Invariant Connections: Classical, Quantum and Applications

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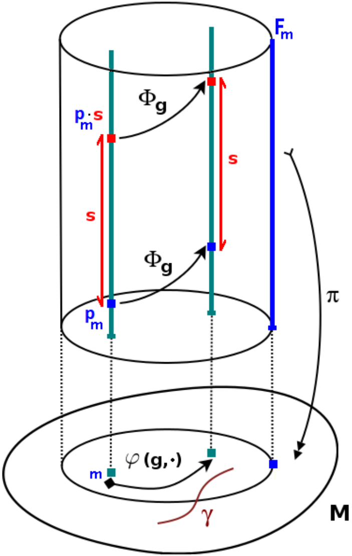
Principal fibre bundle (P, π, M, S) $(S = SU(2),\ M = \Sigma\text{-Cauchy surface})$

- $\mathcal{A}$ – smooth connections on P
- $\mathcal{P}$ – some smooth curves in M
- $\mathfrak{P}$ – cylindrical functions on $\mathcal{A}$ w.r.t. $\mathcal{P}$

  $\rightarrow$ C^*-subalgebra generated by $f_0 \circ h_\gamma$

Quantum Configuration Space

$\overline{\mathcal{A}} := \mathrm{Spec}(\mathfrak{P})$ - generalized connections



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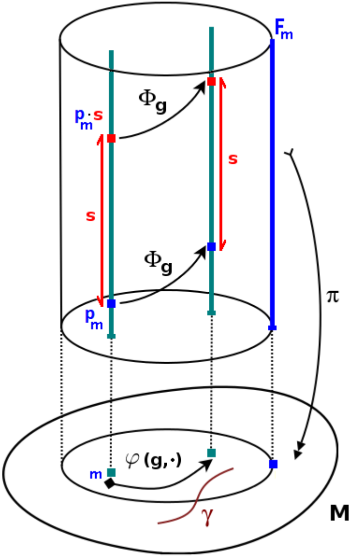
$\Phi\colon G \times P \rightarrow P$ left action with

$$\Phi(g, p \cdot s) = \Phi(g, p) \cdot s$$

for all $g \in G$, $p \in P$, $s \in S$.



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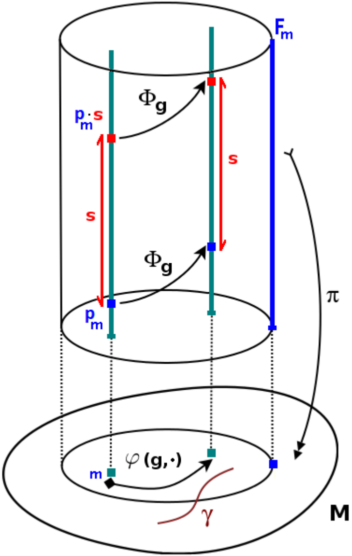

Induced Actions:

- $\theta\colon G \times \mathcal{A} \rightarrow \mathcal{A},\ (g, \omega) \mapsto \Phi^*_{g^{-1}}\omega$
- $\varphi\colon G \times M \rightarrow M,\ (g, m) \mapsto (\pi \circ \Phi)(g, p_m)$ where $p_m \in F_m$

Reduction Based on Invariant Connections

Symmetry (G, Φ) on P with S compact

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Choose $\mathcal{P}$

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$$\omega \mapsto [f \mapsto f(\omega)]$$

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Compute G-invariant connections

$$\begin{aligned} \mathcal{A}_G &= \{\omega \in \mathcal{A} \,|\, \Phi_g^* \omega = \omega \text{ for all } g \in G\} \\ &= \{\omega \in \mathcal{A} \,|\, \mathrm{Stab}_\theta(\omega) = G\} \end{aligned}$$

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Compute $\overline{\iota_{\mathcal{A}}(\mathcal{A}_G)}$

$$\overline{\mathcal{A}_G} := \overline{\iota_{\mathcal{A}}(\mathcal{A}_G)} \cong \mathrm{Spec}\left(\overline{\mathfrak{P}|_{\mathcal{A}_G}}\right)$$

$P = \mathbb{R}^3 \times SU(2)$

- $\boldsymbol{\mathcal{P}}$ **:** $\mathcal{P}_\omega$, $\mathcal{P}_\mathrm{l}$ - embedded analytic, linear curves
- **Group:** $G = E := \mathbb{R}^3 \rtimes_\varrho SU(2)$ for $\varrho \colon SU(2) \to SO(3)$ universal covering map
- **Action:** $\Phi((v,\sigma),(x,s)) := (v,\sigma) \cdot_\varrho (x,s) = (v + \varrho(\sigma)(x), \sigma s)$
- **Invariant connections:** $\mathcal{A}_E \cong \mathbb{R}$ (Wang's theorem)

  Pullbacks by section $x \mapsto (x,e)$:

$$\omega^c \colon \vec{v}_x \mapsto c \sum_{k=1}^{3} \vec{v}_x^{\,k} \tau_k \quad \text{for} \quad \vec{v}_x \in T_x\mathbb{R}^3,$$

  where $\tau_k = -\mathrm{i}\sigma_k$ for Pauli matrices $\sigma_1, \sigma_2, \sigma_3$.

Two Cosmological Quantum Configuration Spaces

$\mathfrak{R} := \overline{\mathfrak{P}|_{\mathcal{A}_E}} \longrightarrow \overline{\mathcal{A}_E} \cong \mathrm{Spec}(\mathfrak{R})$ $i\colon \mathbb{R} \cong \mathcal{A}_E \hookrightarrow \mathcal{A}$ inclusion map

	ABL [2003]	Fleischhack [2010]
$\mathcal{P}$	linear curves	embedded analytic curves
$\mathfrak{R}$	$C_{\mathrm{AP}}(\mathbb{R})$	$C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$
$\overline{\mathcal{A}_E}$	$\mathbb{R}_{\mathrm{Bohr}}$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$
Embedding $\bar{i}$ of $\overline{\mathcal{A}_E}$	no extension $\bar{i}\colon \mathbb{R}_{\mathrm{Bohr}} \to \overline{\mathcal{A}}_\omega$	i extends to embedding $\bar{i}\colon \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}} \to \overline{\mathcal{A}}_\omega$
	$\mathcal{H}^\omega_{\mathrm{kin}}$-full theory	$\rho\colon (0,1) \to \mathbb{R}$ homeomorphism
Measures	Haar measure μ_{B}	Proj. Measures * $\mu_{\rho,t}$ (MH13)
Hilbert Spaces	$\mathrm{L}^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{B}}) \hookrightarrow \mathcal{H}^\omega_{\mathrm{kin}}$ (Engle)	$\mathrm{L}^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{B}}) + \mathrm{L}^2(\mathbb{R}, \lambda)$ $\mathrm{L}^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\rho,t})$ for $t \in (0,1)$

$^*\mu_{\rho,t}(A) = t\,(\rho_*\lambda)(A \cap \mathbb{R}) + (1-t)\,\mu_{\mathrm{B}}(A \cap \mathbb{R}_{\mathrm{Bohr}}) \quad A \in \mathfrak{B}(\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}})$

Calculation of Invariant Connections $(P = \mathbb{R}^3 \times SU(2))$

Classical results:

Wang's theorem: (φ transitive on M)
$\Rightarrow$ p admits bijection between $\mathcal{A}_G$ and linear maps $\mathfrak{g} \to \mathfrak{s}$ fulfilling two conditions concerning the φ-stabilizer of $\pi(p)$.

- Homog. Isotropic LQC: $\mathcal{A}_E \cong \mathbb{R}$
- Homog. LQC (no stab.): $\mathcal{A}_G \cong \{L\colon \mathfrak{g} \to su(2) \mid L \text{ linear}\} \cong \mathbb{R}^{3\cdot\dim[\mathfrak{g}]}$

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Harnad, Shnider, Vinet: (special submfd. P' of P – $G_{\pi(p')}$ same $\forall p' \in P'$)
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Problems: (usually ignored)

Theorems do not work for different orbit types:

- Isotropic case: $G = I := SU(2)$ with $\Phi(\sigma, (x, s)) = (\varrho(\sigma)(x), \sigma s)$
- Scale Invariance: $G = \Lambda := \mathbb{R}_{>0}$ with $\Phi(\lambda, (x, s)) = (\lambda \cdot x, s)$

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Idea: Replace p, P' by Φ-covering of P, i.e. collection $\{P_\alpha\}_{\alpha \in I}$ of submanifolds of P such that:

- Each φ-orbit intersects $\bigcup_{\alpha \in I} \pi(P_\alpha)$.
- $T_pP = T_pP_\alpha + \mathrm{d}_e\Phi_p(\mathfrak{g}) + Tv_pP$ for $p \in P_\alpha$, $\alpha \in I$ – P_α is patch

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Theorem (*MH: math-ph arXiv:1310.0318v1*)

Φ-covering $\{P_\alpha\}_{\alpha \in I}$ admits bijection between $\mathcal{A}_G$ and families $\{\psi_\alpha\}_{\alpha \in I}$ of smooth maps $\psi_\alpha \colon \mathfrak{g} \times TP_\alpha \to \mathfrak{s}$ such that $\psi_\alpha|_{\mathfrak{g}\times T_{p_\alpha}P_\alpha}$ is linear for all $p_\alpha \in P_\alpha$ and (Generalized Wang conditions)

1. $\widetilde{g}(p_\beta) + \vec{w}_{p_\beta} - \widetilde{s}(p_\beta) = dL_q\vec{w}_{p_\alpha} \implies \psi_\beta(\vec{g}, \vec{w}_{p_\beta}) - \vec{s} = \rho(q) \circ \psi_\alpha(\vec{w}_{p_\alpha})$
2. $\psi_\beta(\mathrm{Ad}(q)(\vec{g}), \vec{0}_{p_\beta}) = \rho(q) \circ \psi_\alpha(\vec{g}, \vec{0}_{p_\alpha})$.

For $p_\alpha \in P_\alpha$, $p_\beta \in P_\beta$ with $p_\beta = q \cdot p_\alpha$ for $q \in G \times S$ as well as $\vec{w}_{p_\alpha} \in T_{p_\alpha}P_\alpha$, $\vec{w}_{p_\beta} \in T_{p_\beta}P_\beta$, $\vec{g} \in \mathfrak{g}$ and $\vec{s} \in \mathfrak{s}$.

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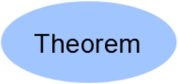
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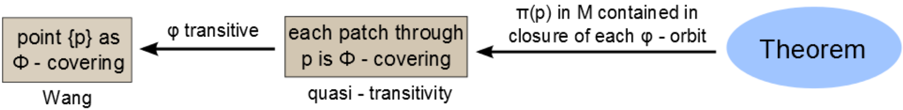
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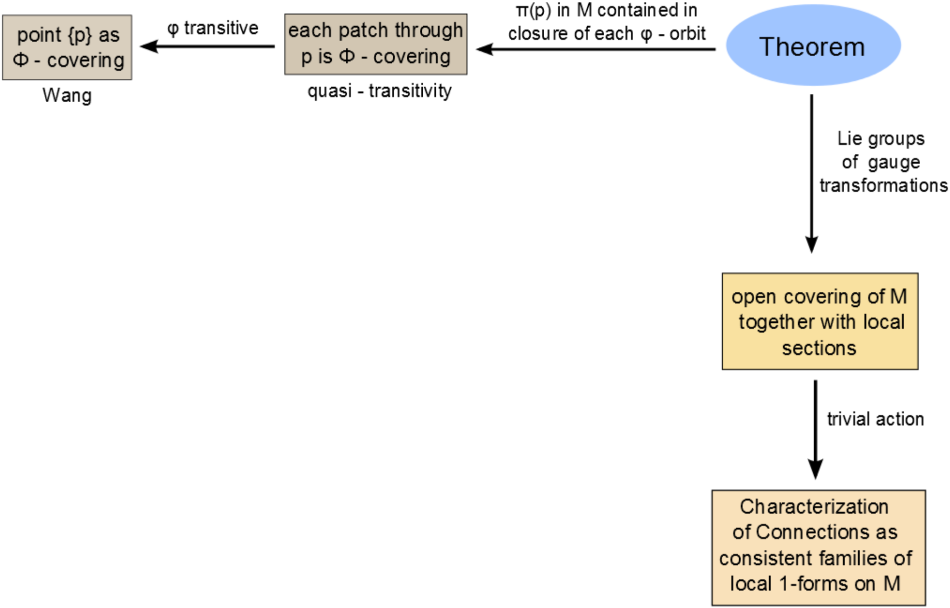
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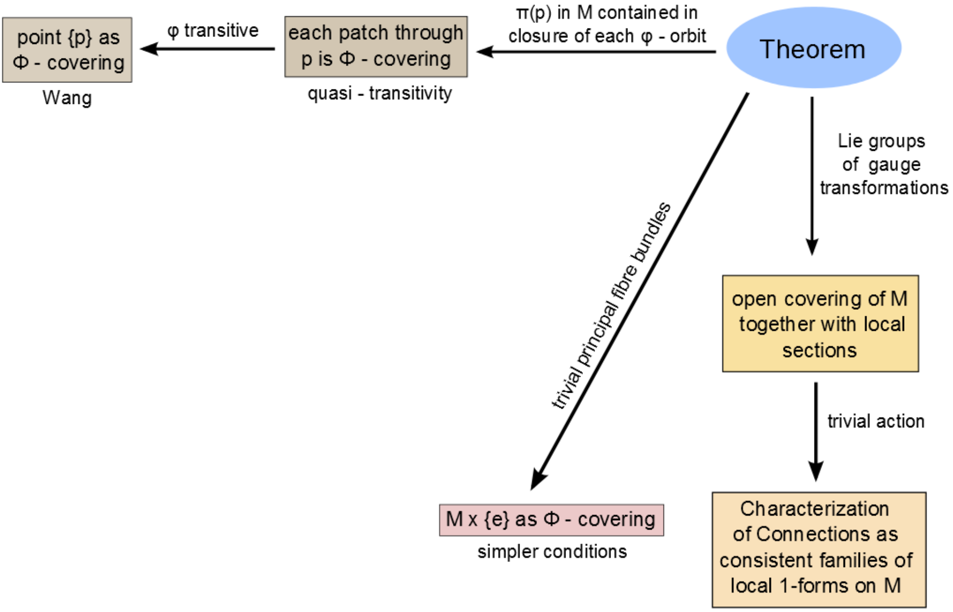
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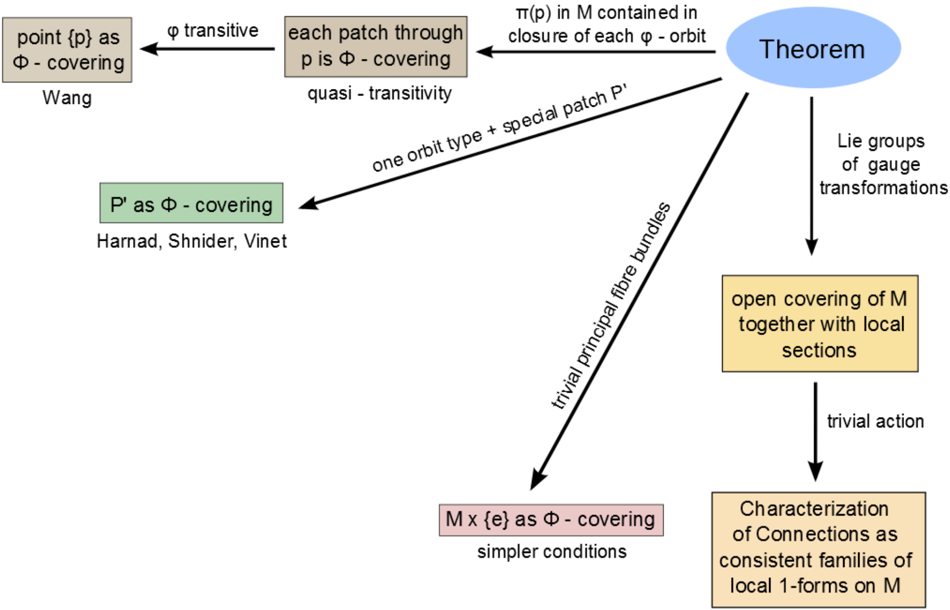
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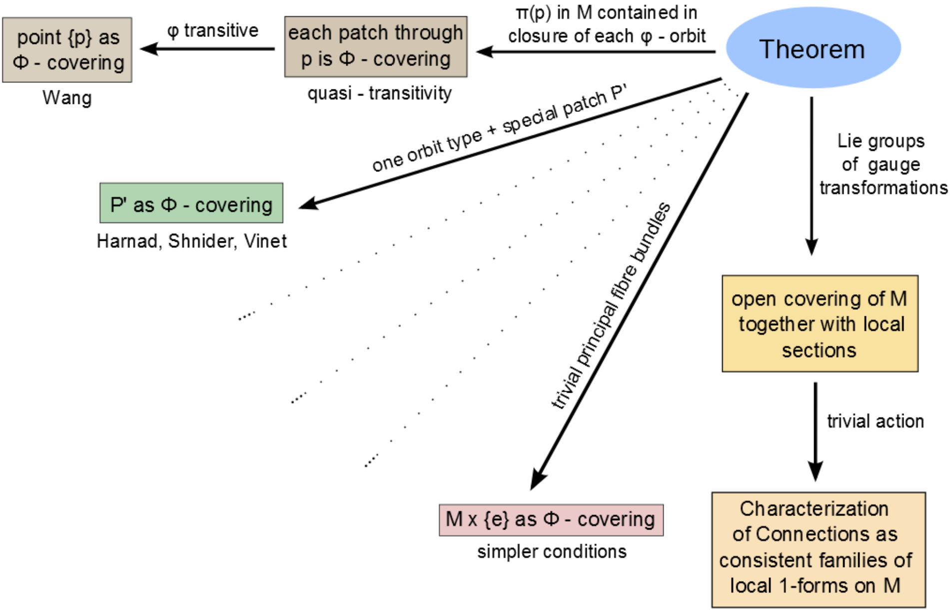
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Non-transitive situations for $P = \mathbb{R}^3 \times SU(2)$

- **Scale invariant connections:** $\mathcal{A}_\Lambda = \{\omega_0\}$ for $\omega_0(\vec{v}_x) = 0$
  For $\mathbb{R}^3\backslash\{0\} \times SU(2)$*:* $\mathcal{A}_\Lambda$ equ. smooth lin. maps $\psi\colon \mathbb{R} \times TS^2 \to su(2)$
- **Isotropic connections $\mathcal{A}_I$:** (pullback by $x \mapsto (x, e)$)

$$\omega^{abc}(\vec{v}_x) := a\big(\|x\|^2\big)\mu(\vec{v}_x) + b\big(\|x\|^2\big)[\mu(x), \mu(\vec{v}_x)] + c\big(\|x\|^2\big)[\mu(x), [\mu(x), \mu(\vec{v}_x)]]$$

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Symmetry	$\mathcal{A}_G$	$\overline{\mathcal{A}_G}$
homogeneous	$\cong \mathbb{R}^9$	$\mathrm{Spec}(\mathfrak{R})$ (hard)
semi-homogeneous	par. by functions	other ?
scale invariance	$\{\omega_0\}$	$\{\omega_0\}$
isotropic	par. by functions	other ?

Alternative Approach – Reduction on Quantum Level

Observation: $\mathcal{P}$ invariant $\Longrightarrow \theta_g^*(\mathfrak{P}) = \mathfrak{P}$ for all $g \in G$ $\quad \theta\colon (g, \omega) \to \Phi_{g^{-1}}^* \omega$

$\mathcal{P}$ invariant $\Longleftrightarrow \varphi_g \circ \gamma \in \mathcal{P} \;\; \forall\, g \in G, \gamma \in \mathcal{P}$

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Proposition (Extension of Group Actions; *arXiv:1307.5303v1*)

$\theta\colon G \times X \to X$ action, $\mathfrak{A} \subseteq B(X)$ C^*-algebra with $\theta_g^*(\mathfrak{A}) = \mathfrak{A} \;\; \forall\, g \in G$

$\Longrightarrow \Theta\colon G \times \mathrm{Spec}(\mathfrak{A}) \to \mathrm{Spec}(\mathfrak{A})$, $(g, \overline{x}) \mapsto \overline{x} \circ \theta_g^*$ unique action with:

- *Θ_g continuous for all $g \in G$,*
- *Θ extends θ: $\quad \Theta_g \circ \iota_X = \iota_X \circ \theta_g$ for all $g \in G$*

G topological:

- *Θ continuous if $\theta_\bullet^* f : G \to \mathfrak{A}$, $g \mapsto \theta_g^* f$ continuous for all $f \in \mathfrak{A}$.*
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Then $\overline{X}_G := \{\overline{x} \in \overline{X} \mid \mathrm{Stab}_\Theta(\overline{x}) = G\}$ closed in $\overline{X}$ and $\overline{X_G} \subseteq \overline{X}_G$.

- $\overline{X} = \mathrm{Spec}(\mathfrak{A})$ $\qquad \iota_X\colon X \to \mathrm{Spec}(\mathfrak{A})$, $x \mapsto [f \mapsto f(x)]$
- $X_G := \{x \in X \mid \mathrm{Stab}_\theta(x) = G\}$ $\qquad \overline{X_G} := \overline{\iota_X(X_G)} \subseteq \overline{X}$

(P, π, M, S) with S compact

Symmetry (G, Φ) on P

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(P, π, M, S) with S compact

MH: gr-qc arXiv:1307.5303v1

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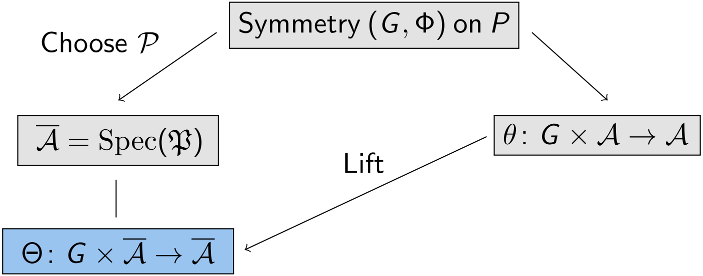

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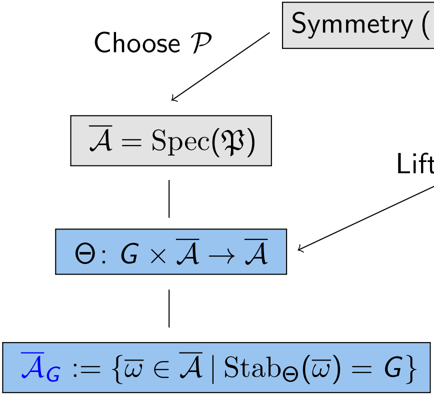

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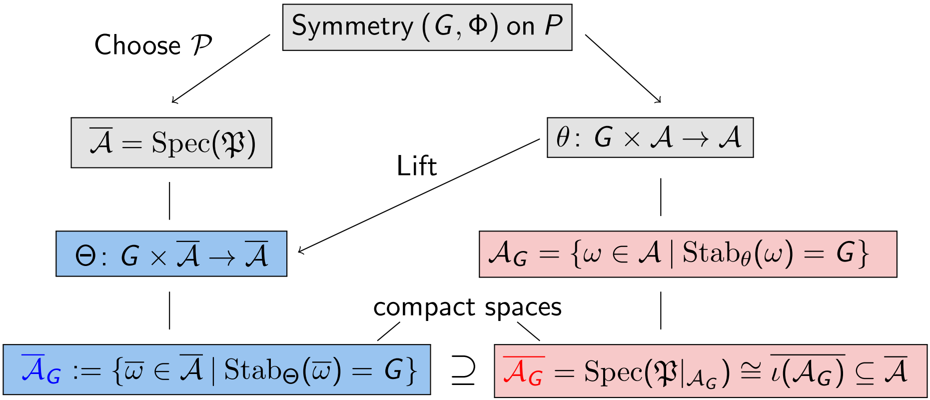

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compact spaces

$\overline{\mathcal{A}}_G := \{\overline{\omega} \in \overline{\mathcal{A}} \mid \mathrm{Stab}_\Theta(\overline{\omega}) = G\} \supseteq \overline{\mathcal{A}_G} = \mathrm{Spec}(\mathfrak{P}|_{\mathcal{A}_G}) \cong \overline{\iota(\mathcal{A}_G)} \subseteq \overline{\mathcal{A}}$

- Lift Θ exists if $\mathcal{P}$ invariant, i.e., $\varphi_g \circ \gamma \in \mathcal{P}$ for all $\gamma \in \mathcal{P}$, $g \in G$.
- Usually fulfilled in LQG where $P = \mathbb{R}^3 \times SU(2)$

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- Have bijection $\kappa\colon \overline{\mathcal{A}} \to \mathrm{Hom}(\mathcal{P}, SU(2))$ $\qquad (\mathcal{P} = \mathcal{P}_{\mathrm{l}}, \mathcal{P}_\omega)$
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- Characterized by simple algebraic relations: $\chi \in \mathrm{Hom}_G(\mathcal{P}, SU(2))$ iff

$$\begin{array}{lrcll} \mathrm{Euclidean:} & \chi(\vec{v} + \varrho(\sigma)(\gamma)) & = & \alpha_\sigma \circ \chi(\gamma) & \forall\, (\vec{v}, \sigma) \in E,\ \gamma \in \mathcal{P}, \\ \mathrm{Rotations:} & \chi(\varrho(\sigma)(\gamma)) & = & \alpha_\sigma \circ \chi(\gamma) & \forall\, \sigma \in SU(2),\ \gamma \in \mathcal{P}, \\ V \subseteq \mathbb{R}^3: & \chi(\vec{v} + \gamma) & = & \chi(\gamma) & \forall\, \vec{v} \in V,\ \gamma \in \mathcal{P}. \end{array}$$

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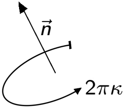

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New elements e.g.:
$\chi(\gamma_c) = \exp\left(2\pi\kappa\langle\vec{n}, \vec{\tau}\rangle\right)$
$\chi(\gamma) = e$ if γ not circular

Ongoing and Future Work

Quantization vs. Reduction:

- General conditions on G and $\mathcal{P}$ showing that $\overline{\mathcal{A}_G} \subsetneq \overline{\mathcal{A}}_G$.

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Future Work:

- Embedding of the resulting Hilbert spaces of square integrable functions into symmetric sector of LQG.
- Reduction of the holonomy-flux algebra and definition of representations on the respective reduced Hilbert spaces.

Conclusions

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Reduction on Quantum Level

- Invariant $\mathcal{P}$ allows for symmetry reduction on quantum level.
- Reduced space embedded in full theory since subset
- $\mathcal{C}_{\text{quant,red}} \leq \mathcal{C}_{\text{red,quant}}$
- Identification $\overline{\mathcal{A}} \cong \mathrm{Hom}(\mathcal{P}, SU(2))$ allows to determine $\mathcal{C}_{\text{red,quant}}$ by simple algebraic relations $\Longrightarrow \mu_{\text{AL}}(\mathcal{C}_{\text{red,quant}}) = 0$
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- Characterization theorem for invariant connections for traditional reduction concept.

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Invariant Connections

- Characterization theorem for invariant connections for traditional reduction concept.

$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$ **as projective limit** $\Longrightarrow$ **Radon measures** $\mu_{\rho,\mathbf{t}}$

- Similarly, for $\mathcal{C}_{\mathrm{red,quant}}$ if φ analytic and proper.
- In particular, for $\overline{\mathcal{A}}_E \supseteq \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$. (compare)

Thank you for your attention !