# Encoding local dynamics without time

Robert Oeckl

Institute for Quantum Gravity
Friedrich-Alexander-Universität Erlangen-Nürnberg
Erlangen, Germany

and

Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
Morelia, Mexico

Tux 2
11 February 2014

# Motivation

"**Dynamics**" signifies traditionally the laws of **evolution in time** of a physical system.

In a **quantum theory of gravity** there is presumably no unique notion of metric spacetime, or even of time. Consequently, there is in general no notion of evolution in time.

# Motivation

"**Dynamics**" signifies traditionally the laws of **evolution in time** of a physical system.

In a **quantum theory of gravity** there is presumably no unique notion of metric spacetime, or even of time. Consequently, there is in general no notion of evolution in time.

In contrast to **classical physics**, the structure of standard **quantum theory** relies heavily on this traditional notion of dynamics.

Thus we require a new perspective on dynamics and with it a new perspective on quantum theory itself. This is what this talk is about.

# Local Dynamics

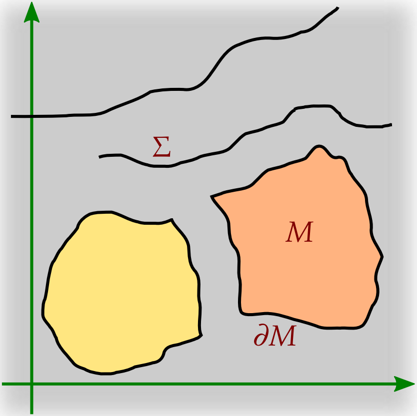
An key characteristic of dynamics as time evolution is its encoding of **locality – in time**.

With Maxwell's theory and the resulting relativistic revolution we have understood that actual physical dynamics conforms to a more general notion of **locality – in spacetime**.

**Locality** should thus be an essential ingredient of any notion of dynamics.

# Local Dynamics – spacetime

A notion of **locality** requires a (weak) **notion of spacetime**.
Spacetime is modeled by a collection of **hypersurfaces** and **regions**.
These are **topological manifolds**, but may carry **additional structure**.




Spacetime regions are the arena for **local physics**.

### "Holography"

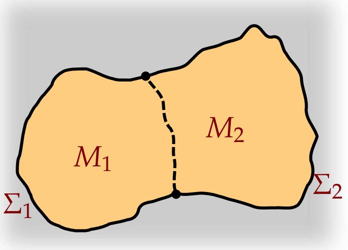
**Information** about local physics is **communicated** between adjacent regions through **interfacing hypersurfaces** (channels).

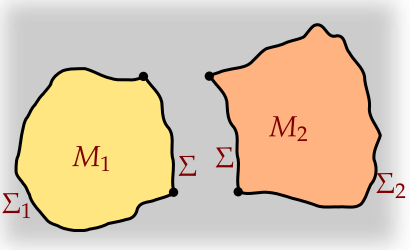
We call this a **spacetime system**.

# Local Dynamics – spacetime operations

There are natural operations that we can perform on the elements of a **spacetime system**:

- We can **decompose** (cut) hypersurfaces into “smaller” pieces
- Given a **region** $M$ we can extract its **boundary** $\partial M$
- Given regions $M_1$, $M_2$ with matching boundary components $\Sigma$ we can **compose** (glue) them to a joint region $M = M_1 \cup M_2$.



# Local Dynamics

What is **dynamics**?

Traditionally, dynamics tells us about the relation between physics at different times.

# Local Dynamics

What is **dynamics**?

Traditionally, dynamics tells us about the relation between physics at different times.

Here, we have no notion of time, but a weak notion of **spacetime**.

Dynamics tells us about the relation between physics in different spacetime regions.

# Local **Dynamics**

What is **dynamics**?

Traditionally, dynamics tells us about the relation between physics at different times.

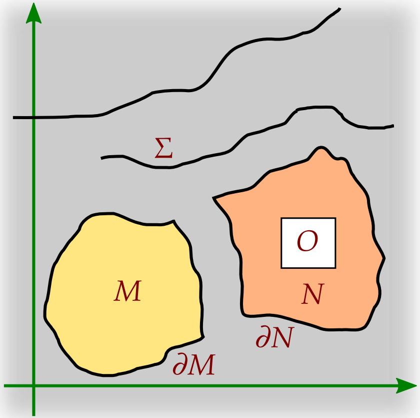
Here, we have no notion of time, but a weak notion of **spacetime**.

Dynamics tells us about the relation between physics in different spacetime regions.

It is also useful to formalize an explicit means of **probing** or **interacting with** local physics. While optional in classical physics this is essential in quantum physics due to the special nature of the observer.

Dynamics tells us about the relation between probes in different spacetime regions.

# Including Probes



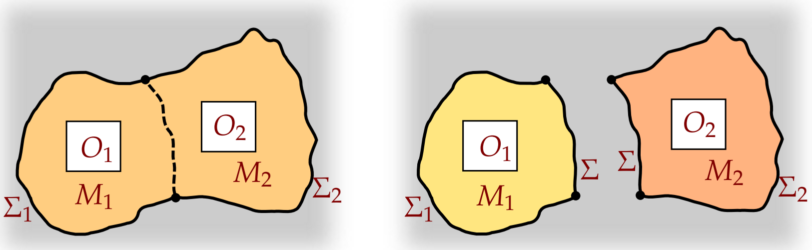

In addition to the usual elements of a spacetime system:

> There are special spacetime regions that contain probes.

# Including Probes

The spacetime composition operation extends to probes:

- Given regions $M_1, M_2$ with matching boundary components we can **compose** (glue) them to a joint region $M = M_1 \cup M_2$.
- If the regions $M_1, M_2$ carry probes $O_1$ and $O_2$ this yields a new probe $O = O_1 \diamond O_2$ in the joint region $M$.



# Towards encoding dynamics

To encode the dynamics we need to **associate mathematical structures** to the elements of the spacetime system, including probe regions. These must facilitate the **extraction of information** about the physics.

- To a **hypersurface** $\Sigma$ we associate a space of **boundary conditions**. This encodes the possible physical information flows between regions adjacent to the hypersurface.
- To a **spacetime region** $M$ we associate an **evaluation map** that pairs with boundary conditions on $\partial M$. This encodes the relation between boundary conditions and the physics in the interior.
- To a spacetime region $M$ with **probe** $O$ we associate a special evaluation map encoding also the interaction with the probe.

# Towards encoding dynamics

Spacetime operations give rise to corresponding **axioms** (rules) for the associated mathematical structures.

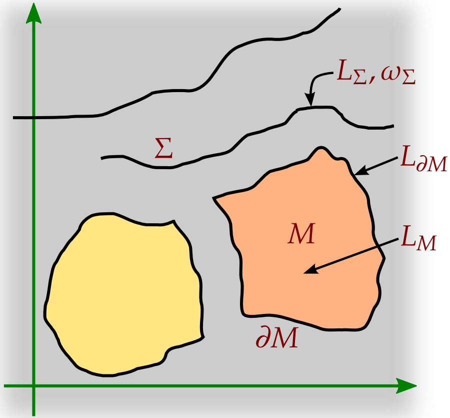
The suitable mathematical structures, their interpretation and the relevant axioms are distinct in

1. **classical physics** vs.
2. **quantum physics**

We consider these in turn.

# Classical physics

In classical Lagrangian field theory[1] we are naturally given the following structures:

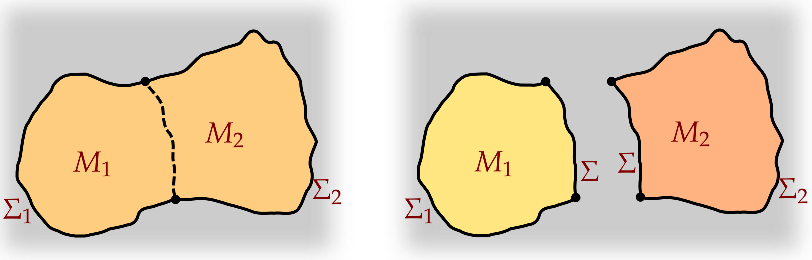



- Per **hypersurface** $\Sigma$ :
  The **space of solutions** near $\Sigma$. This is a **symplectic manifold** $(L_\Sigma, \omega_\Sigma)$.
- Per **region** $M$ :
  The **space of solutions** in $M$. Forgetting the interior yields a map $L_M \to L_{\partial M}$. Under this map $L_M$ is a **Lagrangian submanifold** $L_M \subseteq L_{\partial M}$.

[1] We consider here the simplest case only, without constraints or gauge symmetries.

# An axiom: composition of solutions

Consider regions $M_1, M_2$ with matching boundary components $\Sigma$ and their **composition** to a joint region $M = M_1 \cup M_2$.




Then we have an **exact sequence**

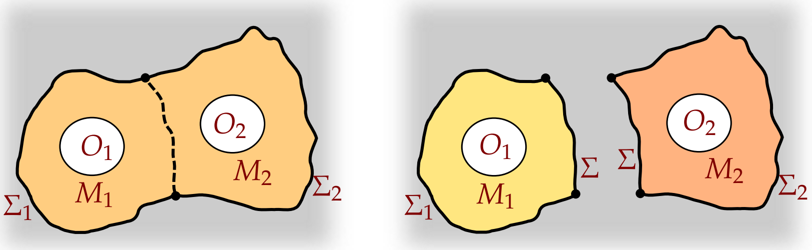
$$L_M \to L_{M_1} \times L_{M_2} \rightrightarrows L_\Sigma$$

This is a relation between the spaces of solutions in $M_1, M_2$ and $M$.

# Observables

In classical physics the role of **probes** is taken by **observables**. An observable in a region $M$ is a function $O : L_M \to \mathbb{R}$.

Consider regions $M_1, M_2$ with matching boundary components $\Sigma$ and their **composition** to a joint region $M = M_1 \cup M_2$.




The joint observable $O = O_1 \diamond O_2$ is the product

$$O(\phi) = O(\phi|_{M_1}) \cdot O(\phi|_{M_2})$$

where $\phi \in L_M$.

# Physical quantities

Physical information can be extracted via evaluation functions as follows.

- A **boundary condition** on $\Sigma$ is given by a boundary solution, i.e., an element of $L_{\partial M}$.
- For a spacetime region $M$ consider the **characteristic function** $e_M : L_{\partial M} \to [0,1]$ where $e_M(\varphi) := 1$ if there exists $\phi \in M$ with $\varphi = \phi|_{\partial M}$ and $e_M(\varphi) := 0$ otherwise.
- For a spacetime region $M$ with **observable** $O$ consider the function $e_M^O : L_{\partial M} \to \mathbb{R}$ given by $e_M^O(\varphi) := O(\phi)$ if $\varphi = \phi|_{\partial M}$ and $e_M^O(\varphi) = 0$ otherwise.

Alternatively, we may consider **statistical ensembles**. That is, a boundary condition is a probability density on the space of boundary solutions. A suitable notion of integration can than be applied to yield evaluations with the functions defined above. (Note that the symplectic form yields a measure.)

# Context and application

Some history:

- the notion of **observable** and the important role of the **symplectic structure** have a very long history
- solutions in bounded manifolds as **Lagrangian submanifolds** are known in the recent literature, see e.g. [expositions by G. Segal]
- For a complete **axiomatic formulation** in the linear case, see [RO 2010]. This served also as a starting point for **quantization**.

The case of **statistical field theory** has not been worked out, however this could be a promising route towards the longstanding problem of a **statistical treatment** of the **general theory of relativity**.

- There are technical challenges concerning measure theory in infinite dimensional spaces.
- It is likely necessary to represent observables as densities and boundary conditions as functions rather than the other way round.
- Evaluation functions will likely not directly yield physical quantities, but quotients of them will. This foreshadows features of the **quantum theory**.

# Quantum theory

The formulation of quantum theory in terms of the present framework is called the

General boundary formulation of quantum theory

So far, there exist two versions of this:

- The **amplitude formalism**
- The **positive formalism**

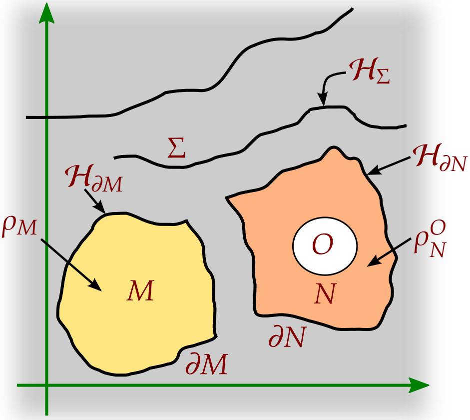
# Amplitude formalism

The **amplitude formalism** is based on an extrapolation of the ingredients of the standard formulation of quantum theory to a spacetime context. That is, **Hilbert spaces** and **quantum observables** are taken to live on spacetime **hypersurfaces** and **regions** respectively.

The mathematical framework that accomplishes this is called **topological quantum field theory** [Witten, Segal, Atiyah, . . . 1988–]. Inspired by the Feynman path integral, this lead to a revolution in algebraic topology, low dimensional topology and knot theory starting in the second half of the 1980's.

- anticipated in [Dirac 1923]
- adopted as central piece of the GBF program [RO 2003]
- complemented with probability interpretation [RO 2005]
- added quantum observables [RO 2010]
- spacetime composition of quantum observables [RO 2012]

# Amplitude formalism – structures

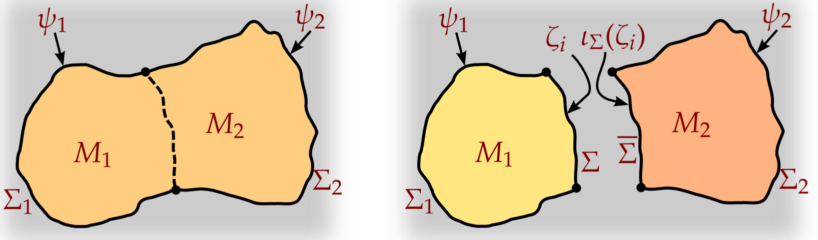



To the geometric structures associate the quantum data,

- per **hypersurface** $\Sigma$ : a **Hilbert space** $\mathcal{H}_\Sigma$,
- per **region** $M$ : a linear **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$,
- per **region** $M$ that contains an **observable** $O$ : a linear **observable map** $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$ .

# Amplitude formalism – TQFT axioms

- Let $\overline{\Sigma}$ denote $\Sigma$ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_{\Sigma} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If $M_1$ and $M_2$ are adjacent regions, then:




$$\rho_{M_1 \cup M_2}(\psi_1 \otimes \psi_2) \cdot c_{M_1,M_2} = \sum_{i \in N} \rho_{M_1}(\psi_1 \otimes \zeta_i)\rho_{M_2}(\iota_\Sigma(\zeta_i) \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\zeta_i\}_{i \in N}$ is an ON-basis of $\mathcal{H}_\Sigma$.

# Amplitude formalism – physical quantities

- A **boundary condition** for a region $M$ is a subspace $\mathcal{S}$ of the boundary Hilbert space $\mathcal{H}_{\partial M}$. These are **partially ordered** by inclusion.
- The amplitude maps $\rho_M$ can be used to define **probability maps**,

$$A_M(\mathcal{S}) := \sum_{i\in I} |\rho_M(\xi_i)|^2, \quad \text{where } \{\xi_i\}_{i\in I} \text{ is an ON-basis of } \mathcal{S}$$

  These play the role of **evaluation maps**.
- A probability map does not directly yield physical quantities, but **quotients** do, giving **conditional probabilities**.
- Observable maps similarly can be used to construct **expectation maps**. Quotients then yield **conditional expectation values**.
- The necessity for quotients of evaluation maps resembles the situation in **classical statistical physics**.

# Some applications

- Conceptual basis for **spin foam approach** to quantum gravity (sometimes secretly so)
- Non-linear models:
  - **Three dimensional quantum gravity** is a TQFT and fits "automatically". [Witten 1988;. . . ]
  - **Quantum Yang-Mills theory** in 2 dimensions for arbitrary regions and hypersurfaces with corners. [RO 2006]
  - **Yang-Mills theory in higher dimensions** is under investigation [H. Diaz 2013]
- New **S-matrix** type asymptotic amplitudes [Colosi, RO 2008; Colosi 2009; Dohse 2011; 2012]
- QFT in **curved spacetime**: dS, AdS and more [Colosi, Dohse 2009–]
- Rigorous and functorial quantization of linear and affine field theories without metric background. [RO 2010; 2011; 2012]
- **Unruh effect**. [Colosi, Rätzel 2012; Bianchi, Haggard, Rovelli 2013]
- Striking results for **fermions**: Hilbert spaces become **Krein spaces** and an **emergent notion of time**. [RO 2012]

# Mixed states and quantum operations

In the standard formulation, if we want to do **statistical physics**, introduce **thermodynamic notions** etc. we need **mixed states** rather than **pure states**. Mixed states live in the space $\mathcal{D}$ of **operators** on the original Hilbert space $\mathcal{H}$. All the dynamics of quantum theory can be expressed directly in terms of $\mathcal{D}$ and operations on it.

On the other hand, the observer in quantum physics has a **special role**, not analogous to the classical one. This cannot be fully captured through the notion of quantum observable. Rather, describing this role requires the more general notion of **quantum operation**. This leads into the arena of **quantum information theory**.

# The positive formalism

Suppose that in the amplitude formalism we systematically replace for each hypersurface $\Sigma$ the Hilbert (or Krein) space $\mathcal{H}_\Sigma$ with the corresponding space of operators $\mathcal{D}_\Sigma$.

At the same time replace the amplitude maps per region $M$ by **probability maps** $A_M : \mathcal{D}_{\partial M} \to \mathbb{C}$,

$$A_M(\sigma) := \sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M(\sigma(\xi_i)) \,.$$

# The positive formalism

Suppose that in the amplitude formalism we systematically replace for each hypersurface $\Sigma$ the Hilbert (or Krein) space $\mathcal{H}_\Sigma$ with the corresponding space of operators $\mathcal{D}_\Sigma$.

At the same time replace the amplitude maps per region $M$ by **probability maps** $A_M : \mathcal{D}_{\partial M} \to \mathbb{C}$,

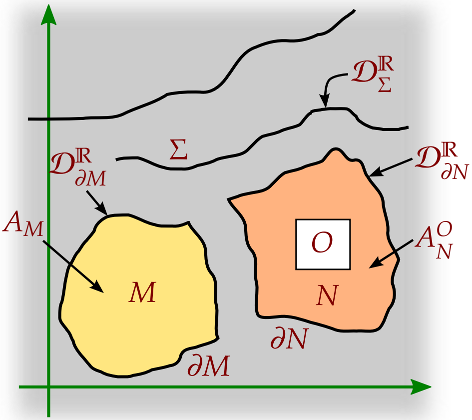
$$A_M(\sigma) := \sum_{i \in I} \overline{\rho_M\left(\xi_i\right)}\, \rho_M\left(\sigma(\xi_i)\right).$$

Remarkably, the new objects satisfy axioms very similar to the old ones.

Now, forget operators and restrict to real parts. Fundamentally, the spaces $\mathcal{D}_\Sigma$ are just **ordered vector spaces** and the probability maps $A_M$ are **positive maps** on them. This is the **positive formalism**. [RO 2012]

# Spacetime assignments

## Positive formalism




To the geometric structures associate the quantum data,

- per **hypersurface** $\Sigma$ : an **ordered vector space** $\mathcal{D}^{\mathbb{R}}_{\Sigma}$,
- per **region** $M$ : a positive **probability map** $A_M : \mathcal{D}^{\mathbb{R}}_{\partial M} \to \mathbb{R}$,
- per **region** $M$ that contains an **operation** $O$ : an **operation map** $A^O_M : \mathcal{D}^{\mathbb{R}}_{\partial M} \to \mathbb{R}$ .

# Positive formalism – physical quantities

- A **boundary condition** for a region $M$ is an element of $\mathcal{D}_\Sigma$. These are **partially ordered**.
- Given boundary conditions $\mathcal{A} \leq \mathcal{S} \in \mathcal{D}_{\partial M}$ the quotient

$$\frac{A_M(\mathcal{A})}{A_M(\mathcal{S})}$$

  is the **conditional probability** for $\mathcal{A}$ to be realized given that $\mathcal{S}$ is realized.
- The expected outcome of a **quantum operation** $O$ in a spacetime region $M$ given a boundary condition $\mathcal{S}$ is given by,

$$\frac{A_M^O(\mathcal{S})}{A_M(\mathcal{S})}.$$

- We can also **compare quantum operations** with one another.

# Advantages of the positive formalism

- Focuses on **operationally relevant information** and eliminates **unphysical data** (phases etc.).
- Probability interpretation simpler and conceptually clearer.
- All models expressed in the amplitude formalism can be translated into the positive formalism (functorially).
- Admits general **quantum operations** and opens the GBF to **quantum information theory**.
- More freedom to formulate quantum theories and quantization schemes.
- Appears necessary to overcome the **state locality problem** in QFT. [RO 2013]
- Probably simplifies and clarifies spin foam approach to quantum gravity.