Symmetry Actions and Invariance Conditions in LQG

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soon to join at Florida Atlantic University

LQG - Configuration Space

- GR as Hamiltonian theory:

$\longrightarrow$	spatial metric	$+$	extrinsic curvature
$\longrightarrow$	$\mathrm{SU}(2)$-connection	$+$	Dreibein

$\mathcal{A}$	$\mathrm{SU}(2)$-connections

LQG - Configuration Space

- GR as Hamiltonian theory:

  $\longrightarrow$ spatial metric $+$ extrinsic curvature

  $\longrightarrow$ $\mathrm{SU}(2)$-connection $+$ Dreibein
- Quantization $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$: $L^2(\overline{\mathcal{A}}, \mu_{\mathrm{AL}})$

$\mathcal{A}$	$\mathrm{SU}(2)$-connections
$\overline{\mathcal{A}}$	generalized connections

Gelfand compactification:

- $\overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{C})$
- $\mathfrak{C} =$ C^*-algebra of cylindrical functions on $\mathcal{A}$ $\qquad \mathfrak{C} \subseteq B(\mathcal{A})$

  generated by $f \circ h_\gamma$ for $\gamma \in \mathcal{P}$ and $f \in C(\mathrm{SU}(2))$
- $\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$ densly embedded if $\mathfrak{C}$ separates points: $\qquad \iota \colon A \mapsto [g \mapsto g(A)]$

Reduction of $\overline{\mathcal{A}}$ (P, π, M, S)

Symmetry: Lie group (G, Φ) in $\mathrm{Aut}(P)$

Classical Reduction

Reduction of $\overline{\mathcal{A}}$ $\quad (P, \pi, M, S)$

Symmetry: Lie group (G, Φ) in $\mathrm{Aut}(P)$

$$\mathcal{A} \xrightarrow{\text{red}} \mathcal{A}_{\mathrm{red}}$$

Classical Reduction

- $\mathcal{A}_{\mathrm{red}} := \{A \in \mathcal{A} \mid \Phi_g^* A = A \ \forall\, g \in G\}$

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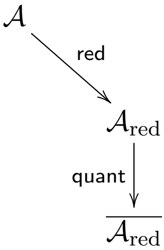

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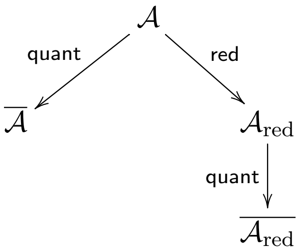

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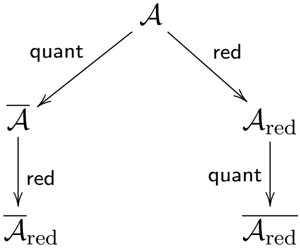

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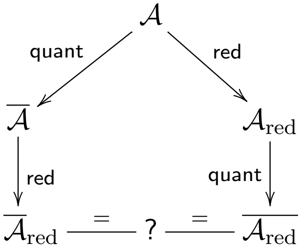

quant. vs. red.

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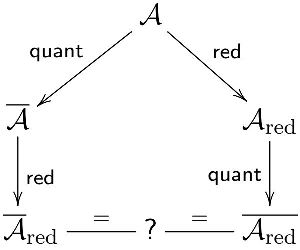

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$$\mathcal{A}_{\mathrm{red}} = \{A \in \mathcal{A} \mid \mathrm{Stab}_\theta(A) = G\}$$

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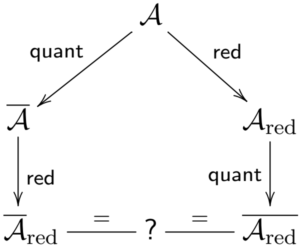

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$\Phi_g \in \mathrm{Aut}(P) \qquad \forall\, g \in G$

- $\theta\colon G \times \mathcal{A} \to \mathcal{A}, \quad (g, A) \mapsto \Phi_{g^{-1}}^* A$

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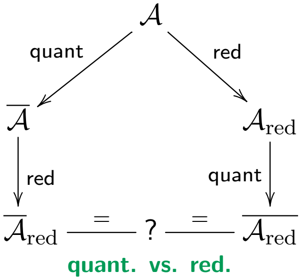

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Idea: Extend θ to $\overline{\mathcal{A}}$ and reduce on quantum level:

$$\overline{\mathcal{A}}_{\mathrm{red}} = \{\, \overline{A} \in \overline{\mathcal{A}} \mid \mathrm{Stab}_\Theta(\overline{A}) = G\}$$

Reduction of $\overline{\mathcal{A}}$

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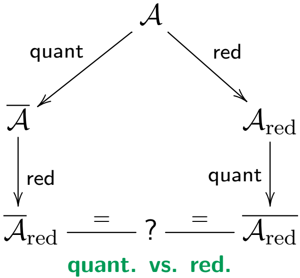

$\Phi_g \in \mathrm{Aut}(P) \qquad \forall\, g \in G$

- $\varphi \colon G \times M \to M, \quad (\pi \circ \Phi)(g, \boldsymbol{p}(m))$

Extension of group actions – Quantum reduction

Observation: $\theta_g^*(\mathfrak{C}) \subseteq \mathfrak{C}$ if $\varphi_g \circ \gamma \in \mathcal{P}$ $\forall\, g \in G,\, \gamma \in \mathcal{P}$

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$\theta_g^*(\mathfrak{C}) \subseteq \mathfrak{C} \implies \exists\, \Theta\colon G \times \overline{X} \to \overline{X}$ unique with: [MH13]

- Θ_g continuous $\forall g \in G$
- $\Theta_g \circ \iota = \iota \circ \theta_g$ $\forall g \in G$

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\Theta_g} & \overline{X} \\ \uparrow \iota & & \uparrow \iota \\ X & \xrightarrow{\theta_g} & X \end{array}$$

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$$\overline{\iota(X_{\mathrm{red}})} \subseteq \overline{X}_{\mathrm{red}}$$

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$\overline{\iota(X_{\mathrm{red}})} \subseteq \overline{X}_{\mathrm{red}}$ $\mathfrak{C}_{\mathbf{red}} := \overline{\mathfrak{C}|_{X_{\mathrm{red}}}}$

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$\overset{\text{unital}}{\Longrightarrow}$ $\overline{X_{\mathrm{red}}} := \mathrm{Spec}(\mathfrak{C}_{\mathbf{red}}) \cong \overline{\iota(X_{\mathrm{red}})} \subseteq \overline{X}_{\mathrm{red}} =$ compact

Quantum Reduction (P, π, M, S)

Symmetry (G, Φ) on P

$\searrow$

$\theta\colon G \times \mathcal{A} \to \mathcal{A}$

Quantum Reduction (P, π, M, S)

Symmetry (G, Φ) on P

choose $\mathcal{P}$

$\overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{C})$

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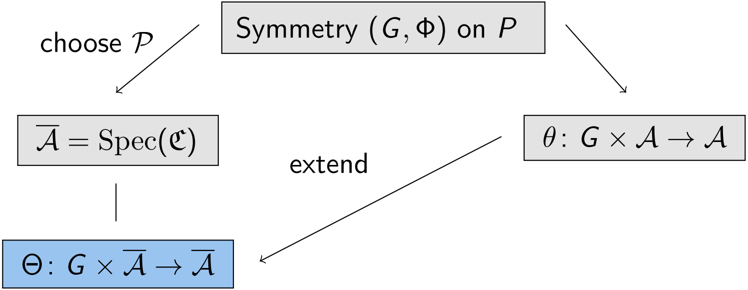
Quantum Reduction (P, π, M, S)



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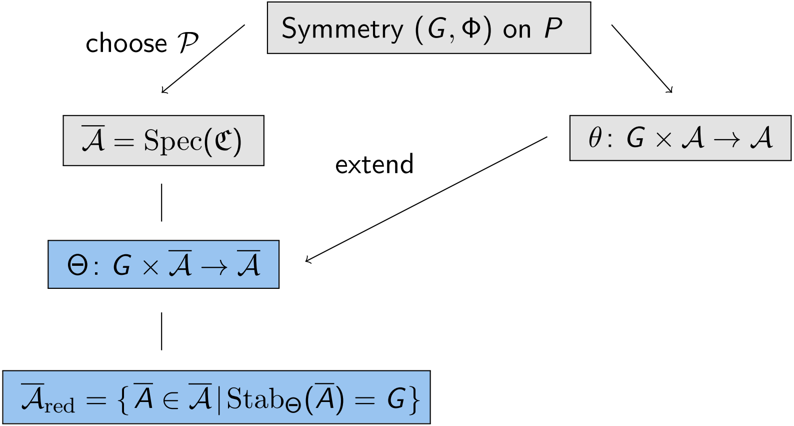

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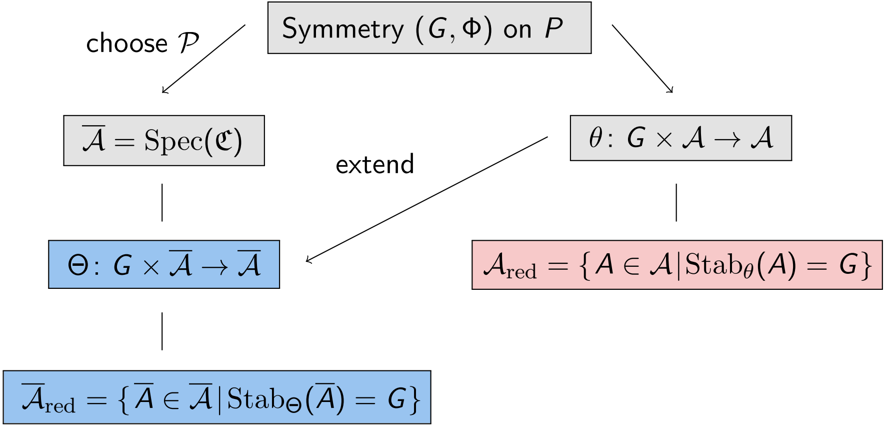

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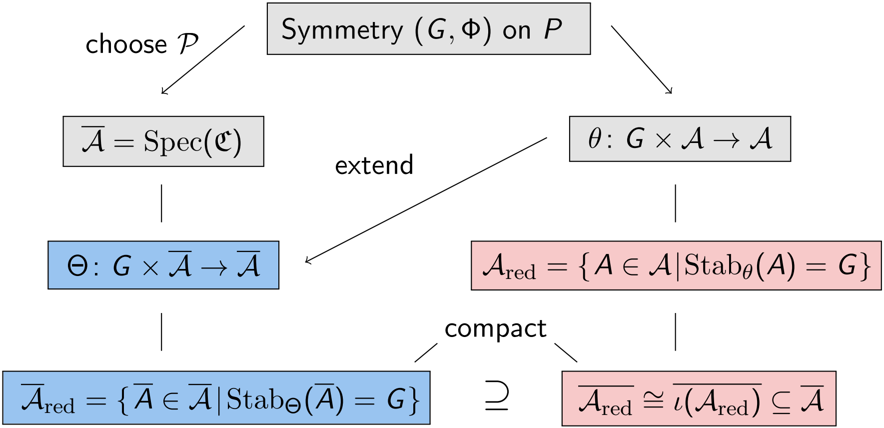

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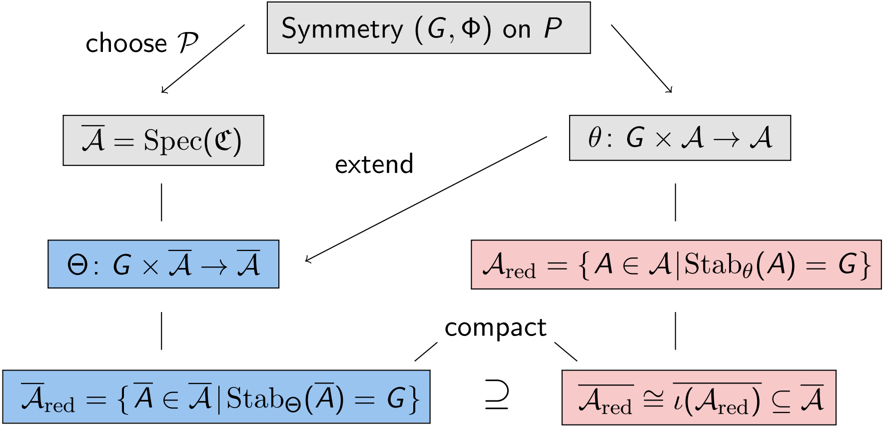

- Θ exists if $\mathcal{P}$ invariant: $\varphi_g \circ \gamma \in \mathcal{P}$ for all $\gamma \in \mathcal{P}$, $g \in G$
- usually fulfilled as φ and $\mathcal{P}$ smooth or analytic

Quantization vs. Reduction

Construct $\overline{A} \in \overline{\mathcal{A}}_{\mathrm{red}} \backslash \overline{\mathcal{A}_{\mathrm{red}}}$ explicitly:

- $\overline{\mathcal{A}} \cong \mathrm{Hom}(\mathcal{P}, S) \qquad \longrightarrow \qquad \overline{\mathcal{A}}_{\mathrm{red}} \cong$ invariant homomorphisms
- Modify $\overline{A}$ along some γ: $\qquad \overline{A}'(\gamma) \notin \overline{\mathcal{A}_{\mathrm{red}}}(\gamma) = \{\overline{A}(\gamma) \mid \overline{A} \in \overline{\mathcal{A}_{\mathrm{red}}}\}$

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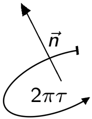
Loop Quantum Cosmology: $P = \mathbb{R}^3 \times \mathrm{SU}(2)$ (MH13/14)

LQC	linear curves	analytic curves
(semi-)homogeneous	$\overline{\mathcal{A}}_{\mathrm{red}} \supsetneq \overline{\mathcal{A}_{\mathrm{red}}}$	$\overline{\mathcal{A}}_{\mathrm{red}} \supsetneq \overline{\mathcal{A}_{\mathrm{red}}}$
homogeneous isotropic	$\overline{\mathcal{A}}_{\mathrm{red}} = \overline{\mathcal{A}_{\mathrm{red}}}$	$\overline{\mathcal{A}}_{\mathrm{red}} \supsetneq \overline{\mathcal{A}_{\mathrm{red}}}$

$$\overline{A}(\gamma) = \begin{cases} \exp(2\pi\tau f(r) \cdot \langle \vec{n}, \vec{\sigma} \rangle) & \text{if } \gamma \text{ circular} \\ \mathbb{1} & \text{else} \end{cases}$$



Invariant Measures

Homogeneous Isotropic LQC:

Φ – euclidean group $\qquad \mathcal{A}_{\mathrm{red}} \cong \mathbb{R} \qquad P = \mathbb{R}^3 \times \mathrm{SU}(2)$

	curves	$\overline{\mathcal{A}_{\mathrm{red}}}$
ABL:	linear	$\mathbb{R}_{\mathrm{Bohr}}$
F:	analytic	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

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Single out $L^2(\mathbb{R}_{\mathbf{Bohr}}, \mu_{\mathbf{Bohr}})$:

$\mathcal{A}_{\text{red}} \cong \mathbb{R} \qquad P = \mathbb{R}^3 \times \text{SU}(2)$

Action:

$+ : \mathbb{R} \times \mathcal{A}_{\text{red}} \to \mathcal{A}_{\text{red}}$

$\mathbf{+} : \mathbb{R} \times \overline{\mathcal{A}_{\text{red}}} \to \overline{\mathcal{A}_{\text{red}}}$

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μ normalized Radon measure $\Longrightarrow$ on $L^2(\overline{\mathcal{A}_{\mathrm{red}}}, \mu)$ [MH14]

$\boldsymbol{+}_t^* \colon \phi \mapsto \phi(t \boldsymbol{+} \cdot)$ unitary $\forall\, t \in \mathbb{R} \iff \mu = \mu_{\mathrm{Bohr}}$

$\Longrightarrow \quad L^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$

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$\Longrightarrow$ $L^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$

ABL: $\boldsymbol{+}_t^*$ exponentiated reduced fluxes on $L^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$

$\{\boldsymbol{+}_t^*\}_{t \in \mathbb{R}}$ strongly continuous 1-parameter group of unitaries

Measures on $\overline{\mathcal{A}}_{\mathrm{red}}$ S compact + connected

- $\mathcal{P}$ = embedded analytic curves (M analytic)
- φ analytic and pointwise proper (φ_m proper)

Measures on $\overline{\mathcal{A}}_{\mathrm{red}}$ $\qquad$ S compact + connected

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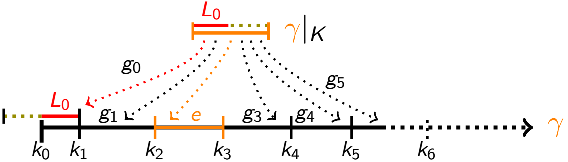
$$\boxed{\mathcal{P}_{\mathrm{cont}}} \longleftarrow \boxed{\mathcal{P}} \longrightarrow \boxed{\mathcal{P}_{\mathrm{disc}}}$$



Measures on $\overline{\mathcal{A}}_{\mathrm{red}}$ — S compact + connected

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$\mathcal{P}_{\mathrm{cont}}$ ⟵ $\mathcal{P}$ ⟶ $\mathcal{P}_{\mathrm{disc}}$

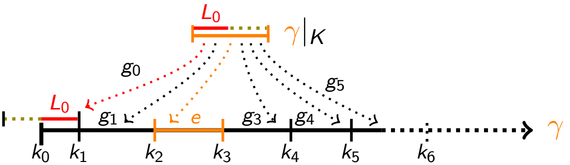
φ trans./proper ↕ normal stabilizers

$\mathcal{P}_{\mathfrak{g}}$ $\quad t \mapsto \varphi_m(\exp(t\vec{g}))$

$\mathcal{P}_{\mathrm{disc}}$ ⟶ $\mathcal{P}_{\mathrm{disc,free}}$, $\mathcal{P}_{\mathrm{disc,stab}}$

$$\Longrightarrow \qquad \overline{\mathcal{A}}_{\mathrm{red}} \cong \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}}$$

(Radon product measure)



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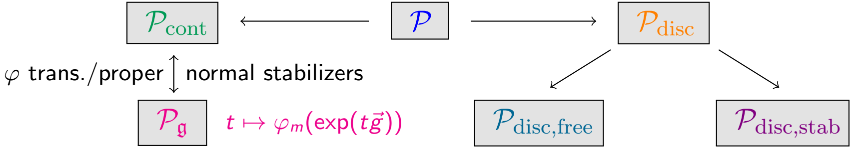

measure for $\mathrm{SU}(2)$, S^1

stable elemente in $\mathfrak{g}$ needed

generalized A-L measure

measure in special situations

$$\Longrightarrow \qquad \overline{\mathcal{A}}_{\mathrm{red}} \cong \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}}$$ (Radon product measure)

(Semi-)homogeneous LQC: $\quad$ (φ proper and free, $\mathrm{SU}(2)$)

$$\overline{\mathcal{A}}_{\mathrm{red}} \cong \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^2\right]^{|\mathrm{P}\mathfrak{g} \times M/G|} \times \overline{\mathcal{A}}_{\mathrm{red}}$$

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$\mathcal{P}_{\mathrm{cont}}$: φ trans./proper $\updownarrow$ normal stabilizers $\mathcal{P}_{\mathfrak{g}}$

$\mathcal{P}_{\mathrm{disc}} \to \mathcal{P}_{\mathrm{disc,free}}$, $\mathcal{P}_{\mathrm{disc}} \to \mathcal{P}_{\mathrm{disc,stab}}$

$\mathcal{P}_{\mathfrak{g}}$: $t \mapsto \varphi_m(\exp(t\vec{g}))$

measure for $\mathrm{SU}(2)$, S^1

stable elemente in $\mathfrak{g}$ needed

$\mathcal{P}_{\mathrm{disc,free}}$: generalized A-L measure

$\mathcal{P}_{\mathrm{disc,stab}}$: measure in special situations

$\Longrightarrow \quad \overline{\mathcal{A}}_{\mathrm{red}} \cong \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}} \times \overline{\mathcal{A}}_{\mathrm{red}}$ (Radon product measure)

Homogeneous Isotropic LQC: $\mathcal{P}_{\mathfrak{g}} \overset{?}{=} \mathcal{P}_{\mathrm{cont}}$

$$\overline{\mathcal{A}}_{\mathrm{red}} \cong \mathbb{R}_{\mathrm{Bohr}} \times \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^1\right]^{|\mathbb{R}\times\mathbb{R}_{>0}|}$$

Invariance up to Gauge

$$\mathrm{Hom}(\mathcal{P}, \mathrm{Iso}_F) \cong \overline{\mathcal{A}}$$

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- $\mathcal{G} := \{\sigma \colon P \to P \mid \pi \circ \sigma = \pi,\ \sigma(p \cdot s) = \sigma(p) \cdot s\}$
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- seems easier to construct measures on this space

Conclusions

Homogeneous Isotropic LQC:

- μ_{Bohr} unique normalized Radon measure on $\mathbb{R}_{\mathrm{Bohr}}$ and $\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$ for which $+_t^*$ unitary on respective Hilbert space.
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Reduction on Quantum Level:

- Usually gives more than quantization of reduced classical space.
  $\longrightarrow$ (Semi-)homogeneous + homogeneous isotropic LQC
- Splitting up $\mathcal{P}$ allows to factorize quantum-reduced space.
  $\longrightarrow$ define measure on each factor separately
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Outlook:

- uniqueness of measures + measures on $\mathrm{Hom}_{\mathrm{red},\mathcal{G}}(\mathcal{P}, \mathrm{Iso}_F)$
- embedding of states + dynamics

Thank you for your attention!