Loop Quantum Gravity

Knot Theory

Chern-Simons Theory

Quantum Curves

Low-dimensional Topology

Supersymmetric Gauge Theory

String/M-Theory

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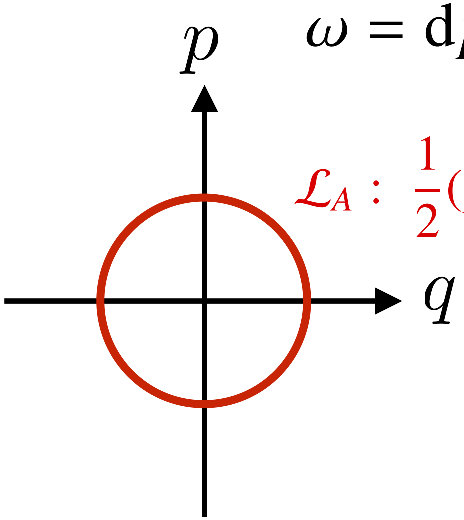
Warm up: Harmonic Oscillator



$$\omega = \mathrm{d}p \wedge \mathrm{d}q$$

$$\mathcal{L}_A : \ \frac{1}{2}(p^2 + q^2) - E = 0$$

Quantization

$$q \mapsto \hat{q}$$

$$p \mapsto \hat{p} = -i\hbar\partial_q$$

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$$

$$\hat{H}\psi(q) = E\psi(q) \quad \longleftrightarrow \quad \left[\frac{1}{2}(\hat{p}^2 + \hat{q}^2) - E\right]\psi(q) = 0$$

Quantized Circle

WKB solution:

$$\psi^{(\pm)}(q) = \exp\left[\frac{i}{\hbar}\int_{\mathcal{L}_A^{(\pm)}}^{q} p(q')\mathrm{d}q' + o(\log\hbar)\right]$$

$$= \exp\left[\frac{i}{\hbar}\int^{q} \pm\sqrt{2E - q'^2}\,\mathrm{d}q' + o(\log\hbar)\right]$$

Quantum Curve

Complex symplectic manifold: $(\mathcal{M}, \omega, J)$ $\qquad \dim_{\mathbb{C}} \mathcal{M} = 2N$

holomorphic symplectic structure (ω) complex structure (J)

Holomorphic coordinates:

$$\omega = \sum_{m=1}^{2N} \mathrm{d}v_m \wedge \mathrm{d}u_m$$

Holomorphic Lagrangian submanifold (Classical curve): $\mathcal{L}_A$

Lagrangian: $\omega|_{\mathcal{L}_A} = 0 \qquad \dim_{\mathbb{C}} \mathcal{L}_A = N$

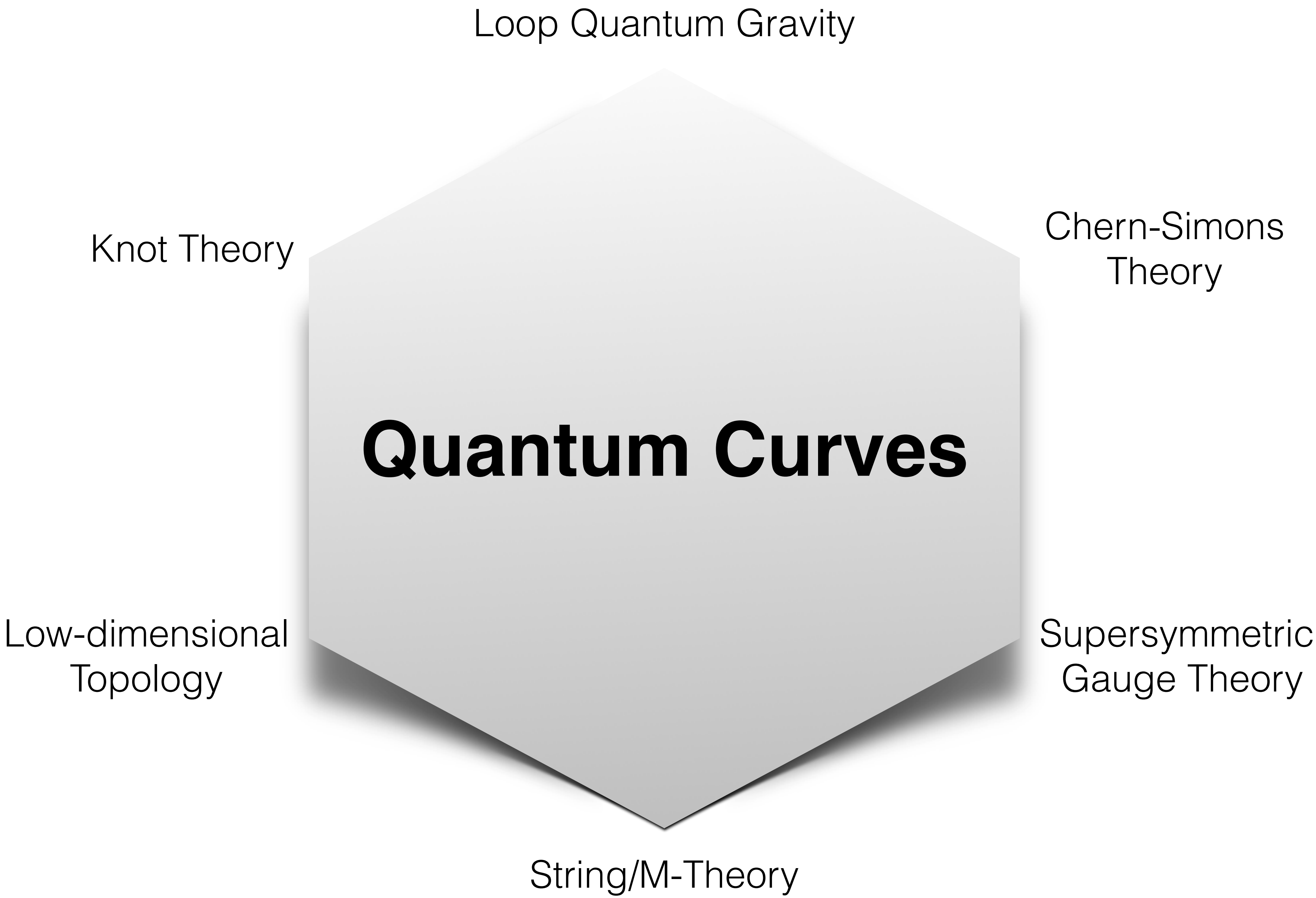
Holomorphic: polynomial eqns $\mathbf{A}_m(e^u, e^v) = 0, \quad m = 1, \cdots, N$

Quantization: $[\hat{u}_m, \hat{v}_n] = i\hbar\delta_{m,n}, \quad \hat{u}_m f(u) = u f(u), \quad \hat{v}_m f(u) = -i\hbar\partial_{u_m} f(u)$

Quantization of $\mathcal{L}_A$**:** $\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \hbar)\, Z(u) = 0, \quad m = 1, \cdots, N$

The goal: find the holomorphic solution $Z(u)$.

Eynard, Orantin 2009
Gukov, Sułkowski 2011
Mulase, Sułkowski 2012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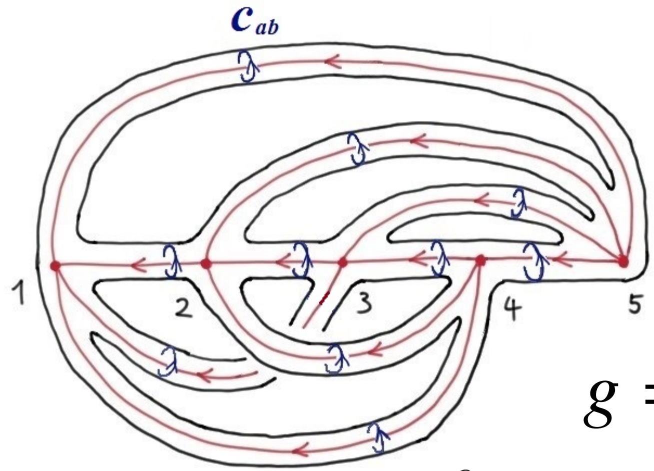
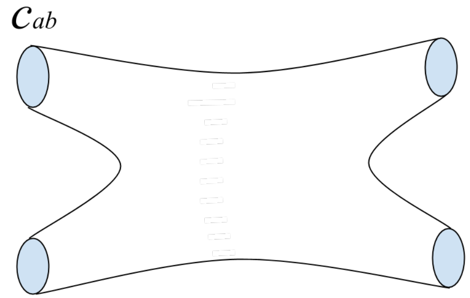
Moduli space of flat connections on 2-surface

Closed 2-surface Σ_g **e.g.**

with a set of closed curves $\{c\}$



$g = 6$



s.t. $\Sigma_g \setminus \{c\} =$ a set of n-holed spheres

Complex symplectic manifold: $\mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$ (A s.t. F_A=0)

- **Hyper-Kahler: complex structures** i (from 2-surface) j (from complex group) $k=ij$

- **Symplectic structure:** $\omega \sim \int_{\Sigma_g} \mathrm{tr}\,[\delta A \wedge \delta A]$ **(Atiyah-Bott-Goldman)**

Holomorphic coordinates (w.r.t j)

Closed 2-surface Σ_g **with the set of closed curves** $\{c\}$

Complex Fenchel-Nielsen (FN) coordinates: $c \mapsto (x_c, y_c) \in (\mathbb{C}^\times)^2$

FN length: x_c **holonomy eigenvalue along the curve** c

FN twist: y_c **"conjugate momenta"**

Holomorphic symplectic structure:

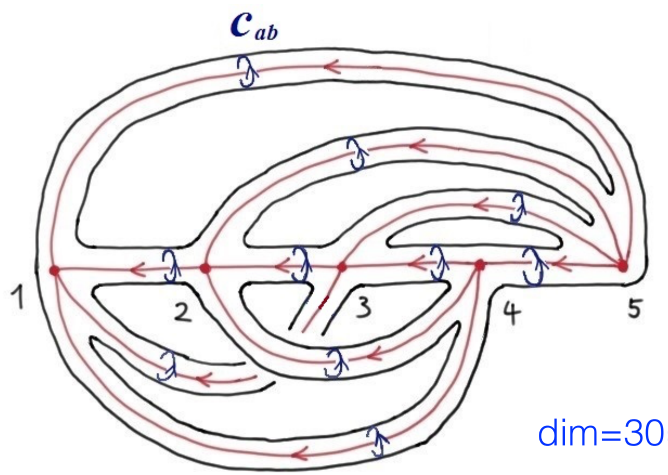
$$\omega = \sum_c \mathrm{d}\ln y_c \wedge \mathrm{d}\ln x_c + \omega_{n\text{-holed sphere}}$$

$\mathcal{M}_{flat}(n\text{-holed } S^2, \mathrm{SL}(2,\mathbb{C}))$
with fixed holonomy
eigenvalues at holes

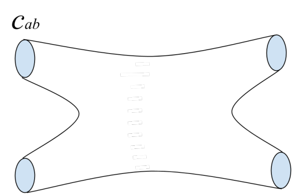
$$\dim_{\mathbb{C}} \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C})) = 6g - 6$$

e.g.

$g = 6$



dim=30=2*10+2*5



dim=2

Holomorphic Lagrangian submanifold

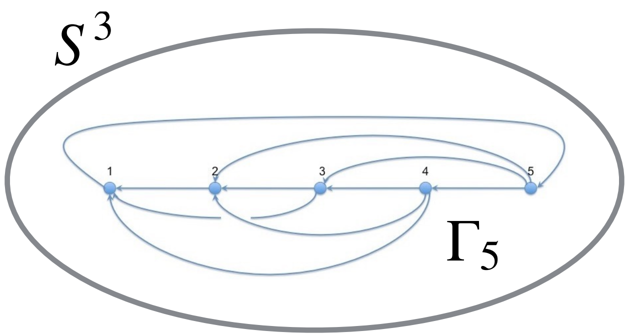
For a 3-manifold M_3 s.t. $\partial M_3 = \Sigma_g$

$$\mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$$

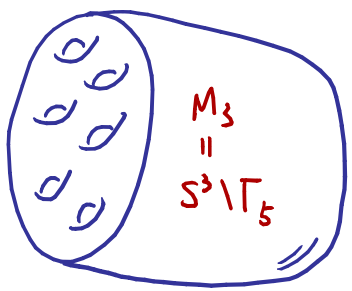
Holomorphic polynomial eqns $\mathbf{A}_m(x_c, y_c; \cdots) = 0, \quad m = 1, \cdots, 3g-3$

We focus on *graph complement 3-manifold* in 3-sphere: removing a tubular open neighborhood of a graph embedded in 3-sphere.

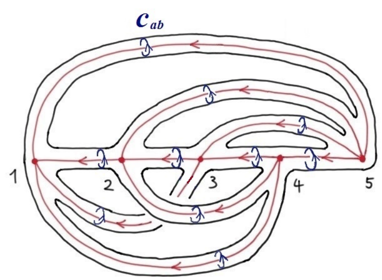
$$M_3 = S^3 \setminus N(\Gamma) \equiv S^3 \setminus \Gamma$$



$\longrightarrow$



$\partial M_3 =$



$\Sigma_{g=6}$

Quantization of flat connections on 3-manifold

$$\mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C})) \qquad \omega = \sum_c \mathrm{d}\ln y_c \wedge \mathrm{d}\ln x_c + \omega_{n\text{-holed sphere}}$$

Holomorphic symplectic coordinates $\quad u_c = \ln x_c, \quad v_c = \ln y_c, \quad \cdots$

$$\hat{u}_c f(u,\cdots) = u_c f(u,\cdots), \qquad \hat{v}_c f(u,\cdots) = -i\hbar\partial_{u_c} f(u,\cdots)$$

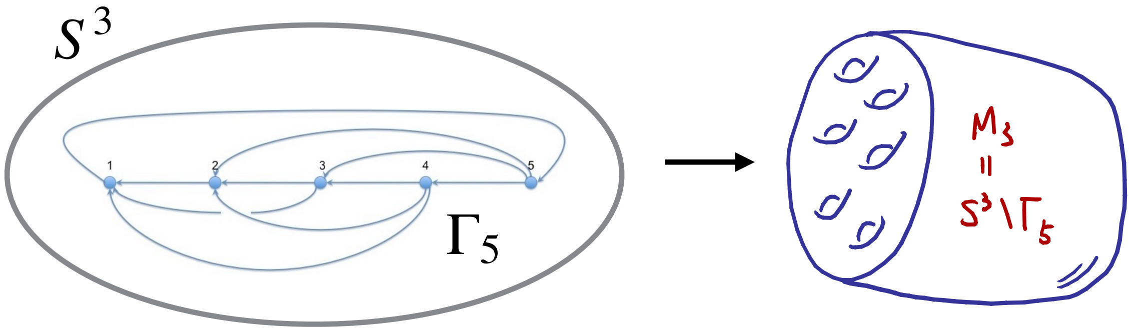
Quantization of $\quad \mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$

$$\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hbar)\, Z(u,\cdots) = 0, \quad m = 1,\cdots,3g-3$$

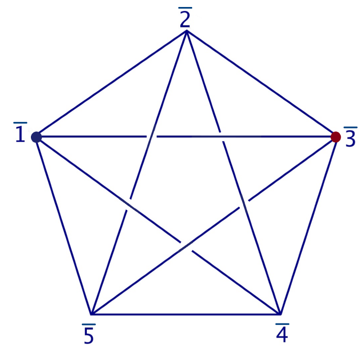
The holomorphic solutions $Z(u,\ldots)$ are the physical states for quantum flat connections on 3-manifold, which quantizes SL(2,C) Chern-Simons theory on 3-manifold.

Dimofte, Gukov, Lenells, Zagier 2009
Dimofte 2011
Gukov, Sułkowski 2011
Gukov, Saberi 2012

Flat Connections in 3d *v.s.* Simplicial Geometry in 4d



Γ_5 graph complement



4-simplex

A class of $\mathrm{SL}(2,\mathbb{C})$ flat connetion on $S^3 \setminus \Gamma_5$

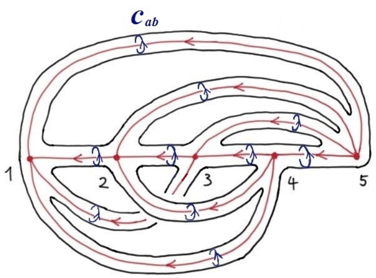
=

Lorentzian 4-simplex geometries with constant curvature Λ

Haggard, MH, Kamiński, Riello 2014

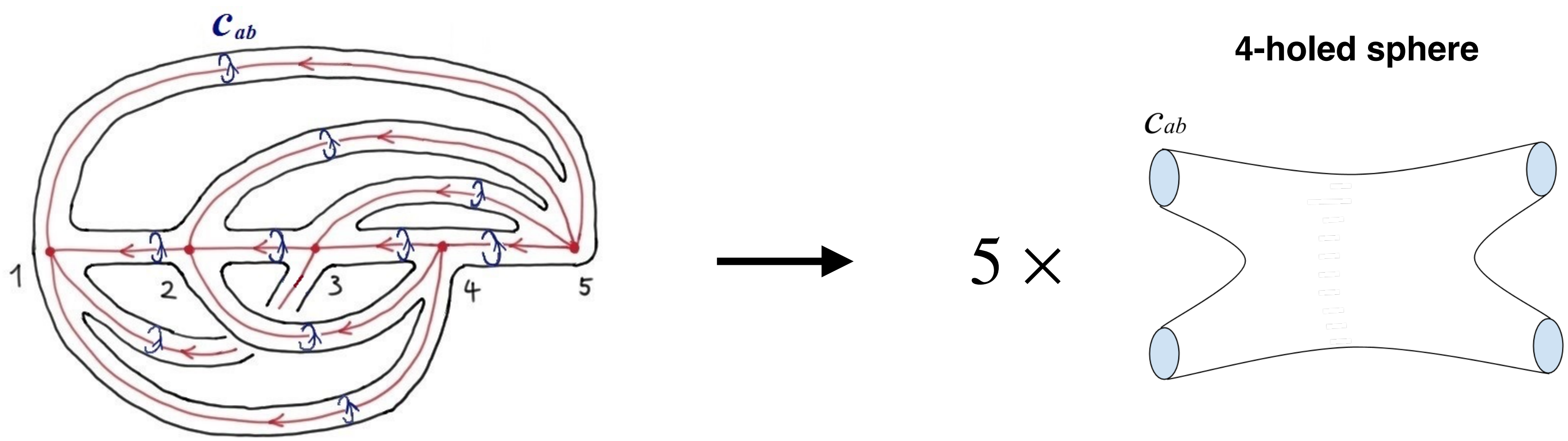
The class of flat connections is specified by the boundary condition

$$\partial M_3 =$$



$\Sigma_{g=6}$

Boundary condition on $\mathcal{M}_{flat}\left(\Sigma_{g=6}, \mathrm{SL}(2,\mathbb{C})\right)$



- We associate each 4-holed sphere with an SU(2) subgroup of SL(2,C)

- We consider the SL(2,C) flat connections on $\Sigma_{g=6}$ that reduce to SU(2) on each 4-holed sphere

- Relates to the simplicity constraint in LQG

Flat Connections in d-1 *v.s.* Discrete Geometry in d

$\pi_1(M_{d-1})$

Fundamental group of a $(d-1)$-manifold M_{d-1} (defected S^{d-1})

$\overset{S}{\simeq}$

$\pi_1(\mathrm{sk}(M_d))$

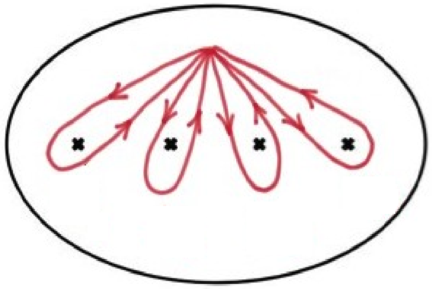
Fundamental group of the 1-skeleton of a d-dim polyhedron M_d

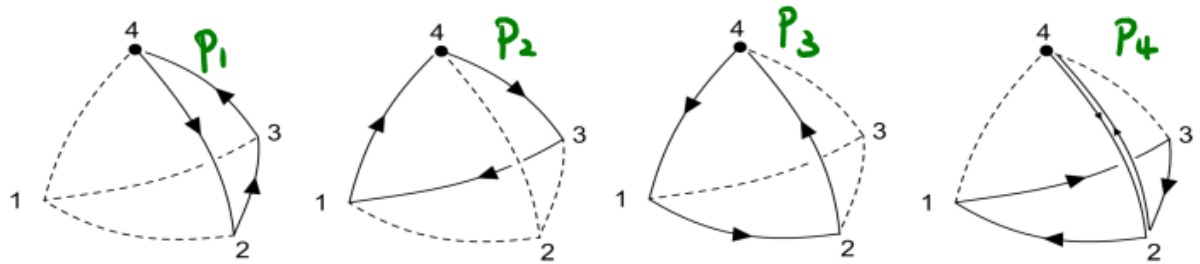
ω_{flat} $\searrow$ $\quad$ $\swarrow$ ω_{spin}

$\mathrm{SO}(d)$ or $\mathrm{SO}(d-1,1)$ **modulo gauge**

$$\omega_{spin} = \omega_{flat} \circ S$$

4-holed sphere *v.s.* tetrahedron





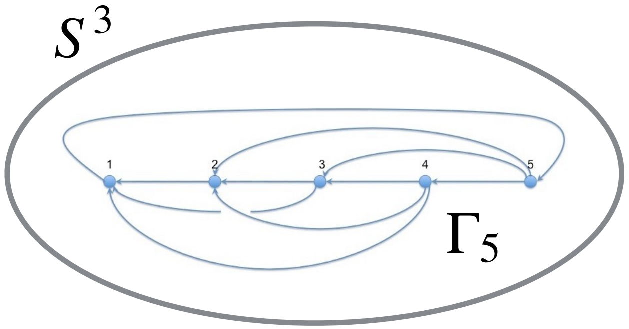
$$\pi_1(\text{4-holed sphere}) \overset{S}{\simeq} \pi_1(\text{sk(Tetra)})$$

$$= \left\langle \ell_1, \cdots, \ell_4 \middle| \ell_4 \ell_3 \ell_2 \ell_1 = e \right\rangle \qquad = \left\langle p_1, \cdots, p_4 \middle| p_4 p_3 p_2 p_1 = e \right\rangle$$

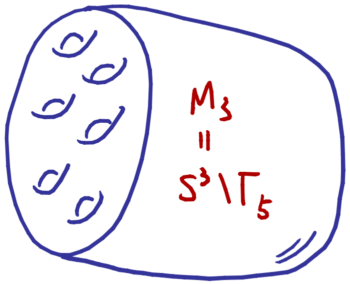
ω_{flat} $\qquad$ ω_{spin}

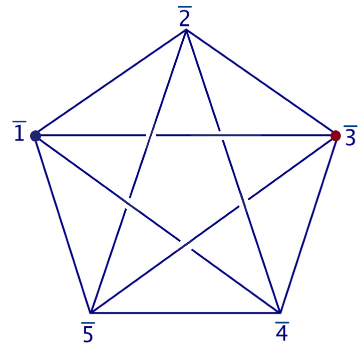
$$\left\langle H_1, \cdots, H_4 \in \mathrm{SO}(3) \middle| H_4 H_3 H_2 H_1 = 1 \right\rangle / \text{conjugation}$$

Γ_5 graph complement *v.s.* 4-simplex



$\longrightarrow$





$$
\begin{aligned}
&\text{vertex 1}: && \mathfrak{l}_{14}\mathfrak{l}^{(1)}_{13}\mathfrak{l}_{12}\mathfrak{l}_{15} = 1,\\
&\text{vertex 2}: && \mathfrak{l}^{-1}_{12}\mathfrak{l}_{24}\mathfrak{l}_{23}\mathfrak{l}_{25} = 1,\\
&\text{vertex 3}: && \mathfrak{l}^{-1}_{23}(\mathfrak{l}^{(2)}_{13})^{-1}\mathfrak{l}_{34}\mathfrak{l}_{35} = 1,\\
&\text{vertex 4}: && \mathfrak{l}^{-1}_{34}\mathfrak{l}^{-1}_{24}\mathfrak{l}^{-1}_{14}\mathfrak{l}_{45} = 1,\\
&\text{vertex 5}: && \mathfrak{l}^{-1}_{25}\mathfrak{l}^{-1}_{35}\mathfrak{l}^{-1}_{45}\mathfrak{l}^{-1}_{15} = 1,\\
&\text{crossing}: && \mathfrak{l}^{(1)}_{13} = \mathfrak{l}_{24}\mathfrak{l}^{(2)}_{13}\mathfrak{l}^{-1}_{24}.
\end{aligned}
$$

$$\overset{S}{\simeq}$$

$$
\begin{aligned}
&\text{tetra 1}: && p_{14}p^{(1)}_{13}p_{12}p_{15} = 1,\\
&\text{tetra 2}: && p^{-1}_{12}p_{24}p_{23}p_{25} = 1,\\
&\text{tetra 3}: && p^{-1}_{23}(p^{(2)}_{13})^{-1}p_{34}p_{35} = 1,\\
&\text{tetra 4}: && p^{-1}_{34}p^{-1}_{24}p^{-1}_{14}p_{45} = 1,\\
&\text{tetra 5}: && p^{-1}_{25}p^{-1}_{35}p^{-1}_{45}p^{-1}_{15} = 1,\\
&\text{“crossing”}: && p^{(1)}_{13} = p_{24}p^{(2)}_{13}p^{-1}_{24}.
\end{aligned}
$$

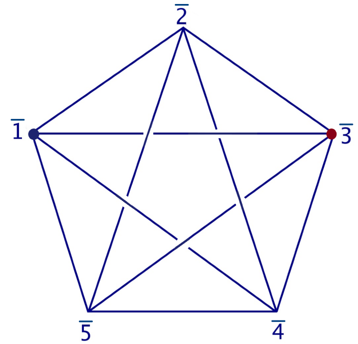
ω_{flat} $\searrow$ $\swarrow$ ω_{spin}

$$\Big\langle H_{ab} \in \mathrm{SO}(3,1) \Big| \quad \cdots \quad \Big\rangle \Big/ \text{conjugation}$$

$\omega_{spin} = \omega_{flat} \circ S$ **are a set of holonomies along closed paths on 1-skeleton**

How much do they know about the geometry?

In general they know very little.



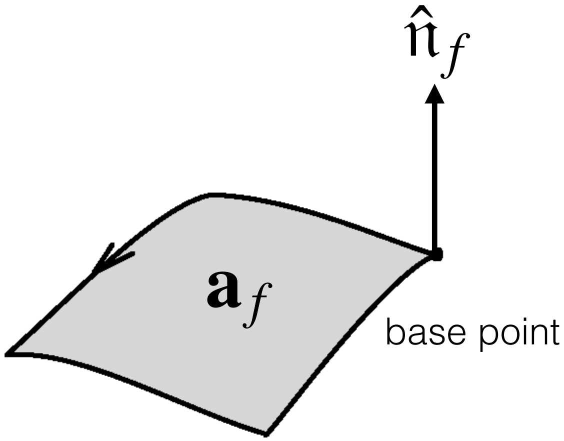
But for constant curvature simplex, whose 2-faces are flatly embedded surfaces:

Lemma: Given 2-surface flatly embedded (K=0) in constant curvature space, the holonomy of spin connection along the boundary of surface:

$$h_{\partial f}(\omega_{spin}) = \exp\left[-i\frac{\Lambda}{6}\mathbf{a}_f \hat{\mathrm{n}}_f \cdot \vec{\sigma}\right]$$

in 3d space

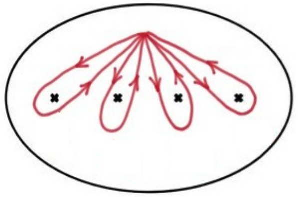
replaced by normal bivector in 4d spacetime

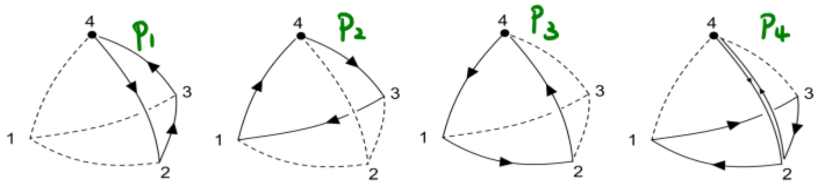


Area and normal data determine the simplex geometry

<u>Theorem</u>: There is 1-to-1 correspondence between

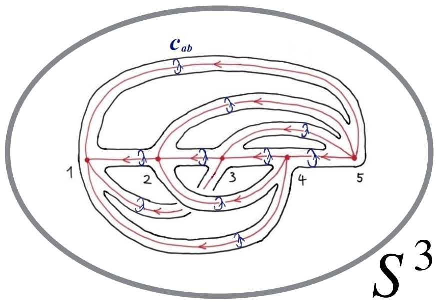
$A \in \mathcal{M}_{flat}(\text{4-holed sphere}, \mathrm{PSU}(2))$ ⟷ A convex constant curvature tetrahedron geometry with $\Lambda > 0$ or $\Lambda < 0$



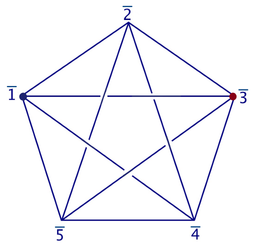


<u>Theorem</u>: There is 1-to-1 correspondence between

$A \in \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{PSL}(2,\mathbb{C}))$ satisfying the boundary condition ⟷ A convex constant curvature 4-simplex geometry with $\Lambda > 0$ or $\Lambda < 0$ (Lorentzian)



$S^3 \setminus \Gamma_5$

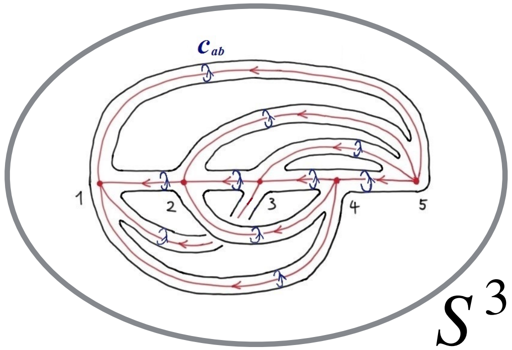


<u>Remark</u>: The above statements hold as far as the geometry is nondegenerate.

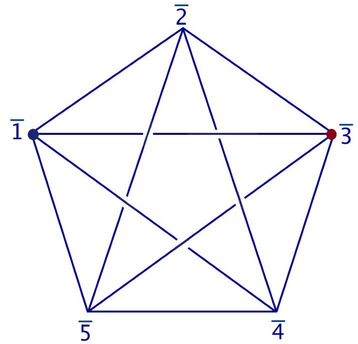
<u>Remark</u>: Flat conn holonomy around defect = Spin conn holonomy around face.

Dictionary between coordinates

Flat connection | **4-simplex geometry**



$S^3 \setminus \Gamma_5$



FN length: x_{ab} **=** $\pm \exp\left[-i\frac{\Lambda}{6}\mathbf{a}_{ab}\right]$ **triangle area**

FN twist: y_{ab} **=** $\pm \exp\left[-\frac{1}{2}\Theta^{\Lambda}_{ab}\right]$ **4d dihedral angle**

4-holed sphere: (x_a, y_a) **=** **shape of tetrahedron**

Parity Pair

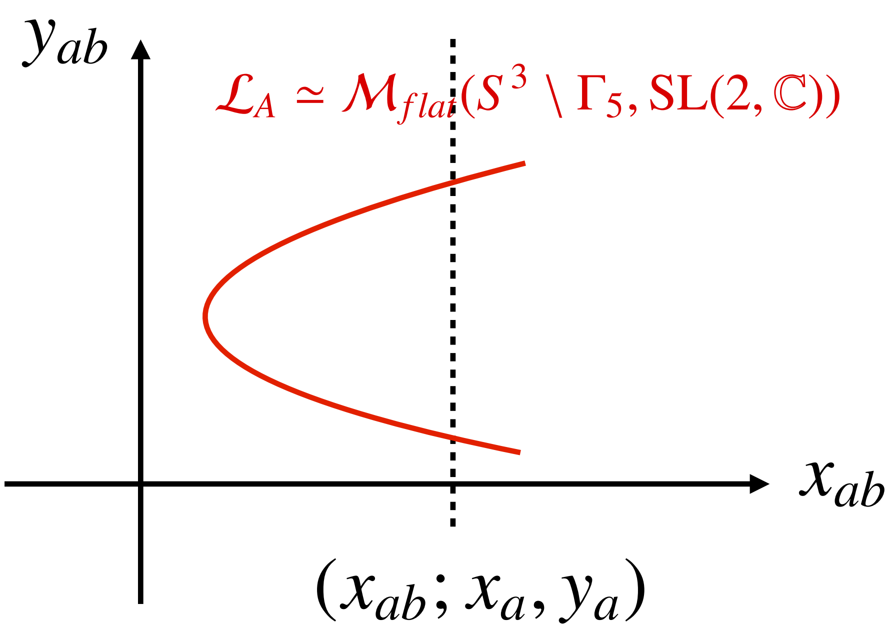
Given a flat connection $A \in \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{SL}(2,\mathbb{C}))$ **satisfying boundary condition,**

It associates a unique $\tilde{A} \in \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{SL}(2,\mathbb{C}))$ **satisfying boundary condition,**

with the same boundary data: $(x_{ab}; x_a, y_a)$

but with different twist variable: $y_{ab} = \pm \exp\left[-\frac{1}{2}\Theta^{\Lambda}_{ab}\right], \quad \tilde{y}_{ab} = \pm \exp\left[\frac{1}{2}\Theta^{\Lambda}_{ab}\right]$

$\mathcal{M}_{flat}(\Sigma_{g=6}, \mathrm{SL}(2,\mathbb{C}))$



2 constant curvature 4-simplex with the same geometry but with opposite 4d orientations

Quantum Theory

Flat connection on $S^3 \setminus \Gamma_5$ **= 4-simplex geometry**

Quantum flat connection on $S^3 \setminus \Gamma_5$ **= Quantum 4-simplex geometry**

Quantization of 4d geometry

↓

Quantization of flat connection on 3-manifold

(Quantization of holomorphic Lagrangian submanifold)

Quantization of flat connections on 3-manifold

$$\mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C})) \qquad \omega = \sum_c \mathrm{d}\ln y_c \wedge \mathrm{d}\ln x_c + \omega_{n\text{-holed sphere}}$$

Holomorphic symplectic coordinates $\quad u_c = \ln x_c, \quad v_c = \ln y_c, \quad \cdots$

$$\hat{u}_c f(u,\cdots) = u_c f(u,\cdots), \qquad \hat{v}_c f(u,\cdots) = -i\hbar\partial_{u_c} f(u,\cdots)$$

Quantization of $\quad \mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$

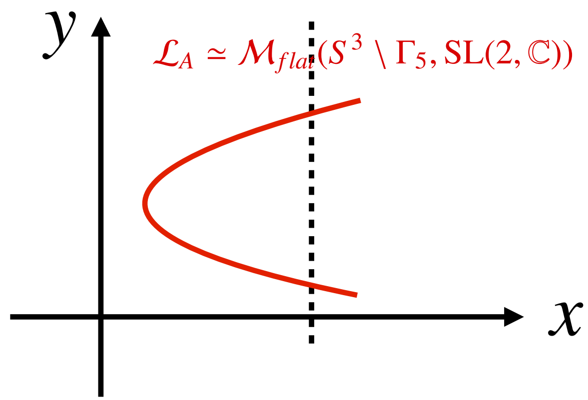
$$\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hbar)\, Z(u,\cdots) = 0, \quad m = 1,\cdots,3g-3$$

The holomorphic solutions $Z(u,\ldots)$ are the physical states for quantum flat connections on 3-manifold, which quantizes SL(2,C) Chern-Simons theory on 3-manifold.

Dimofte, Gukov, Lenells, Zagier 2009
Dimofte 2011
Gukov, Sułkowski 2011
Gukov, Saberi 2012

$\mathbf{A}_m(x_c, y_c; \cdots) = 0, \quad m = 1, \cdots, 3g-3$

$\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hbar)\, Z(u, \cdots) = 0, \quad m = 1, \cdots, 3g-3$



Dimofte, Gukov, Lenells, Zagier 2009
Witten 2010

WKB solutions: holomorphic 3d block

$$Z^{(\alpha)}\left(M_3 \middle| u\right) = \exp\left[\frac{i}{\hbar} \int_{\substack{(u_0,v_0) \\ \mathfrak{C} \subset \mathcal{L}_{\mathbf{A}}}}^{\left(u, v^{(\alpha)}\right)} \vartheta + o(\log \hbar)\right]$$

$$\vartheta = \sum_c v_c \mathrm{d}u_c + \vartheta_{n\text{-holed spheres}}$$

Liouville 1-form

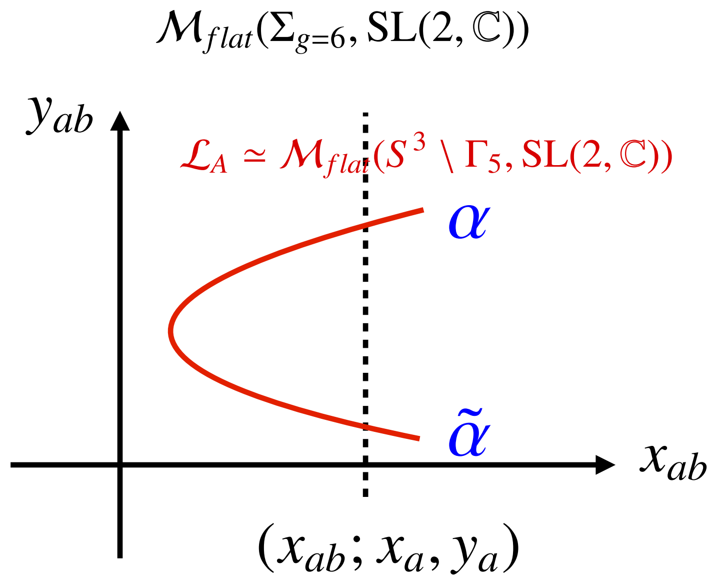
α labels the branches of Lagrangian submanifold. Thus (u,α) corresponds to a unique SL(2,C) flat connection on M_3

$Z^{(\alpha)}\left(M_3 \middle| u\right)$ **has ambiguities: (1)** $Z^{(\alpha)}\left(M_3 \middle| u\right) \mapsto Z^{(\alpha)}\left(M_3 \middle| u\right) \exp\left(-\frac{2\pi}{\hbar} u\right) \qquad (v \sim v + 2\pi i)$

(2) starting point of contour → overall phase.

Wave function of 4-geometry

- **Quantize** $\mathcal{L}_A \simeq \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{SL}(2,\mathbb{C}))$
- **Impose the boundary condition**
- **Consider the branch α s.t. (u,α) corresponds to a constant curvature 4-simplex geometry**
- **α associates with its parity partner $\tilde{\alpha}$ s.t. (u,α) and $(u,\tilde{\alpha})$ are parity pair**



Holomorphic 3d block defined at branch α with the reference at branch $\tilde{\alpha}$ is a state for quantum 4-simplex geometry

$$Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) = \exp\left[\frac{i}{\hbar}\int_{\substack{(u,v^{(\tilde{\alpha})})\\ \mathfrak{C}\subset\mathcal{L}_{\mathbf{A}}}}^{(u,v^{(\alpha)})} \vartheta + o(\log\hbar)\right]$$

Quantum Geometry = Quantum Gravity

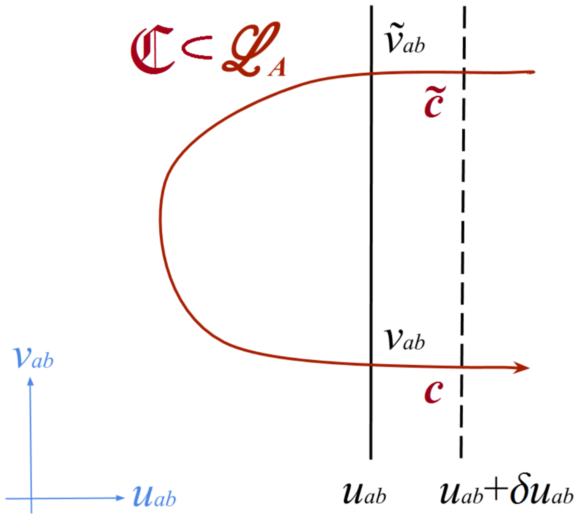
Semiclassical limit of $Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) \longrightarrow$ **Discrete Einstein gravity in 4d**

Semiclassical limit $\hbar \to 0$

$$Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) \sim \exp\left[\frac{i}{\hbar} S^{\Lambda}_{Regge} + o(\log \hbar)\right]$$

Discrete 4d Einstein-Hilbert action on a constant curvature 4-simplex:

$$S^{\Lambda}_{Regge} = \sum_{a<b} \mathbf{a}_{ab} \Theta^{\Lambda}_{ab} - \Lambda \mathrm{Vol}^{\Lambda}_{4}$$



$$I^{\alpha}_{\tilde{\alpha}} = \int_{\substack{(u,v^{(\tilde{\alpha})}) \\ \mathfrak{C}\subset\mathcal{L}_{\mathbf{A}}}}^{(u,v^{(\alpha)})} \sum_{a<b} v_{ab} \mathrm{d}u_{ab}$$

$$Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) = \exp\left[\frac{i}{\hbar} I^{\alpha}_{\tilde{\alpha}} + o(\log\hbar)\right]$$

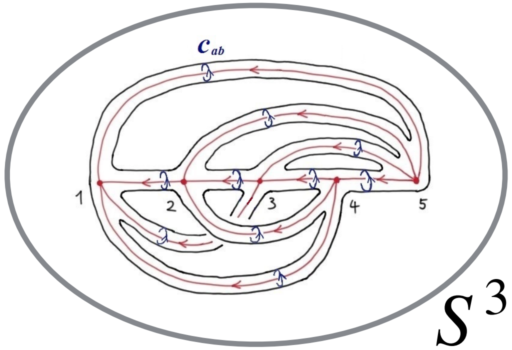
Variation of boundary data $[u_{ab}; u_a, v_a] \mapsto [u_{ab} + \delta u_{ab}; u_a + \delta u_a, v_a + \delta v_a]$

$$\delta I^{\alpha}_{\tilde{\alpha}} = \int_{c\cup\tilde{c}} \sum_{a<b} v_{ab} \mathrm{d}u_{ab} \sim \text{“symplectic area of the square”}$$

$$= \delta u_{ab}\left[v_{ab} - \tilde{v}_{ab}\right] + o\left((\delta u)^2\right)$$

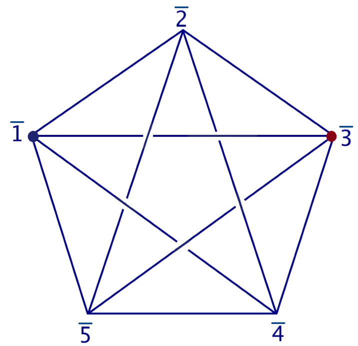
Dictionary between coordinates

Flat connection



$S^3 \setminus \Gamma_5$

4-simplex geometry



FN length: $x_{ab} = e^{u_{ab}}$ **=** $\pm \exp\left[-i\frac{\Lambda}{6}\mathbf{a}_{ab}\right]$ **triangle area**

FN twist: $y_{ab} = e^{-\frac{2\pi}{t}v_{ab}}$ **=** $\pm \exp\left[-\frac{1}{2}\Theta^{\Lambda}_{ab}\right]$

$\tilde{y}_{ab} = e^{-\frac{2\pi}{t}\tilde{v}_{ab}}$ **=** $\pm \exp\left[+\frac{1}{2}\Theta^{\Lambda}_{ab}\right]$

4d dihedral angle

t **is CS coupling**

$$\delta I^{\alpha}_{\tilde{\alpha}} = \left(\frac{\Lambda t}{12\pi i}\right)\sum_{a<b}\delta\mathbf{a}_{ab}\Theta_{ab} + \left(\frac{\Lambda t}{6}\right)\mathbb{Z}\sum_{a<b}\delta\mathbf{a}_{ab}$$

Integrate by using Schafli identity $\sum_{a<b}\mathbf{a}_{ab}\,\delta\Theta_{ab} = \Lambda\,\delta\mathrm{Vol}_4^{\Lambda}$

Suarez-Peiro 2000
Haggard, Hedeman, Kur, Littlejohn 2014

integration const.

$$Z^{(\alpha)}\left(S^3\setminus\Gamma_5\middle|u\right) = \exp\left[\frac{i}{\hbar}\left(\frac{\Lambda t}{12\pi i}\right)\left(\sum_{a<b}\mathbf{a}_{ab}\Theta_{ab} - \Lambda\mathrm{Vol}_4^{\Lambda}\right) + C^{\alpha}_{\tilde{\alpha}} + \frac{i}{\hbar}\left(\frac{\Lambda t}{6}\right)\mathbb{Z}\sum_{a<b}\mathbf{a}_{ab} + \cdots\right]$$

Lorentzian Regge action in 4d — ambiguity (1) of holomorphic block

To obtain an oscillatory phase: consider full SL(2,C) Chern-Simons theory with both holomorphic and anti-holomorphic contribution

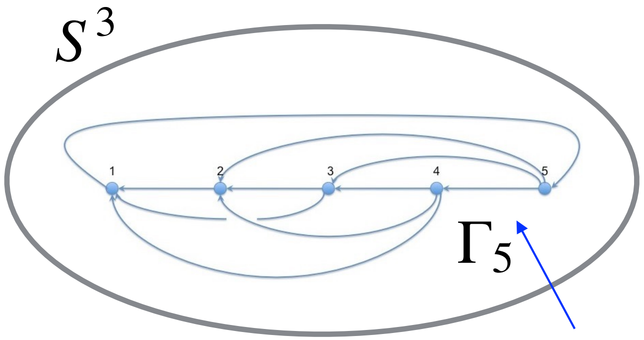
$$Z^{(\alpha)}\left(S^3\setminus\Gamma_5\middle|u\right)Z^{(\bar{\alpha})}\left(S^3\setminus\Gamma_5\middle|\bar{u}\right)$$
$$= \exp\left[\frac{i}{\hbar}2\mathrm{Re}\left(\frac{\Lambda t}{12\pi i}\right)\left(\sum_{a<b}\mathbf{a}_{ab}\Theta_{ab} - \Lambda\mathrm{Vol}_4^{\Lambda}\right) + C^{\alpha}_{\tilde{\alpha}} + \frac{i}{\hbar}2\mathrm{Re}\left(\frac{\Lambda t}{6}\right)\mathbb{Z}\sum_{a<b}\mathbf{a}_{ab} + \cdots\right].$$

- **Gravitational coupling:** $G_{\mathrm{N}} = \left|\frac{3}{2\mathrm{Im}(t)\Lambda}\right|$ — t **is CS coupling**
- **Independent of ambiguity:** $2\mathrm{Re}\left(\frac{\Lambda t}{6}\right)\sum_{a<b}\mathbf{a}_{ab} \in 2\pi\hbar\mathbb{Z}$ **fulfilled by LQG** $\mathbf{a} \sim j$

Interesting: $t \in i\mathbb{R}$ **no quantization condition needed**

In LQG term, corresponding to the limit: Barbero-Immirzi parameter —> infinity

Relation with Loop Quantum Gravity



Wilson lines with unitary representation of SL(2,C)

Haggard, MH, Kamiński, Riello 2014

SL(2,C) CS theory on S^3 with certain Wilson graph operator

$$Z_{\Gamma_5} = \int [DAD\bar{A}]\, e^{\frac{i}{\hbar} CS[S^3|A,\bar{A}]}\, \Gamma_5[A,\bar{A}]$$

Wilson graph operator imposes the right boundary condition on $\partial(S^3 \setminus \Gamma_5) = \Sigma_{g=6}$

$$x_{ab} = \exp\left[\frac{2\pi i\hbar}{t}(1+i\gamma)j_{ab}\right], \quad \gamma = \frac{\mathrm{Im}(t)}{\mathrm{Re}(t)}, \quad j_{ab} \in \mathbb{Z}/2$$

Barbero-Immirzi parameter

- **semiclassical limit = double-scaling limit** $j \to \infty,\ \hbar \to 0,\ j\hbar$ fixed
- Z_{Γ_5} **has the same semiclassical limit as the 3d block**

$$Z_{\Gamma_5} \sim Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) Z^{(\overline{\alpha})}\left(S^3 \setminus \Gamma_5 \middle| \bar{u}\right)$$

gives classical Einstein-Regge action as the leading order.

Deformation of EPRL Spinfoam Amplitude

Haggard, MH, Kamiński, Riello 2014

$$
\begin{array}{ccc}
Z_{\Gamma_5} & \xrightarrow{\hbar\to 0,\ j\to\infty,\ j\hbar \text{ fixed}} & e^{\frac{i}{\ell_P^2} S^{\Lambda}_{Regge}} + e^{-\frac{i}{\ell_P^2} S^{\Lambda}_{Regge}} \\
\Big\downarrow t\to\infty & & \Big\downarrow \Lambda\to 0 \\
Z_{EPRL} & \xrightarrow{j\to\infty} & e^{\frac{i}{\ell_P^2} S_{Regge}} + e^{-\frac{i}{\ell_P^2} S_{Regge}}
\end{array}
$$

Barrett, Dowdall, Fairbairn, Hellmann, Pereira 2010

Promote CS 3d block to be a wave-function/spinfoam-amplitude of 4d LQG

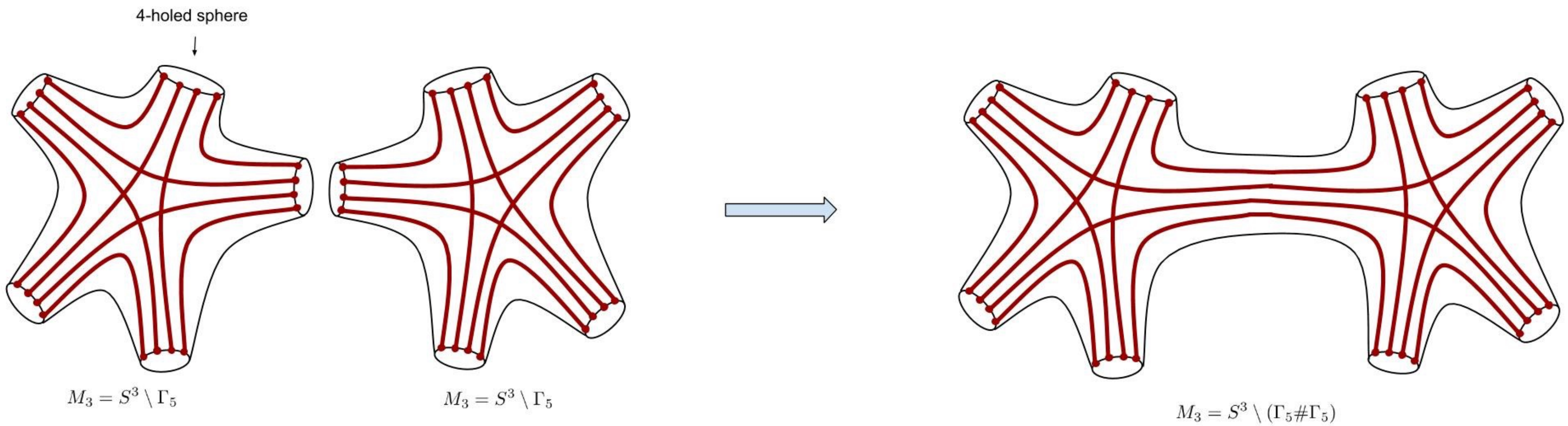
$$
Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) Z^{(\overline{\alpha})}\left(S^3 \setminus \Gamma_5 \middle| \bar{u}\right)
$$

Identify/generalize spin-network data to flat connection data on closed 2-surface + the quantization condition

Rovelli, Vidotto 2015
see Francesca Vidotto's talk

Generalize to 4d Simplicial Complex

3-manifold obtained from gluing graph complements through 4-holed sphere

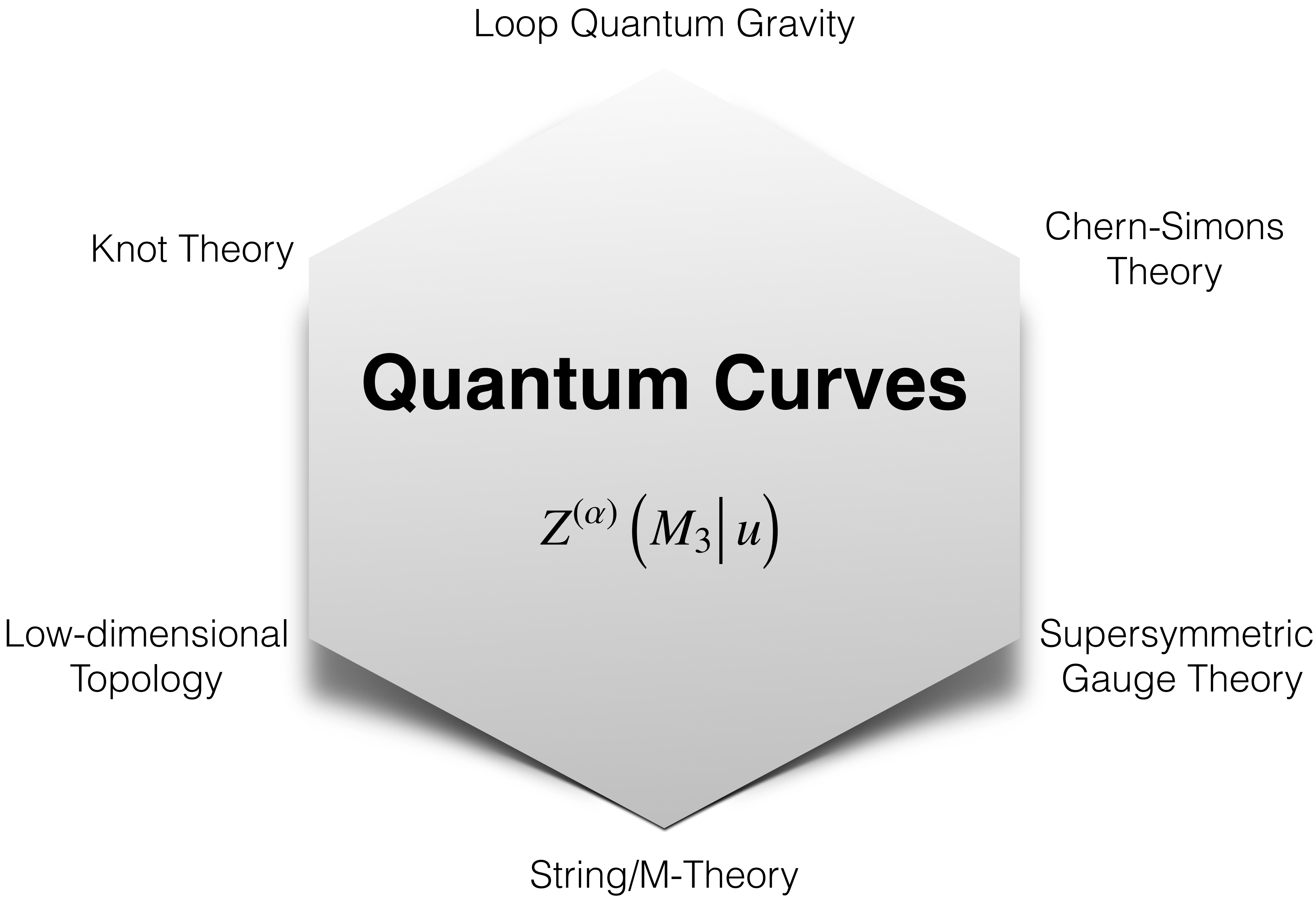


Flat connections on 3-manifold = Simplicial geometry on 4-manifold

$$Z^{(\alpha)}\left(M_3 \middle| u\right) \sim \exp\left[\frac{i}{\hbar} S^{\Lambda}_{Regge} + o(\log \hbar)\right]$$

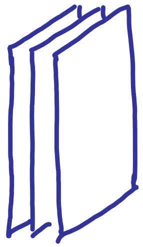
↑

Einstein-Regge action on the entire simplicial complex


Loop Quantum Gravity
Knot Theory
Chern-Simons Theory
Quantum Curves
$Z^{(\alpha)}\left(M_3 \middle| u\right)$
Low-dimensional Topology
Supersymmetric Gauge Theory
String/M-Theory

3d-3d correspondence

M-theory in 11d:



N

M5-brane ⟶ **IR dynamics: 6d SCFT with gauge group** G

(6-dim) **16 supercharges (maximal SUSY)**

Compactify M5 on $M_3 \times S_b^3$ 3d ellipsoid

(or $S^2 \times_q S^1$ or $\mathbb{R}^2 \times_q S^1$)

$\mathrm{G}_{\mathbb{C}}$ CS on M_3 $\Leftrightarrow$ 3d $\mathcal{N}=2$ SUSY gauge theory T_{M_3} (SCFT with 4 Q's)

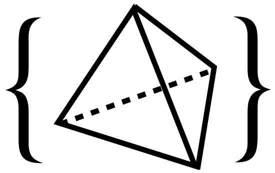
- $Z_{CS}(M_3) = Z_{T_{M_3}}^{\mathcal{N}=2}(S_b^3)$
- $\mathcal{M}_{flat}(M_3, \mathrm{G}_{\mathbb{C}}) \simeq \mathcal{M}_{SUSY}(T_{M_3})$
- $Z^{(\alpha)}(M_3) = Z_{T_{M_3}}^{\mathcal{N}=2}(\mathbb{R}^2 \times_q S^1)$ with boundary SUSY ground state α

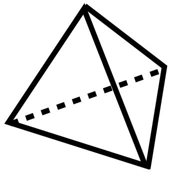
Dimofte, Gaiotto, Gukov 2011
C. Beem, T. Dimofte, S. Pasquetti 2012
Cordova, Jafferis 2013
Lee, Yamazaki 2013
Chung, Dimofte, Gukov, Sułkowski 2014

Dimofte-Gaiotto-Gukov (DGG) Construction

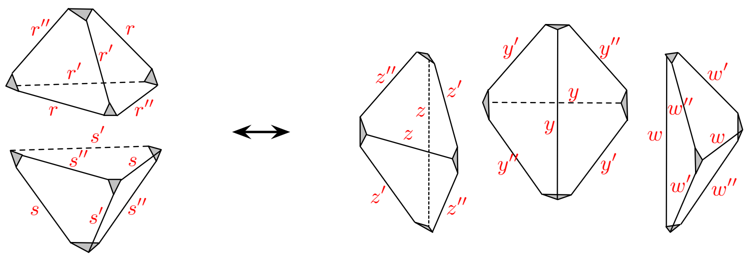
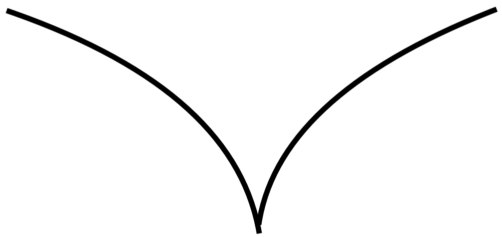
T_{DGG,M_3} **3d $\mathcal{N}$= 2 SCFT with Abelian gauge group** $\mathrm{U}(1)^n$ Dimofte, Gaiotto, Gukov 2011

(Gauge theories labelled by 3-manifolds)

M_3 **Ideal triangulation** $\longrightarrow$ 

 $T_\Delta =$ **3d $\mathcal{N}$= 2 chiral multiplet ; gluing → gauging + superpotential**

→ **Pachner move = 3d mirror symmetry**

 **=** 

- $\mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \hookleftarrow \mathcal{M}_{SUSY}(T_{DGG,M_3})$
- $Z'_{CS}(M_3) = Z_{DGG,M_3}(S^3_b)$
- $Z^{(\alpha)}(M_3) = Z_{DGG,M_3}(\mathbb{R}^2 \times_q S^1)$ with boundary SUSY ground state α

Dimofte, Gaiotto, Gukov 2011
C. Beem, T. Dimofte, S. Pasquetti 2012

4d LQG and 3d SCFT

LQG vacua = Simplicial geometries = Flat conn on M_3 = SUSY vacua in T_{M_3}

Spinfoam Amplitude = CS partition function of M_3 = SUSY partition function of T_{M_3}

$$\sim \exp\left[iS^{\Lambda}_{Regge} + \cdots\right]$$

4d LQG and 3d SCFT

LQG vacua = Simplicial geometries = Flat conn on M_3 = SUSY vacua in T_{M_3}

Spinfoam Amplitude = CS partition function of M_3 = SUSY partition function of T_{M_3}

$$\sim \exp\left[iS^{\Lambda}_{Regge} + \cdots\right]$$

The end

Thanks for your attention !