Asymptotic silence in quantum gravity

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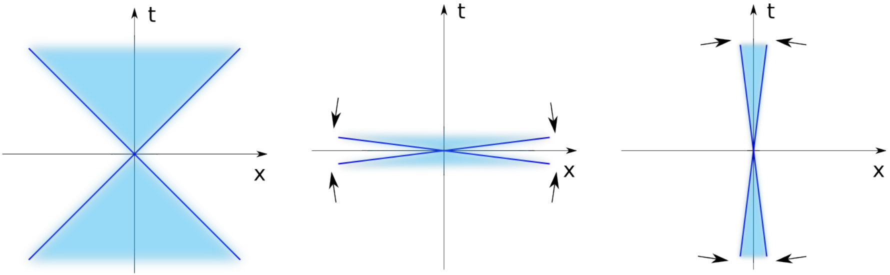
1 March, 2013

$c \to 0$



Minkowski

Newton ($c \to \infty$)

Carroll ($c \to 0$)

The asymptotic silence is obtained while taking the speed of light $c \to 0$, which is known as the Carrollian limit[1].

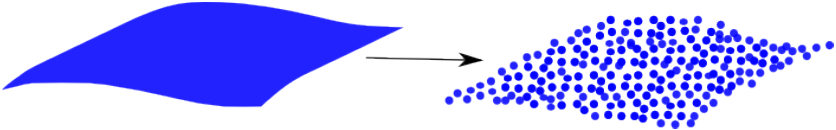
,,A slow sort of country ..., ... now, here, you see, it takes all the running you can do to stay in the same place..." Lewis Carroll

[1] J-M. Levy-Leblond, *Annales de l'I.H.P.*, section A, tome 3, No 1 (1965).

BKL conjecture

The state of asymptotic silence appears in various contexts. Perhaps the best known is the so-called Belinsky-Khalatnikov-Lifshitz (BKL) conjecture[2]:

"Near to a singularity spatially separated points decouple, and the role of most forms of matter is negligible."



In the BKL scenario, each of the "points" is described by the anisotropic cosmological solution.

[2] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, *Adv. Phys.* **19** (1970) 525; **31** (1982) 639

$G_N \to \infty$

The less intuitive case is the strong coupling limit of the gravitational interactions, when $G_N \to \infty$. Relation between this limit and the asymptotic silence can be understood by analyzing the Hamiltonian formalism of general relativity, where

$$H_G[N, N^a] = S[N] + D[N^a] = \int d^3x (NS + N^a D_a).$$

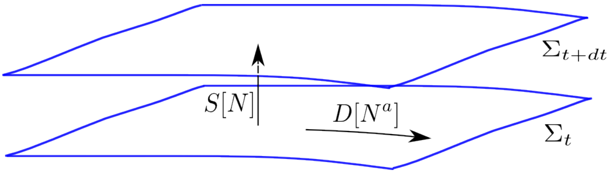
Namely, the scalar constraint can be schematically written as

$$S = G_N \cdot \ kinetic + \frac{1}{G_N} \cdot \ potential.$$

Here, only the potential term contains spatial derivatives, which relate the neighboring points. Therefore, while talking $G_N \to \infty$ only the kinetic term survives, and the theory becomes ultralocal.

Hypersurface deformation algebra

The constraints generate gauge transformations:



The algebra of constraints:

$$
\begin{aligned}
\{D[N_1^a], D[N_2^a]\} &= D[N_1^b \partial_b N_2^a - N_2^b \partial_b N_1^a], \\
\{S[N], D[N^a]\} &= -S[N^a \partial_a N], \\
\{S[N_1], S[N_2]\} &= sD\left[g^{ab}(N_1 \partial_b N_2 - N_2 \partial_b N_1)\right],
\end{aligned}
$$

where $s = 1$ corresponds to the Lorentzian signature and $s = -1$ to the Euclidean one. Due to the factor g^{ab} the algebra of constraint is not a Lie algebra.

In the ultralocal limit[3] ($G_N \to \infty$) the algebra of constraints simplifies to the Lie algebra:

$$
\begin{aligned}
\{D[N_1^a], D[N_2^a]\} &= D[N_1^b \partial_b N_2^a - N_2^b \partial_b N_1^a], \\
\{S[N], D[N^a]\} &= -S[N^a \partial_a N], \\
\{S[N_1], S[N_2]\} &= 0.
\end{aligned}
$$

Surprisingly, the number of the local symmetry generators is the same as in GR. The same holds for Hořava-Lifshitz[4] gravity in the $z \to 0$ limit of the dynamical exponent, where the anisotropic scaling

$$\mathbf{x} \to b\mathbf{x} \quad \text{and} \quad t \to b^z t.$$

Cosmological evolution can be interpreted as a flow from $z = 0$ ($d_s \to \infty$) in the early universe to $z = 1$ ($d_s = 4$) observed now.

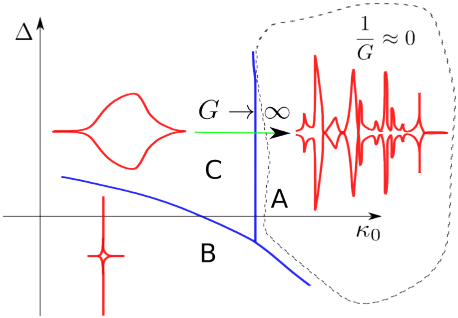
[3] C. J. Isham, *Proc. Roy. Soc. Lond. A* **351** (1976) 209.

[4] P. Hořava, *Phys. Rev. D* **79** (2009) 084008.

Causal Dynamical Triangulation

Possible confirmation of the asymptotic silence is coming also from Causal Dynamical Triangulation (CDT) approach to quantum gravity (Ambjorn, Jurkiewicz, Loll).

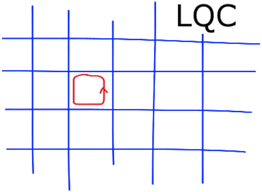
Phase diagram of CDT:



As observed from the numerical computations, universe breaks up into several independent components, when the effective gravitational coupling constant G increases.

Loop quantum gravity

- Loop quantum gravity (LQG) is based on Hamiltonian formulation of GR.
- Loop quantum cosmology (LQC) is a regular lattice model of LQG.



- Physical area of a loop $\mathrm{Ar}_{\square} = \bar{p}\bar{\mu}^2$, where $\bar{p} = a^2$ and a is a scale factor. In general $\bar{\mu} \propto \bar{p}^{\delta}$, where $-1/2 \leq \delta \leq 0$. For the so-called $\bar{\mu}-$scheme: $\bar{\mu} = \sqrt{\frac{\Delta}{\bar{p}}}$, where $\Delta = 2\sqrt{3}\pi\gamma l_{\mathrm{Pl}}^2$ is the area gap derived from LQG.

- At the effective level, quantum gravity effects can be studied by introducing appropriate corrections to the classical constraints:

$$\mathcal{C}_{tot} \to \mathcal{C}^{Q}_{tot},$$

where $\mathcal{C}_{tot} = \mathcal{C}_G + \mathcal{C}_M$.
- In LQC one usually consider two kinds of such quantum corrections:
    - Inverse volume corrections.
    - Holonomy corrections $\left(\bar{k} \to \frac{\sin(n\bar{\mu}\gamma\bar{k})}{n\bar{\mu}\gamma}\right)$.
- Problems appear for inhomogeneous models:
- The procedure of introducing quantum corrections suffers from ambiguities.
- In general, the algebra of modified constraints is not closed:

$$\{\mathcal{C}^{Q}_{I}, \mathcal{C}^{Q}_{J}\} = g^{K}{}_{IJ}(A^{j}_{b}, E^{a}_{i})\mathcal{C}^{Q}_{K} + \mathcal{A}_{IJ}.$$

- Can we introduce quantum corrections in the anomaly-free manner (i.e. such that $\mathcal{A}_{IJ} = 0$)?
- It turns out that it is possible at least for perturbative inhomogeneities (application to cosmology) for:
  - Inverse volume corrections (gauge invariant: Bojowald, Hossain, Kagan, Shankaranarayanan - 2008)
  - Holonomy corrections (gauge invariant: Cailleteau, Mielczarek, Barrau, Grain - 2012, fixed gauge: Wilson-Ewing - 2012)
- We found that, for perturbative inhomogeneities with holonomy corrections:
  - There is a unique way of modifying constraints such that the algebra is closed.
  - Additionally, the conditions of anomaly-freedom are fulfilled if and only if $\mathrm{Ar}_{\Box}$ =const, which corresponds to "new quantization scheme" ($\delta = -1/2$).

See T. Cailleteau, J. Mielczarek, A. Barrau, J. Grain, Class. Quantum Grav. **29** (2012) 095010 and my PhD dissertation.

Algebra of constraints (off-shell):

$$
\begin{aligned}
\{D_{tot}[N_1^a], D_{tot}[N_2^a]\} &= 0, \\
\left\{S^Q_{tot}[N], D_{tot}[N^a]\right\} &= -S^Q_{tot}[\delta N^a \partial_a \delta N], \\
\left\{S^Q_{tot}[N_1], S^Q_{tot}[N_2]\right\} &= \beta D_{tot}\left[\frac{\bar{N}}{\bar{p}}\partial^a(\delta N_2 - \delta N_1)\right].
\end{aligned}
$$

The algebra is closed but deformed with respect to the classical case due to presence of the factor

$$\beta = \cos(2\bar{\mu}\gamma\bar{k}) = 1 - 2\frac{\rho}{\rho_c} \in [-1, 1] \quad \text{where} \quad \rho_c = \frac{3}{8\pi G \Delta \gamma^2} \sim \rho_{\text{Pl}}.$$

What is the interpretation? Classically, we have

$$\{S_{tot}[N_1], S_{tot}[N_2]\} = sD\left[\frac{\bar{N}}{\bar{p}}\partial^a(\delta N_2 - \delta N_1)\right],$$

where $s = 1$ corresponds to the Lorentzian signature and $s = -1$ to the Euclidean one.

- The effective algebra of constraints shows that space is Euclidean for $\rho > \rho_c/2$, while Lorentzian geometry emerges for $\rho < \rho_c/2$. Signature change at $\rho = \rho_c/2$. Spacetime becomes 4D Euclidean space for $\rho > \rho_c/2$.
- It is interesting to notice that this model naturally have properties of the Hartle-Hawking no-boundary proposal (Hartle, Hawking - 1983).
- At $\rho = \rho_c/2$ the Ultralocal Gravity is recovered $\{S_{tot}[N_1], S_{tot}[N_2]\} = 0$ since $\beta = 0 \rightarrow$ the stage of asymptotic silence.
- The $\beta-$deformation appears also from the inverse-volume corrections. However, in that case $\beta > 0$. The asymptotic silence can be realized but not the signature change.
- However, the signature change was observed also for spherically symmetric models with holonomy corrections[5].

[5] M. Bojowald and G. M. Paily, *Phys. Rev. D* **86** (2012) 104018.

Equations of motion:

Scalar pertubations. One can derive modified Mukhanov equation:

$$\frac{d^2}{d\eta^2}v - \beta\nabla^2 v - \frac{z''}{z}v = 0,$$

where $z := \sqrt{\bar{p}}\frac{\dot{\varphi}}{H}$. Spatial curvature $\mathcal{R} = v/z$.

Vector perturbations. For the considered model with a scalar field vector modes are pure gauge.

Tensor perturbations. Equation of motion for the gravitational waves is the following:

$$\frac{d^2}{d\eta^2}h_{ab} + 2\left(aH - \frac{1}{2\beta}\frac{d\beta}{d\eta}\right)\frac{d}{d\eta}h_{ab} - \beta\nabla^2 h_{ab} = 0.$$

Inflationary scalar power spectrum:

$$\mathcal{P}_{\mathrm{S}}(k) = A_S \left(\frac{k}{aH}\right)^{n_{\mathrm{S}}-1},$$

$$A_S = \frac{1}{\pi\epsilon}\left(\frac{H}{m_{\mathrm{Pl}}}\right)^2 \left(1 + 2\frac{V}{\rho_c}\right),$$

$$n_{\mathrm{S}} = 1 + 2\eta - 6\epsilon(1 - V/\rho_c).$$

Inflationary tensor power spectrum:

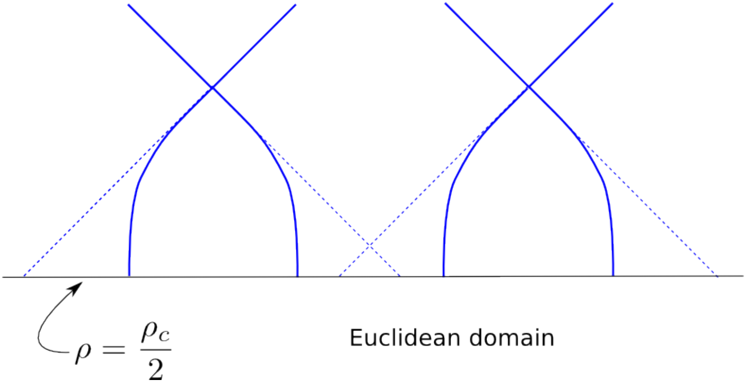
$$\mathcal{P}_{\mathrm{T}}(k) = A_T \left(\frac{k}{aH}\right)^{n_{\mathrm{T}}},$$

$$A_T = \frac{16}{\pi}\left(\frac{H}{m_{\mathrm{Pl}}}\right)^2 \left(1 + 3\frac{V}{\rho_c}\right),$$

$$n_{\mathrm{T}} = -2\epsilon(1 - 3V/\rho_c).$$

Corrections are tiny: $\mathcal{O}(V/\rho_{\mathrm{c}}) \sim 10^{-12}$.

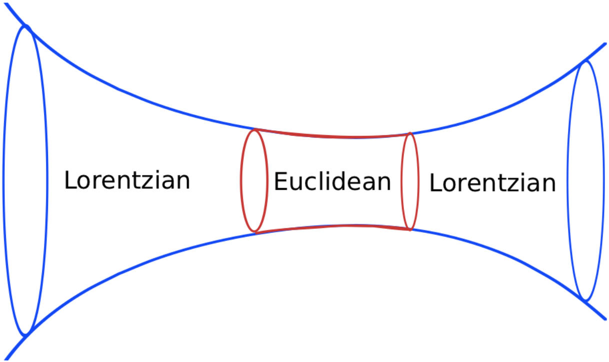
Light cones



Effective speed of light:

$$c_{eff} = \sqrt{\beta} = \sqrt{1 - 2\frac{\rho}{\rho_c}}.$$

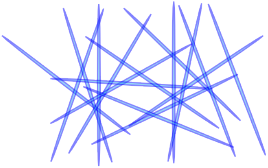
The asymptotic silence ($c_{eff} \to 0$) is realized while $\rho \to \frac{\rho_c}{2}$.



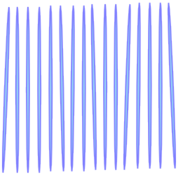
- Is there quantum tunneling through the Euclidean phase from the contraction to expansion phase?
- Suppression of the spatial derivatives while $\{S^Q, S^Q\} \to 0$. Possible support for the BKL conjecture.
- Silent initial conditions ($\langle\phi(0)\phi(r)\rangle \to 0$) at $\rho = \rho_c/2$?

Spontaneous symmetry breaking?

Let us consider a model of metamaterial, composed of nanowires (or ferromagnet, liquid crystal, etc.).



$T > T_C$



$T < T_C$

The $SO(3)$ symmetry is broken to $SO(2)$ at temperatures below T_C. An order parameter is the local magnetization and the corresponding Goldstone bosons manifest as spin waves.

In the direction of magnetization, the dielectric permittivity becomes negative leading to emergence of a new effective time variable [6]

[6] I. I. Smolyaninov and E. E. Narimanov, *Phys. Rev. Lett.* **105** (2010) 067402.

So, at the level of the equations of motion for the electric field propagating in the considered material, the original $SO(3)$ symmetry of Laplace operator is replaced by $SO(1,2)$.

In case of gravity, we observe that the symmetry of equations for the fields change from $SO(4)$ in the Euclidean region to $SO(1,3)$ in the Lorentzian regime. One can speculate that this transition is a result of the symmetry breaking at the level of the fundamental structure of spacetime.

In particular, one can suppose that the original $SO(4)$ spacetime symmetry is broken into $SO(3)$, where the residue $SO(3)$ is the rotational symmetry of triads. Time can be therefore seen as the order parameter of the symmetry broken phase. Interestingly, in such a picture, Goldstone bosons associated with the broken symmetry must appear. Such particles could naturally serve as inflatons, which are required to explain the inflationary stage after the Planck epoch.

Loop deformation of the Poincaré algebra

In the limit of vanishing space-time curvaure, the loop-deformed algebra of constraints, reduces to the loop-deformed Poincaré algebra. This corresponds to linear deformations of the hypersurface[7]:

$$N(x) = \Delta t + v_a x^a, \quad N^a(x) = \Delta x^a + R^a{}_b x^b, \text{ and } g_{ab} = \delta_{ab}.$$

The obtained deformation is a special case of a class of generalizations of the Poincaré algebra[8].

The fact that only the $\{S^Q, S^Q\}$ bracket is deformed imposes constraints on the possible deformations of the Poincaré algebra. In particular, undeformed form of $\{D, S^Q\} = S^Q$ implies that $[K_i, P_j] = i\delta_{ij} P_0$ remains undeformed as well.

[7] M. Bojowald and G. M. Paily, arXiv:1212.4773 [gr-qc]

[8] D. Kovacevic, S. Meljanac, A. Pachol and R. Strajn, Phys. Lett. B **711** (2012) 122

Poincaré algebra for the metric $\eta_{\mu\nu} = (-s, 1, 1, 1)$:

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -is\epsilon_{ijk} J_k \\
[J_i, P_j] &= i\epsilon_{ijk} P_k \\
[K_i, P_j] &= i\delta_{ij} P_0 \\
[J_i, P_0] &= 0 \\
[K_i, P_0] &= isP_i \\
[P_i, P_j] &= 0 \\
[P_i, P_0] &= 0
\end{aligned}
$$

Loop-deformed Poincaré algebra for metric signature $(-, +, +, +)$:

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
\left[J_i, \tilde{K}_j\right] &= i\epsilon_{ijk} \tilde{K}_k \\
\left[\tilde{K}_i, \tilde{K}_j\right] &= -i\beta\epsilon_{ijk} J_k \\
[J_i, P_j] &= i\epsilon_{ijk} P_k \\
\left[\tilde{K}_i, P_j\right] &= i\delta_{ij} P_0 \\
[J_i, P_0] &= 0 \\
\left[\tilde{K}_i, P_0\right] &= i\beta P_i \\
[P_i, P_j] &= 0 \\
[P_i, P_0] &= 0
\end{aligned}
$$

where $\tilde{K}_i$ is deformed generator of boost $\tilde{K}_i = X_i P_0 - X_0 P_i \beta$. For $s \to 0$ or $\beta \to 0$ the Carroll group is recovered.

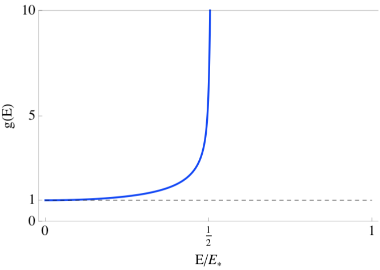
Casimir operator:

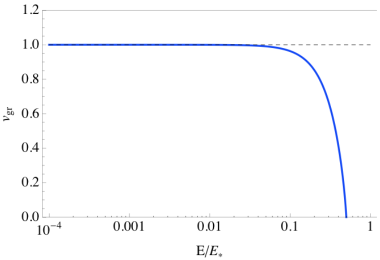
$$\mathcal{C}_1 = E_*^2 \int_0^{P_0/E_*} \frac{2y}{\beta(y)} dy - (P_i)^2.$$

Let us consider the cosine type deformation $\beta(x) = \cos(\pi x)$, where $x = P_0/E_* \in [0,1]$. We denote $E = P_0$, and $E_* \sim E_{\rm Pl}$.

Dispersion relation of photon:
$E^2 g(E) - p^2 = 0$

Group velocity: $v_{\rm gr} = \beta\sqrt{g(E)}$





Summary and outlook

- Different approaches to quantum gravity meet at the asymptotic silence.
- Our model unifies various ideas, such as signature change and asymptotic silence.
- The presented results contribute to the growing evidence, showing that the asymptotic silence may indeed have something to do with the state of spacetime under very extreme conditions. However, the final vote belongs to experiment.
- Silent initial conditions at $\rho = \rho_c/2$. Predictions of the spectra of primordial perturbations. Comparison with the CMB data. (in progress with A. Barrau and L. Linsefors)
- Deformation of the Poincaré algebra. Modifications of the photon dispersion relation. Phenomenology available.
- Paradigm shift in LQC.

Supplement 1: Quantum state of the asymptotic silence

We redefine

$$A^i_a \to G_N A^i_a$$

such that phase space structure

$$\left\{E^a_j(\mathbf{x}), A^i_b(\mathbf{y})\right\} = 8\pi\gamma\delta^a_b\delta^i_j\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

does not depend on G_N. In the limit $G_N \to \infty$ and $\Lambda \sim G_N^2 \to \infty$ the constraints simplify to:

$$\begin{aligned}
C_i &\to \frac{2}{\gamma}\epsilon_{ijk}A^j_a E^a_k, \\
C_a &\to {\color{blue}G_N}\frac{2}{\gamma}\epsilon^i{}_{jk}A^j_a A^k_b E^b_i, \\
C &\to -\frac{1}{\gamma^2}{\color{blue}G_N}\frac{\epsilon^{ij}{}_k E^a_i E^b_j}{\sqrt{\det E}}\left(\epsilon^k{}_{lm}A^l_a A^m_b - \frac{1}{3}\lambda\gamma^2\epsilon_{abc}E^{ck}\right),
\end{aligned}$$

where $\lambda \equiv \frac{\Lambda}{G_N^2} = \text{const}$ is the considered limit.

Canonical quantization:

$$\begin{aligned} A_a^i &\rightarrow \hat{A}_a^i \Psi = A_a^i \Psi, \\ E_i^a &\rightarrow \hat{E}_i^a \Psi = 8\pi\gamma\hbar i \frac{\delta}{\delta A_a^i} \Psi, \end{aligned}$$

such that $\left[\hat{E}_j^a(\mathbf{x}), \hat{A}_b^i(\mathbf{y})\right] = i8\pi\gamma\hbar\delta_b^a\delta_j^i\delta^{(3)}(\mathbf{x}-\mathbf{y})$. The state annihilated by the constraints

$$\begin{aligned} \hat{C}_i \Psi[A] &= 0, \\ \hat{C}_a \Psi[A] &= 0, \\ \hat{C} \Psi[A] &= 0, \end{aligned}$$

has the following form:

$$\Psi[A] = \mathcal{N} \exp\left[\frac{i}{4\pi\gamma^3\lambda\hbar} \int_\Sigma \mathrm{tr}(A \wedge A \wedge A)\right],$$

which resembles the famous Kodama state.

Supplement 1: Scalar perturbations

- The constraints $\mathcal{C}_I$ are subject of perturbative expansion:

$$\mathcal{C}_I = \mathcal{C}_I^{(0)} + \mathcal{C}_I^{(1)} + \mathcal{C}_I^{(2)} + \dots$$

- Holonomy corrections are introduced by the replacement

$$\bar{k} \to \mathbb{K}[n] := \frac{\sin(n\bar{\mu}\gamma\bar{k})}{n\bar{\mu}\gamma},$$

in the classical constraints, where $n \in \mathbb{Z}$.

- Because the constraints are quantum-modified ($\mathcal{C}_I \to \mathcal{C}_I^Q$), there is a worry that the corresponding Poisson algebra will not be closed:

$$\{\mathcal{C}_I^Q, \mathcal{C}_J^Q\} = f^K{}_{IJ}(A_b^j, E_i^a)\mathcal{C}_K^Q + \mathcal{A}_{IJ}.$$

For consistency (closure of algebra), $\mathcal{A}_{IJ}$ is required to vanish.

The holonomy-modified Hamiltonian constraint can be written as:

$$S_G^Q[N] = \frac{1}{2\kappa}\int_\Sigma d^3x \left[\bar{N}(\mathcal{H}_G^{(0)} + \mathcal{H}_G^{(2)}) + \delta N \mathcal{H}_G^{(1)}\right], \text{ where}$$

$$\begin{aligned}
\mathcal{H}_G^{(0)} &= -6\sqrt{\bar{p}}(\mathbb{K}[1])^2, \\
\mathcal{H}_G^{(1)} &= -4\sqrt{\bar{p}}\left(\mathbb{K}[s_1] + \alpha_1\right)\delta_j^c \delta K_c^j - \frac{1}{\sqrt{\bar{p}}}\left(\mathbb{K}[1]^2 + \alpha_2\right)\delta_c^j \delta E_j^c \\
&+ \frac{2}{\sqrt{\bar{p}}}(1+\alpha_3)\partial_c \partial^j \delta E_j^c, \\
\mathcal{H}_G^{(2)} &= \sqrt{\bar{p}}(1+\alpha_4)\delta K_c^j \delta K_d^k \delta_k^c \delta_j^d - \sqrt{\bar{p}}(1+\alpha_5)(\delta K_c^j \delta_j^c)^2 \\
&- \frac{2}{\sqrt{\bar{p}}}\left(\mathbb{K}[s_2] + \alpha_6\right)\delta E_j^c \delta K_c^j - \frac{1}{2\bar{p}^{3/2}}\left(\mathbb{K}[1]^2 + \alpha_7\right)\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j \\
&+ \frac{1}{4\bar{p}^{3/2}}\left(\mathbb{K}[1]^2 + \alpha_8\right)(\delta E_j^c \delta_c^j)^2 \\
&- \frac{1}{2\bar{p}^{3/2}}(1+\alpha_9)\delta^{jk}(\partial_c \delta E_j^c)(\partial_d \delta E_k^d).
\end{aligned}$$

Here, $\alpha_i(\bar{p}, \bar{k})$ are the counter-terms ($\alpha_i(\bar{p}, \bar{k}) \to 0$ for $\bar{\mu} \to 0$).

Diffeomorphism constraint takes the classical form

$$
\begin{aligned}
D_G[N^a] &= \frac{1}{\kappa}\int_\Sigma d^3x\delta N^c\left[\bar{p}\partial_c(\delta^d_k\delta K^k_d)-\bar{p}(\partial_k\delta K^k_c)\right. \\
&- \left.\bar{k}\delta^k_c(\partial_d\delta E^d_k)\right].
\end{aligned}
$$

The scalar matter diffeomorphism constraint is

$$
D_M[N^a]=\int_\Sigma \delta N^a\bar{\pi}(\partial_a\delta\varphi).
$$

The scalar matter Hamiltonian can be expressed as follows

$$
S^Q_M[N]=S_M[\bar{N}]+S_M[\delta N],
$$

where

$$
\begin{aligned}
S_M[\bar{N}] &= \int_\Sigma d^3x\bar{N}\left[\left(\mathcal{H}^{(0)}_\pi+\mathcal{H}^{(0)}_\varphi\right)+\left(\mathcal{H}^{(2)}_\pi+\mathcal{H}^{(2)}_\nabla+\mathcal{H}^{(2)}_\varphi\right)\right], \\
S_M[\delta N] &= \int_\Sigma d^3\delta N\left[\mathcal{H}^{(1)}_\pi+\mathcal{H}^{(1)}_\varphi\right].
\end{aligned}
$$

$$
\mathcal{H}^{(0)}_{\pi} = \frac{\bar{\pi}^2}{2\bar{p}^{3/2}} \qquad \mathcal{H}^{(1)}_{\pi} = \frac{\bar{\pi}\delta\pi}{\bar{p}^{3/2}} - \frac{\bar{\pi}^2}{2\bar{p}^{3/2}}\frac{\delta^j_c\delta E^c_j}{2\bar{p}}
$$

$$
\mathcal{H}^{(0)}_{\varphi} = \bar{p}^{3/2}V(\bar{\varphi}) \qquad \mathcal{H}^{(1)}_{\varphi} = \bar{p}^{3/2}\left[V_{,\varphi}(\bar{\varphi})\delta\varphi + V(\bar{\varphi})\frac{\delta^j_c\delta E^c_j}{2\bar{p}}\right]
$$

$$
\mathcal{H}^{(2)}_{\pi} = \frac{1}{2}\frac{\delta\pi^2}{\bar{p}^{3/2}} - \frac{\bar{\pi}\delta\pi}{\bar{p}^{3/2}}\frac{\delta^j_c\delta E^c_j}{2\bar{p}} + \frac{1}{2}\frac{\bar{\pi}^2}{\bar{p}^{3/2}}\left[\frac{(\delta^j_c\delta E^c_j)^2}{8\bar{p}^2} + \frac{\delta^k_c\delta^j_d\delta E^c_j\delta E^d_k}{4\bar{p}^2}\right],
$$

$$
\mathcal{H}^{(2)}_{\nabla} = \frac{1}{2}\sqrt{\bar{p}}(1+{\color{red}\alpha_{10}})\delta^{ab}\partial_a\delta\varphi\partial_b\delta\varphi,
$$

$$
\begin{aligned}
\mathcal{H}^{(2)}_{\varphi} &= \frac{1}{2}\bar{p}^{3/2}V_{,\varphi\varphi}(\bar{\varphi})\delta\varphi^2 + \bar{p}^{3/2}V_{,\varphi}(\bar{\varphi})\delta\varphi\frac{\delta^j_c\delta E^c_j}{2\bar{p}} \\
&+ \bar{p}^{3/2}V(\bar{\varphi})\left[\frac{(\delta^j_c\delta E^c_j)^2}{8\bar{p}^2} - \frac{\delta^k_c\delta^j_d\delta E^c_j\delta E^d_k}{4\bar{p}^2}\right].
\end{aligned}
$$

Brackets

The total Hamiltonian and diffeomorphism constraints are:

$$\begin{aligned} S^Q_{tot}[N] &= S^Q_G[N] + S^Q_M[N], \\ D_{tot}[N^a] &= D_G[N^a] + D_M[N^a]. \end{aligned}$$

The Poisson bracket between two total diffeomorphism constraints is vanishing:

$$\{D_{tot}[N^a_1], D_{tot}[N^a_2]\} = 0.$$

The bracket between the total Hamiltonian and diffeomorphism constraints can be decomposed as follows:

$$\begin{aligned} \left\{S^Q_{tot}[N], D_{tot}[N^a]\right\} &= \left\{S^Q_M[N], D_{tot}[N^a]\right\} + \left\{S^Q_G[N], D_G[N^a]\right\} \\ &+ \left\{S^Q_G[N], D_M[N^a]\right\}. \end{aligned}$$

$$\left\{S_M^Q[N], D_{tot}[N^a]\right\} = -S_M^Q[\delta N^a \partial_a \delta N].$$

$$\left\{S_G^Q[N], D_G[N^a]\right\} = -S_G^Q\left[\delta N^a \partial_a \delta N\right] + \mathcal{B}\ D_G[N^a]$$
$$+\frac{\sqrt{\bar{p}}}{\kappa}\int_\Sigma d^3x \delta N^a (\partial_a \delta N)\mathcal{A}_1 + \frac{\bar{N}\sqrt{\bar{p}}\bar{k}}{\kappa}\int_\Sigma d^3x \delta N^a (\partial_i \delta K_a^i)\mathcal{A}_2$$
$$+\frac{\bar{N}}{\kappa\sqrt{\bar{p}}}\int_\Sigma d^3x \delta N^i (\partial_a \delta E_i^a)\mathcal{A}_3 + \frac{\bar{N}}{2\kappa\sqrt{\bar{p}}}\int_\Sigma d^3x (\partial_a \delta N^a)(\delta E_i^b \delta_b^i)\mathcal{A}_4.$$

The functions $\mathcal{A}_1, \ldots, \mathcal{A}_4$ are the first anomalies coming from the effective nature of the Hamiltonian constraint.

$$\left\{S_G^Q[N], D_M[N^a]\right\} = 0.$$

The Poisson bracket between the two total Hamiltonian constraints can be decomposed in the following way:

$$\begin{aligned}\left\{S^{Q}_{tot}[N_1], S_{tot}[N_2]\right\} &= \left\{S^{Q}_{G}[N_1], S^{Q}_{G}[N_2]\right\} + \{S_M[N_1], S_M[N_2]\} \\ &+ \left[\left\{S^{Q}_{G}[N_1], S_M[N_2]\right\} - (N_1 \leftrightarrow N_2)\right].\end{aligned}$$

Here, the Poisson bracket between two matter Hamiltonians is

$$\left\{S^{Q}_{M}[N_1], S^{Q}_{M}[N_2]\right\} = (1+\alpha_{10}) D_M \left[\frac{\bar{N}}{\bar{p}} \partial^a (\delta N_2 - \delta N_1)\right].$$

The appearance of the front-factor $(1+\alpha_{10})$ will allow us to close the algebra of total constraints.

$$
\begin{aligned}
\left\{S_G^Q[N_1], S_G^Q[N_2]\right\} = {} & (1+\alpha_3)(1+\alpha_5) D_G \left[\frac{\bar{N}}{\bar{p}} \partial^a (\delta N_2 - \delta N_1)\right] \\
& + \frac{\bar{N}}{\kappa} \int_\Sigma d^3x \partial^a (\delta N_2 - \delta N_1)(\partial_i \delta K_a^i)(1+\alpha_3)\mathcal{A}_5 \\
& + \frac{\bar{N}}{\kappa \bar{p}} \int_\Sigma d^3x (\delta N_2 - \delta N_1)(\partial^i \partial_a \delta E_i^a)\mathcal{A}_6 \\
& + \frac{\bar{N}}{\kappa} \int_\Sigma d^3x (\delta N_2 - \delta N_1)(\delta_i^a \delta K_a^i)\mathcal{A}_7 \\
& + \frac{\bar{N}}{\kappa \bar{p}} \int_\Sigma d^3x (\delta N_2 - \delta N_1)(\delta_a^i \delta E_i^a)\mathcal{A}_8
\end{aligned}
$$

The $\mathcal{A}_5, \ldots, \mathcal{A}_8$ are the next four anomalies. Moreover, the diffeomorphism constraint is multiplied by the factor $(1+\alpha_3)(1+\alpha_5)$.

$$
\begin{aligned}
&\left\{S_G^Q[N_1], S_M[N_2]\right\} - (N_1 \leftrightarrow N_2) = \\
&= \frac{1}{2}\int_\Sigma d^3x \bar{N}(\delta N_2 - \delta N_1)\left(\frac{\bar{\pi}^2}{2\bar{p}^3} - V(\bar{\varphi})\right)(\partial_c\partial^j\delta E^c_j)\mathcal{A}_9 \\
&\quad +3\int_\Sigma d^3x \bar{N}(\delta N_2 - \delta N_1)\left(\frac{\bar{\pi}\delta\pi}{\bar{p}^2} - \bar{p}V_\varphi(\bar{\varphi})\delta\varphi\right)\mathcal{A}_{10} \\
&\quad +\int_\Sigma d^3x \bar{N}(\delta N_2 - \delta N_1)(\delta^c_j\delta K^c_j)\left(\frac{\bar{\pi}^2}{2\bar{p}^3} - V(\bar{\varphi})\right)\bar{p}\mathcal{A}_{11} \\
&\quad +\frac{1}{2}\int_\Sigma d^3x \bar{N}(\delta N_2 - \delta N_1)(\delta^j_c\delta E^c_j)\left(\frac{\bar{\pi}^2}{2\bar{p}^3}\right)\mathcal{A}_{12} \\
&\quad +\frac{1}{2}\int_\Sigma d^3x \bar{N}(\delta N_2 - \delta N_1)(\delta^j_c\delta E^c_j)V(\bar{\varphi})\mathcal{A}_{13}
\end{aligned}
$$

The functions $\mathcal{A}_9, \ldots, \mathcal{A}_{13}$ are the last five anomalies.

Anomaly freedom

The requirement of anomaly freedom is equivalent to the conditions $\mathcal{A}_i = 0$ for $i = 1, \ldots, 13$. Furthermore $(1 + \alpha_3)(1 + \alpha_5) = (1 + \alpha_{10})$. These conditions uniquely determine form of the counter terms α_i for $i = 1, \ldots, 10$.

Moreover, we have

$$
\begin{aligned}
\mathcal{A}_7 &= 2(1 + 2\delta)(\beta\mathbb{K}[1]^2 - \mathbb{K}[2]^2), \\
\mathcal{A}_8 &= \bar{k}(1 + 2\delta)(\mathbb{K}[2]^2 - \beta\mathbb{K}[1]^2).
\end{aligned}
$$

The anomaly freedom conditions for those terms, $\mathcal{A}_7 = 0$ and $\mathcal{A}_8 = 0$, are fulfilled if and only if $\delta = -1/2$. The choice $\delta = -1/2$ is called the $\bar{\mu}-$scheme ('new quantization scheme').

Our results show that the $\bar{\mu}-$scheme is embedded in the structure of the theory and this gives a new motivation for this particular choice of quantization scheme.

The counter-terms allowing the algebra to be anomaly-free are uniquely determined, and are given by:

$$
\begin{aligned}
\alpha_1 &= \mathbb{K}[2] - \mathbb{K}[s_1], \\
\alpha_2 &= 2\mathbb{K}[1]^2 - 2\bar{k}\mathbb{K}[2], \\
\alpha_3 &= 0, \\
\alpha_4 &= \beta - 1, \\
\alpha_5 &= \beta - 1, \\
\alpha_6 &= 2\mathbb{K}[2] - \mathbb{K}[s_2] - \bar{k}\beta, \\
\alpha_7 &= -4\mathbb{K}[1]^2 + 6\bar{k}\mathbb{K}[2] - 2\bar{k}^2\beta, \\
\alpha_8 &= -4\mathbb{K}[1]^2 + 6\bar{k}\mathbb{K}[2] - 2\bar{k}^2\beta, \\
\alpha_9 &= 0, \\
\alpha_{10} &= \beta - 1,
\end{aligned}
$$

where

$$\beta := \cos(2\bar{\mu}\gamma\bar{k}) = 1 - 2\frac{\rho}{\rho_c}.$$