Uniqueness of the Fock Quantization and Signature Change in Cosmology

Guillermo A. Mena Marugán

IEM-CSIC (Laura Castelló Gomar)

Tux 12 February 2014

Ambiguities in QFT

Ambiguities in QFT

- The quantization of a classical system is not univocally defined. Even in linear field theory, one finds **infinitely many** Fock quantizations.

- For a Klein-Gordon scalar field in Minkowski spacetime, there exists esentially only ONE quantization with Poincaré invariant vacuum.

- For STATIONARY spacetimes, one can select one quantization with certain requirements on the energy.

- For more general cases, one loses symmetry. Recently, **UNIQUENESS** has been reached in some nonstationary scenarios by appealing to the unitarity of the dynamics, rather than to invariance.

Uniqueness criteria for the Fock description

1) **INVARIANCE** under the spatial symmetries of the field equations.

2) **UNITARY** implementability of the **DYNAMICS** in a finite time interval.

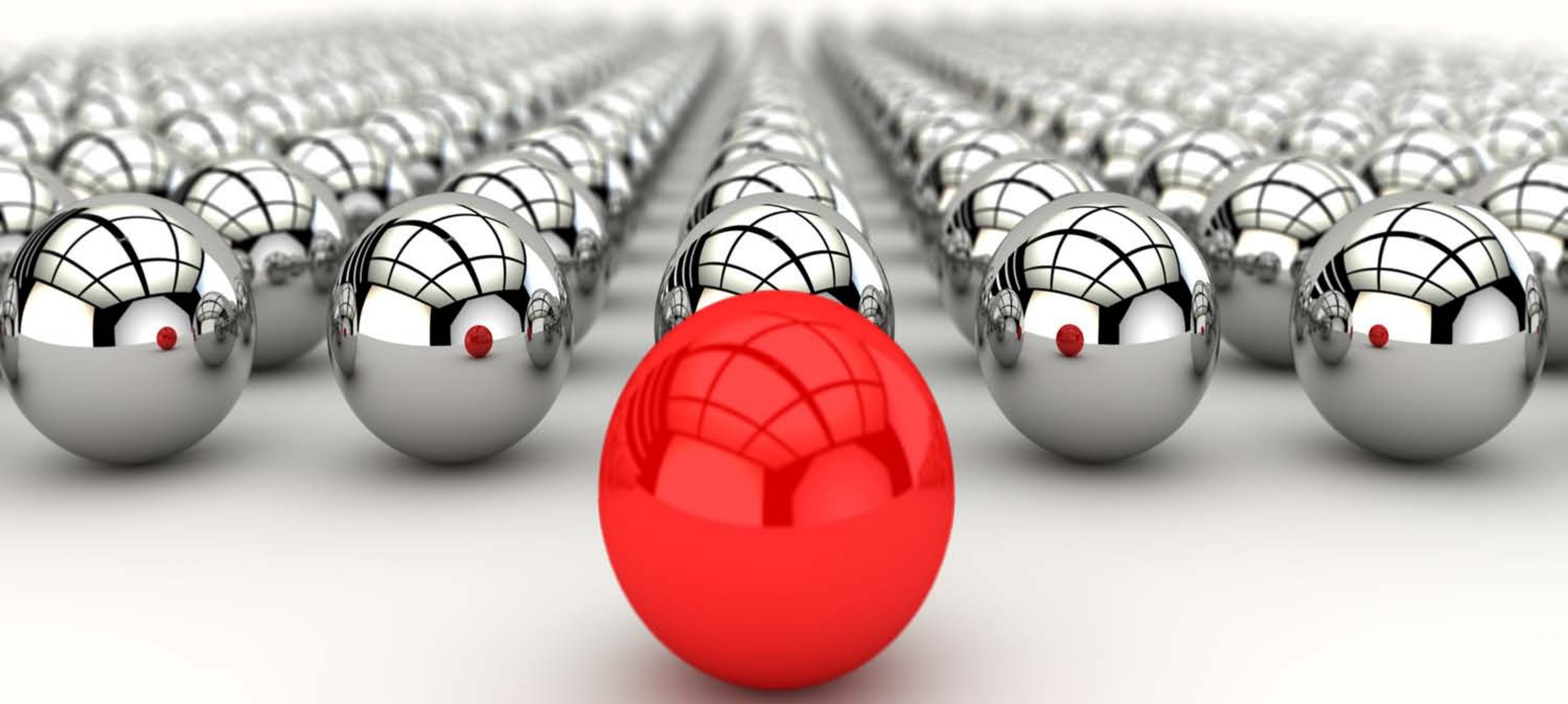
- Klein-Gordon field in ultrastatic spactime with **time-dependent** mass:

$$\ddot{\varphi} - \Delta \varphi + m^2(t)\varphi = 0 .$$

- Our criteria select a a UNIQUE Fock representation for the CCR's, for any (smooth) mass function.

- The uniqueness result is valid for any spatial topology, and at least in any spatial dimension no larger than three.

Uniqueness criteria for the Fock description



Uniqueness criteria for the Fock description

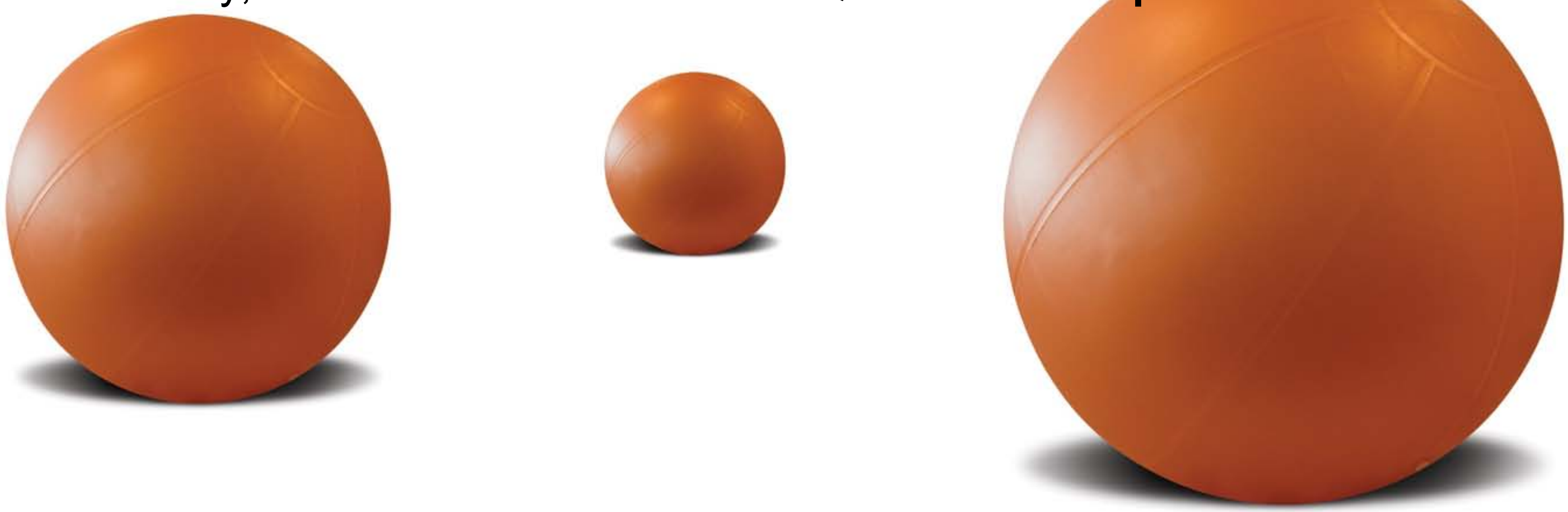
SPATIAL SYMMETRY INVARIANCE and UNITARY DYNAMICS

$$\ddot{\varphi} - \Delta \varphi + m^2(t)\varphi = 0 .$$

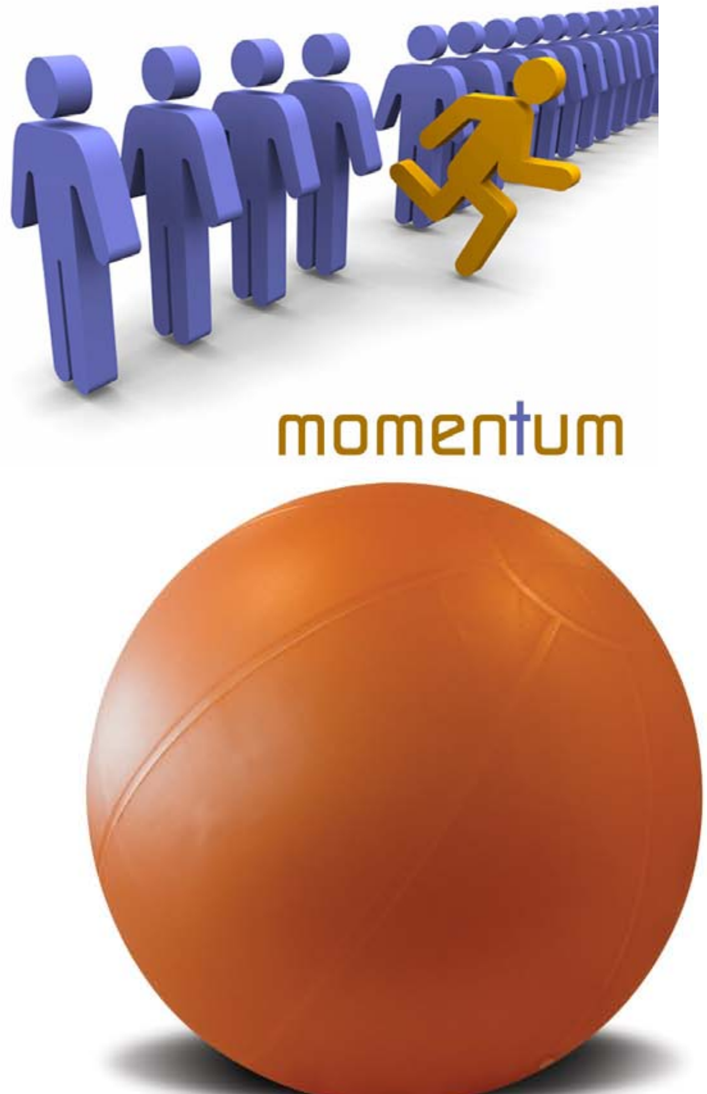
- There is a natural ambiguity in the **separation of the background** from the field. In cosmology, this introduces time-dependent canonical field transformations.

$$\phi = f(t)\varphi , \qquad P_\phi = \frac{1}{f(t)} P_\varphi + g(t)\varphi .$$

- Remarkably, our criteria select also a **UNIQUE canonical pair for the field**.



Uniqueness criteria for the Fock description



Motivation

- We want to generalize the class of field equations for which we can apply our UNIQUENESS results.

- This would allow us to extend the range of applicability of our criteria.

- In this way, we would cover more general situations in cosmology, obtaining robust quantizations.

- In particular, we would like to study situations with “signature change”. This kind of scenarios have received a lot of attention in LQC recently.

Motivation: Signature change

- Has been repeatedly studied in Quantum Cosmology: think e.g. of the ***tunneling from nothing*** or the **no-boundary** proposals.

- It also appears in the context of LQC.

- Quantum modifications may lead to a deformed algebra of constraints.

- The corresponding effective equations may behave as if the fiels propagated in a background with signature change.

Motivation

- Can we deal with field equations that involve processes of signature change?
- What is the **spacetime interpretation** when these processes are present?

- Can be set initial conditions in scenarios with signature change?
- Can this be made compatible with the uniqueness criteria?

Fock quantization with unitary dynamics

- Klein-Gordon real scalar field in ultrastatic spacetime $I \times M$, with I **any time interval** and M **compact**:

$$\ddot{\varphi} - \Delta \varphi + m^2(t)\varphi = 0.$$

- The mass has a *second* derivative, integrable in all compact subintervals.
- P_φ : Canonical field momentum, equal to the densitized time derivative.
- $\{\Psi_{nl}\}$: Modes of the **Laplace-Beltrami operator**, with eigenvalue $-\omega_n^2$.
  l : degeneration index. g_n : degeneration number.
- We expand the field in modes: $\varphi(\vec{x}, t) = \sum q_{nl}(t) \Psi_{nl}(\vec{x})$.

Fock quantization with unitary dynamics

- The modes **decouple** dynamically:

$$\ddot{q}_{nl} + \left[\omega_n^2 + m^2(t)\right] q_{nl} = 0 , \qquad p_{nl} = \dot{q}_{nl} .$$

The dynamics is insensitive to the degeneration.

- We choose the Fock representation selected by the **complex structure** J_0 which is naturally associated to the massless case:

$$a_{nl} = \frac{1}{\sqrt{2\omega_n}} \left(\omega_n q_{nl} + i\, p_{nl} \right) .$$

- J_0 is invariant under the spatial symmetries.

Fock quantization with unitary dynamics

- The evolution is a Bogoliubov transformation of the form:

$$a_{nl}(t)=\alpha_n(t,t_0)a_{nl}(t_0)+\beta_n(t,t_0)a^*_{nl}(t_0).$$

- An asymptotic analysis of the dynamics, proves that beta is of the order

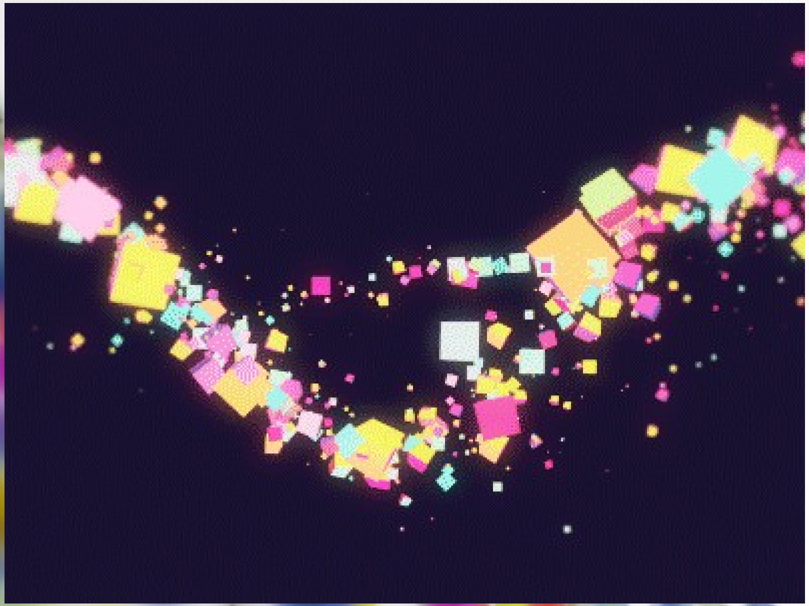
$$\beta_n=O\left(\omega_n^{-2}\right).$$

- The dynamics is unitarily implementable iff $\sum_n g_n|\beta_n(t,t_0)|^2<\infty.$

- Asymptotically, the degeneration is of order $g_n=O\left(\omega_n^{d-1}\right).$

- Therefore, the evolution is implementable as a unitary transformation in three or less spatial dimensions d .

Fock quantization with unitary dynamics



- With similar techniques one can prove the uniqueness of the representation--up to unitary transformations that respect the symmetry invariance-- as well as of the field description.
- Note that the production of particles is finite.

Extensions

- Time-dependent **scalings** of the field: $\phi = f(t)\varphi$.

- We have considered finite dynamical transformations.
- Unitary implementability is valid for any **time reparametrization:**

$$U(t, t_0) \xrightarrow{t(T)}$$

$$\tilde{U}(T, T_0) = U\left[t(T), t(T_0) = t_0\right].$$

$$t'(T) \neq 0, \infty.$$

Generalizations of the field equation

Generalizations of the field equation

- Allowing for time-dependent scalings and time reparametrizations:

$$\ddot{\phi}+c(t)\dot{\phi}+d(t)\Delta\phi+\tilde{m}^2(t)\phi=0,$$

$$\phi=f(t)\varphi \qquad \downarrow \qquad dT=g(t)\,dt, \qquad g(t)\neq 0,$$

$$\varphi''-\Delta\varphi+m^2(T)\varphi=0.$$

- Up to time reversal, there is a **bijective correspondence**:

$$g(t)=s\sqrt{d(t)}, \qquad s=\pm.$$

$$f(t)=c(t)\left[d(t)\right]^{-1/4}\exp\left(-\frac{1}{2}\int^t c\right).$$

Generalizations of the field equation

- We cover all the field equations of generalized Klein-Gordon type with time-dependent coefficients and spatial dependence contained only in the Laplace-Beltrami operator.

$$\ddot{\phi}+c(t)\dot{\phi}+d(t)\Delta\phi+\tilde{m}^2(t)\phi=0.$$

- We find obstructions only **IF** the Laplace-Beltrami coefficient $d(t)$ **vanishes**, and problems if it becomes **negative**.

This result allows us to extend the applicability of our criteria for the uniqueness in the choice of a Fock description.

Generalizations of the field equation

- Recalling that P_φ is the densitized time derivative of the field φ, we conclude that original **field momentum** is:

$$P_\phi = \sqrt{h}\left(A(T)\phi + \frac{1}{f^2(T)}\dot{\phi} \right),$$

$$P_\varphi = f(t) P_\phi - \sqrt{h}\left(f(t) A[T(t)] + \frac{\dot{f}(t)}{f^2(t)} \right)\phi.$$

$A(T)$ is an arbitrary function.

Generalizations of the field equation

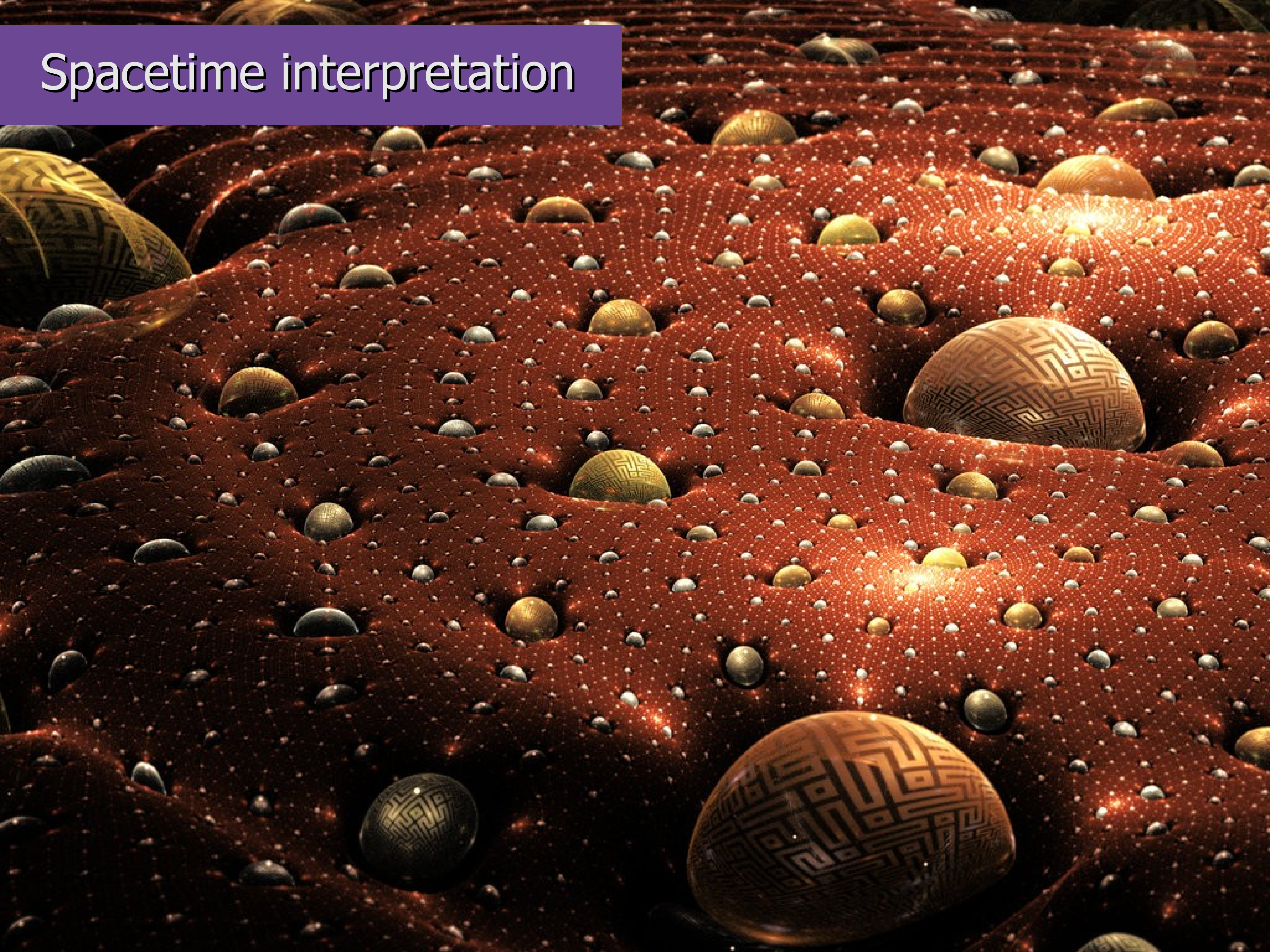
- The relation between the **masses** of the two descriptions is:

$$m^2[T(t)] = \frac{\tilde{m}^2(t)}{d(t)} - \frac{\ddot{d}(t)}{4\mathrm{d}^2(t)} + \frac{5[\dot{d}(t)]^2}{16\mathrm{d}^3(t)} - \frac{\dot{c}(t)}{2\mathrm{d}(t)} - \frac{c^2(t)}{4\mathrm{d}(t)}.$$

- The mass $m(t)$ explodes if $d(t)$ **vanishes**.

- This mass satisfies the conditions for our uniqueness results, e.g., if $\tilde{m}(t)$ does and c and d have a third and a fourth derivative, respectively, integrable in compact intervals.

Spacetime interpretation



Spacetime interpretation

- Let us consider **conformally ultrastatic spacetimes** with metric:

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,h_{ij}(x)\,dx^i\,dx^j .$$

- The considered field equations are the corresponding Klein-Gordon equations (of mass $\bar{m}$) under the **bijective correspondence**:

$$a^4(t) = d(t)\exp\left[\int^t 2c(\tilde{t})\,d\tilde{t}\right],$$

$$N^4(t) = d^3(t)\exp\left[\int^t 2c(\tilde{t})\,d\tilde{t}\right],$$

$$\ddot{\phi} + c(t)\dot{\phi} + d(t)\Delta\phi + \tilde{m}^2(t)\phi = 0 .$$

Here, $\tilde{m}^2 = N^2\bar{m}^2 .$

Spacetime interpretation

$$a^4(t) = d(t)\exp\left[\int^t 2c(\tilde{t})\,d\tilde{t}\right], \qquad N^4(t) = d^3(t)\exp\left[\int^t 2c(\tilde{t})\,d\tilde{t}\right],$$

- With this spacetime interpretation, the right scaling of the field is

$$\phi \propto \frac{\varphi}{a(t)}.$$

- If $d(t)$ **approaches zero**:

  - The **scale factor** and **the lapse** tend to zero.
  - Since $\tilde{m}^2 = N^2 \bar{m}^2$, **the mass** tends to zero as well.
  - The lapse function approaches zero **faster** than the scale factor.

Spacetime interpretation

- The spacetime metric adopts the form:

(D is a constant)

$$ds^2 = \left[-d(t)\,dt^2 + h_{ij}(x)\,dx^i\,dx^j\right] D\sqrt{|d(t)|}\exp\int_{t_d}^{t} c\,.$$

The metric **degenerates completely** when $d(t)$ vanishes.

- From this perspective, vanishing $d(t)$ is more than a signature change. It involves a **singularity** where the scalar curvature explodes as $d^{-7/2}$.

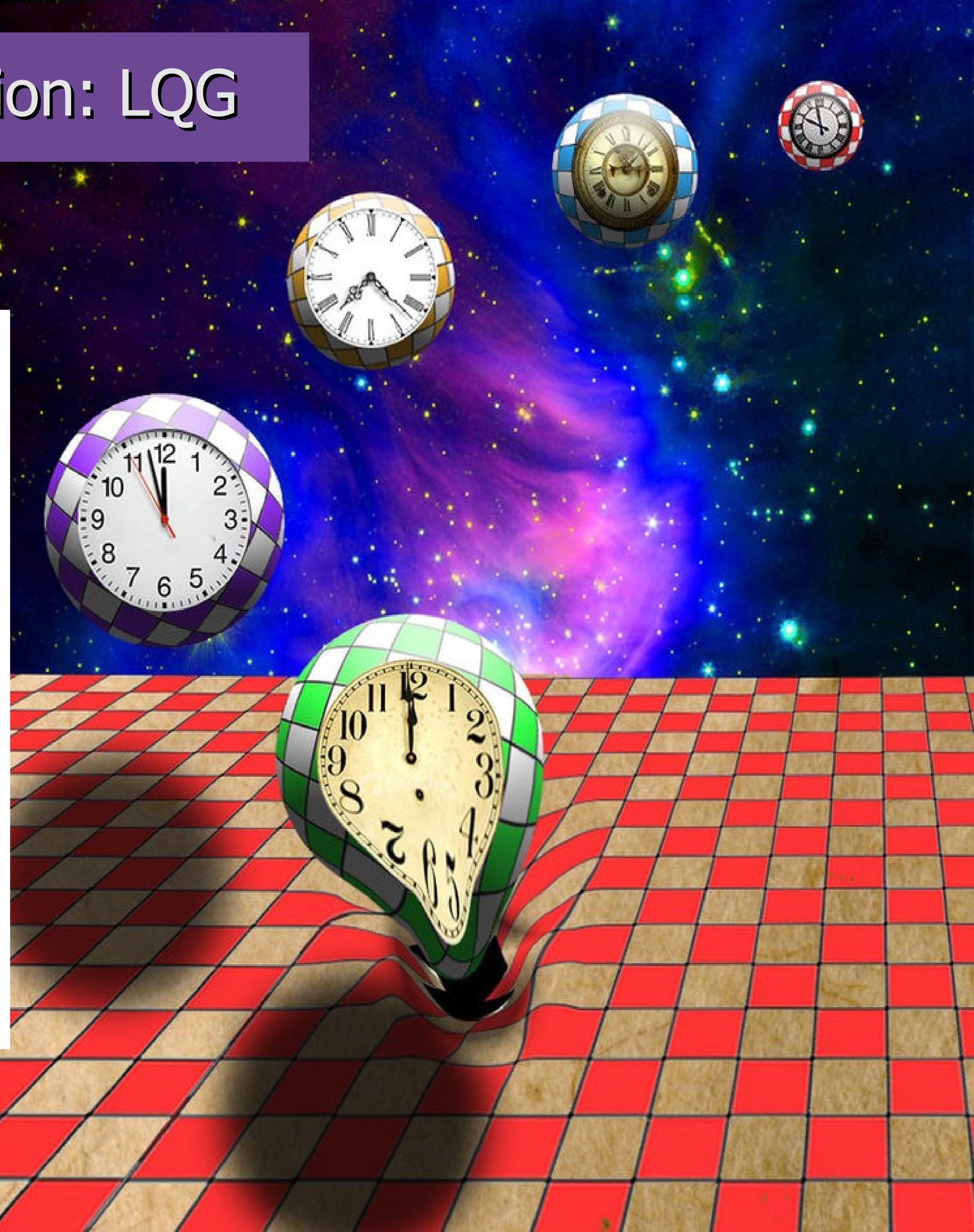
- If we set $d(t_d)=0$, the metric becomes Euclidean in the region where $d(t)$ becomes negative.

Spacetime interpretation: LQG

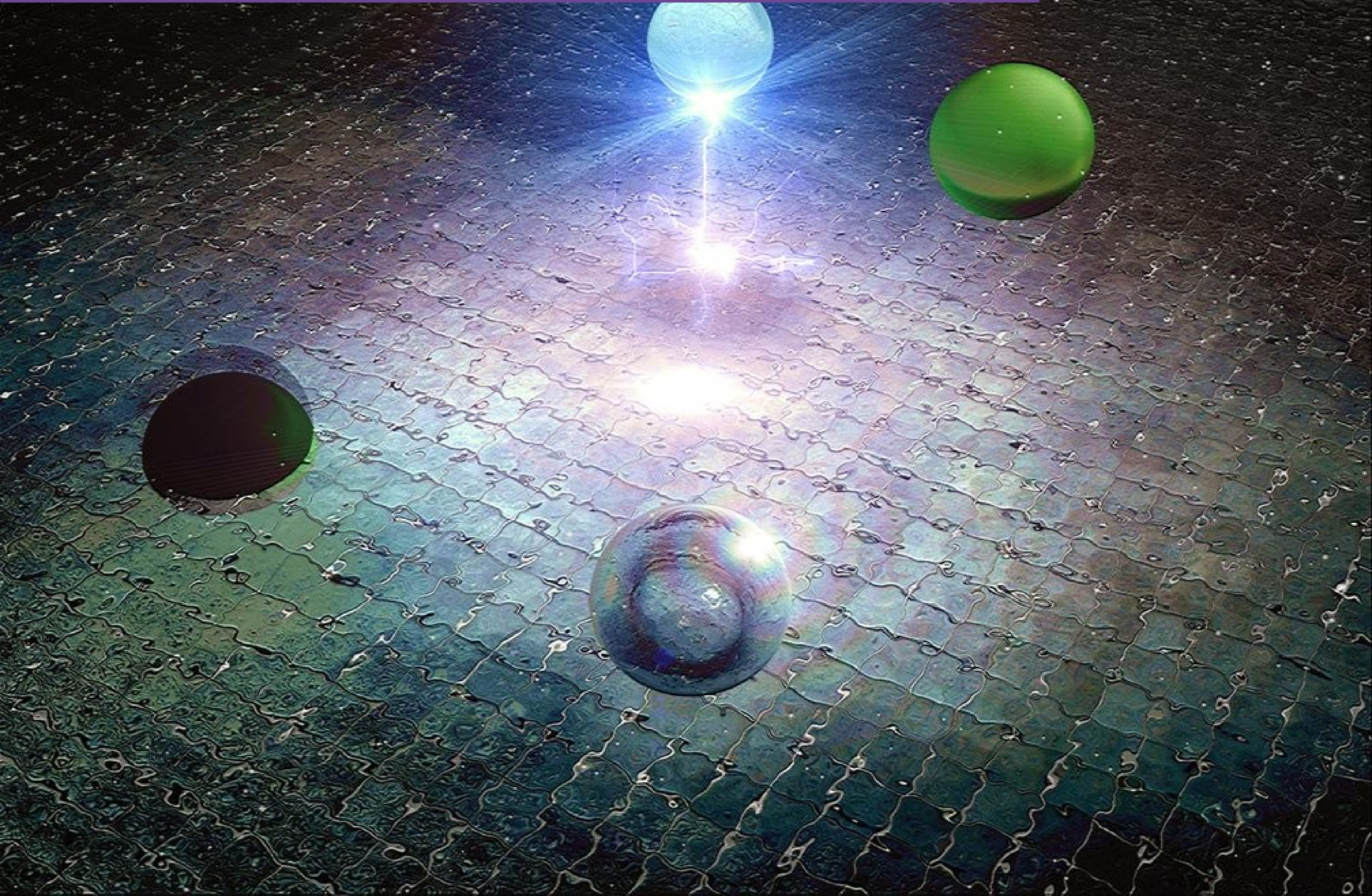
- For these geometries, the Ashtekar-Barbero variables behave as:

$$E \sim a^2 \sim \sqrt{|d|} \rightarrow 0\,,$$
$$A \sim K \sim d^{-9/4} \rightarrow \infty\,.$$

They become ill defined in the process of signature change.



Vacuum dynamics with signature change



Vacuum dynamics with signature change

- Can we fix ***initial conditions*** *for the vacuum* in the elliptic regime and obtain a meaningful vacuum in the conventional region?

➔ The field equation is well defined for $\phi \propto \frac{\varphi}{a}$ and the choice of lapse $N^2 = \varepsilon a^6, \ \varepsilon = \pm 1$ (for Lorentzian and Euclidean sectors).

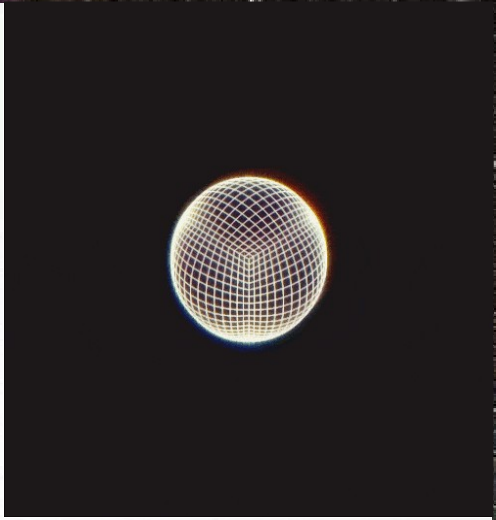
$$\ddot{\phi} = -\varepsilon \left[a^4 \Delta + a^6 m^2 \right] \phi .$$

➔ Our uniqueness criteria for φ provide, under **scaling** and **change of time**, a unique choice of positive and negative frequencies for ϕ.

$$\left\{ \varphi_n^{\pm}(T) \Psi_{nl}(\vec{x}) \right\} \longrightarrow \left\{ \phi_n^{\pm}(\tau) \Psi_{nl}(\vec{x}) \right\}.$$

$$\left[dT^2 = \varepsilon a^4 d\tau^2 = d(t) dt^2 \right]$$

Vacuum dynamics with signature change



- Assume that we can perform a **Wick rotation**: analytic continuation of the solutions.

$$\phi_n^{\pm(E)}(\tau) = \lim_{\tilde{\tau} \to i\tau} \phi_n^{\pm}(\tilde{\tau}).$$

- In the Euclidean region, the solutions are linear combinations of these, with **coefficients** $c_{nl}^{\pm(E)}$.

- When $d(\tau)$ **vanishes** (at $\tau = 0$), we impose as **matching conditions** the continuity of the field ϕ and its time derivative $\partial_\tau \phi$.

- For $\tau > 0$, the field is a linear combination of the Lorentzian modes, with coefficients $c_{nl}^{\pm}$.

Vacuum dynamics with signature change

- The matching conditions imply:

$$\begin{pmatrix} \phi_n^{+(E)}(0) & \phi_n^{-(E)}(0) \\ \partial_\tau \phi^{+(E)}{}_n(0) & \partial_\tau \phi^{-(E)}{}_n(0) \end{pmatrix} \begin{pmatrix} c_{nl}^{+(E)} \\ c_{nl}^{-(E)} \end{pmatrix} = \begin{pmatrix} \phi_n^{+}(0) & \phi_n^{-}(0) \\ \partial_\tau \phi^{+}{}_n(0) & \partial_\tau \phi^{-}{}_n(0) \end{pmatrix} \begin{pmatrix} c_{nl}^{+} \\ c_{nl}^{-} \end{pmatrix}.$$

- Using that the modes are orthonormal with the Klein-Gordon product and the definition $I_n^{(rs)} = \lim_{\tau \to 0} \langle \phi_n^{r(E)}(-|\tau|), \phi_m^{s}(|\tau|) \rangle$,

$$\Rightarrow \quad \begin{pmatrix} c_{nl}^{+} \\ c_{nl}^{-} \end{pmatrix} = \begin{pmatrix} -I_n^{(+-)} & -I_n^{(--)} \\ I_n^{(++)} & I_n^{(-+)} \end{pmatrix} \begin{pmatrix} c_{nl}^{+(E)} \\ c_{nl}^{-(E)} \end{pmatrix}.$$

Vacuum dynamics with signature change

- Starting only with "*positive frequency*" contributions in the Euclidean sector, so that $c_{nl}^{-(E)} = 0$, we obtain:

$$c_{nl}^{+} = -I_{n}^{(+-)}, \qquad c_{nl}^{-} = I_{n}^{(++)}.$$

In the Lorentzian region we have **positive and negative** frequencies.

There is **particle creation**.

Vacuum dynamics with signature change

- If we employ a **WKB approximation** for the computation (with due care to handle some subtleties), we obtain:

$$c_{nl}^{-} = I_n^{(++)} = -\frac{1+i}{2}\exp\left(\omega_n \Lambda\right), \qquad \Lambda = \int_0^{|\tau_0|} \bar{a}^2(\tilde{\tau})\, d\tilde{\tau},$$

$$\bar{a}^2(-\tau) = \lim_{\tilde{\tau}\to i\tau} a^2(\tilde{\tau}).$$

The corresponding particle production depends on the *background* only through Λ and the production is **exponential**.

Conclusions

- The criteria of spatial symmetry and unitary dynamics select a **unique Fock representation** and a **canonical pair**.
- With **time reparametrizations** and **field scalings**, the results can be extended to Klein-Gordon equations with **time-dependent coefficients**.
- These field equations are the Klein-Gordon equations of fields in **conformally ultrastatic spacetimes**, in a **bijective** correspondence.
- In a process of signature change, the metric **degenerates** completely and the Ashtekar-Barbero connection is **ill defined**.
- Assuming a Wick rotation, we can set initial conditions in the Euclidean region. The evolution generally leads to a **particle production**.

Conclusions