dk \sum_{J,M}[s_{\mathbf{k}}^{I} u_{\mathbf{k}}^{I} + s_{\mathbf{k}}^{I\dagger} u_{\mathbf{k}}^{I*} + s_{\mathbf{k}}^{II} u_{\mathbf{k}}^{II} + s_{\mathbf{k}}^{II\dagger} u_{\mathbf{k}}^{II*}]$

$s_{\mathbf{k}}^{I\dagger}$ creates *spherons*,
excitations localized within
the S^2 entangling surface



The sphere vacuum satisfies $s_{\mathbf{k}}^{I}|0\rangle_S = s_{\mathbf{k}}^{II}|0\rangle_S = 0$.



This vacuum has no in-out entanglement due to mode localization

Spherical entanglement spectrum

Curved space, traceless (improved) stress tensor

$$T_{\mu\nu} = \frac{2}{3}\nabla_\mu\bar{\varphi}\nabla_\nu\bar{\varphi} - \frac{1}{6}g_{\mu\nu}(\nabla^\rho\bar{\varphi}\nabla_\rho\bar{\varphi}) - \frac{1}{3}\bar{\varphi}\nabla_\mu\nabla_\nu\bar{\varphi} + \frac{1}{3}g_{\mu\nu}\bar{\varphi}\Box\bar{\varphi} + \frac{1}{6}[\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}]\bar{\varphi}^2.$$

Entanglement Hamiltonian

$$C_{in} = \int_B T_{\mu\nu}\xi^\mu d\Sigma^\nu.$$

Entanglement spectrum

zero pt. ener.

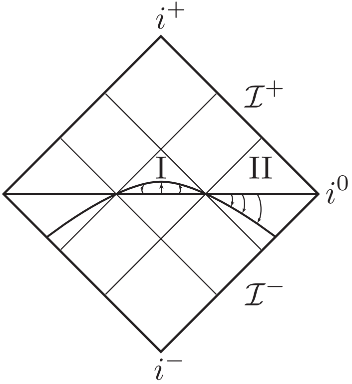
$$C_{in} = \int dk \sum_{J,M} k(s_{\mathbf{k}}^{I\dagger} s_{\mathbf{k}}^{I} + \frac{1}{2}[s_{\mathbf{k}}^{I}, s_{\mathbf{k}}^{I\dagger}]).$$

Schmidt decomposition of Minkowski vacuum

Total Hamiltonian

$$C = C_{in} - C_{out}$$

$$C = \int dk \sum_{J,M} k(s_{\mathbf{k}}^{I\dagger} s_{\mathbf{k}}^{I} + s_{\mathbf{k}}^{II\dagger} s_{\mathbf{k}}^{II})$$



$$\boxed{|0\rangle_M = \int dk \sum_{J,M} e^{-\epsilon_k} u_{\mathbf{k}}^{I} \otimes u_{\mathbf{k}}^{II}.}$$

Outline

Entanglement and the entanglement spectrum

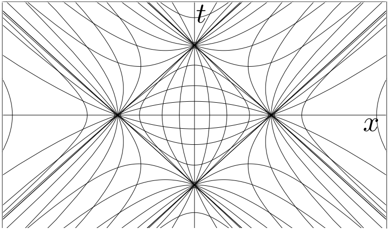
The entanglement spectrum of a sphere

Interest for quantum gravity

The conformal Killing field $\left(\frac{\partial}{\partial\lambda}\right)^{\mu}$ is the sum of a boost generator and a special conformal transformation

Expand around $r = R - \epsilon_r$

$$
\begin{aligned}
\partial_\lambda &= \epsilon_r \partial_t + t\partial_{\epsilon_r} - \frac{1}{2}\left(\frac{\epsilon_r^2 + t^2}{R}\right)\partial_t - \frac{\epsilon_r t}{R}\partial_{\epsilon_r} \\
&= K + S
\end{aligned}
$$



The first term gives rise to the universal area contribution; what is the contribution of the second term?

Jacobson's thermodynamic derivation of Einstein's equations can be achieved directly in loop gravity [Chirco et al, gr-qc/1401.5262]

Jacobson: Assume

$$S = \alpha A,$$

α universal. Then

$$\delta S = \alpha \delta A,$$

$\delta S = \delta E / T$ implies

$$\delta E = \alpha T \delta A.$$

Put in Unruh Temp

$$\boxed{\delta E = \alpha \frac{\hbar a}{2\pi} \delta A,}$$

$\alpha = 1/4\hbar G$ the FGP relation

Loops: Single facet state

$$\rho_f = e^{-2\pi K_f},$$

$$K_f = \vec{K} \cdot \vec{n}.$$

Entanglement entropy,

$$\delta S = 2\pi \delta \langle K_f \rangle.$$

Linear simplicity gives

$$\delta S = \tfrac{\delta \langle A_f \rangle}{4G\hbar},$$

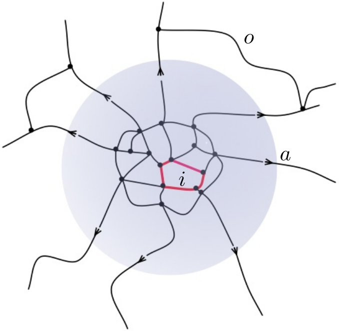
and a boost observer has

$$\delta E = \tfrac{a}{8\pi G} \delta \langle A_f \rangle.$$

$\rightsquigarrow$ with diffeos a sufficient condition for Einstein's eqns.

What is the relationship between the statistical and entanglement derivations of black hole entropy?

Alejandro Perez is forging a bridge; it appears they are the same!



$$|\Psi\rangle = \sum_{a,i,e} C_{aie} |\psi^a_{\text{int}}\rangle \otimes |\psi^a_{\text{ext}}\rangle$$

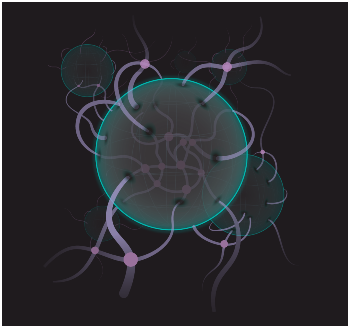
He takes tensor network form

$$|\Psi\rangle = \sum_{a,i,e} \alpha_a \beta_{ai} \gamma_{ae} |\psi^a_{\text{int}}\rangle \otimes |\psi^a_{\text{ext}}\rangle$$

and argues that the density matrix, tracing over the interior, reproduces the microcanonical picture.

Investigation of a spherical region suggests an intriguing perspective on spin networks.

Individual nodes are like the interior of the sphere with no entanglement between a given node and its neighbors.



Can we choreograph entanglement to yield the Minkowski vacuum?

♦ A wealth of condensed matter research on entanglement and tensor networks to draw from.

Conclusions

Looking to evaluate vacuum proposals and provide design criteria to recover the Minkowski vacuum.

Ultra local correlations are sufficient to recover the Jacobson argument in the small. Does this survive to larger scales?

Entanglement may help to understand the unity of the statistical and entanglement approaches to black hole entropy.

Numerous possibilities

- More general entangling surfaces
- Continue to draw from condensed matter
- Anomalies

Credits

Spherical spin network: Z. Merali, "The origins of space and time," Nature News, Aug. 28, 2013

Special thanks to the Emerging Fields Institute and our organizers (N. Bodendorfer, J. Lewandowski, H. Sahlmann) for making this conference possible. Also to the Cracow School of Theoretical Physics, LIII Course, 2013 for a lovely school and setting while this work was begun. And to the Perimeter Institute for their gracious hosting of visitors and support during the continuation of this work.

Sphere modes analytically

Metric $g = -d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\tilde{\Omega}^2$ is static $\kappa = -1$ FRW.

Separate: $u_{\mathbf{k}}(x) = \chi_k(\lambda)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta,\phi)$ with

$$\chi_k(\lambda) = (2k)^{-\frac{1}{2}}\, e^{-ik\lambda}$$

$$\Pi^-_{kJ}(\sigma) = N(k,J)\,\mathsf{sh}^J\sigma \left(\frac{d}{d\,\mathsf{ch}\,\sigma}\right)^{1+J} \cos\left(k\sigma\right)$$

$$Y^M_J(\theta,\phi) \qquad \text{spherical harmonics}$$

$$M = -J, -J+1, \ldots, J; \qquad J = 0, 1, \ldots; \qquad 0 < k < \infty.$$

Sphere modes: $\bar{\varphi} = \Omega(x)^{-1}\varphi$, (recall $\Omega = R/(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)$)

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta,\phi)\, e^{-ik\lambda}.$$