Loop quantization of spherically symmetric vacuum spacetimes

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Motivation

1) Spherically symmetric spacetimes:
   a) Black hole physics: local singularity, evaporation (Hawking radiation)
   b) Gravitational collapse (with a matter field)
2) Previous attempts
   a) Kuchař's quantization (superposition masses)
   b) Interior of the black hole (Kantowski-Sachs)
   c) Exterior of the black hole (gauge fixing)

Classical system and Ashtekar variables

1) The Ashtekar variables adapted to a spherically symmetric spacetime, are given by

$$
\begin{aligned}
A &= A_a^i \tau_i dx^a = A_x(x)\tau_3 dx + [A_1(x)\tau_1 + A_2(x)\tau_2]d\theta \\
&\quad + [A_1(x)\tau_2 - A_2(x)\tau_1]\sin\theta d\phi + \tau_3 \cos\theta d\phi, \\
E &= E_i^a \tau^i \partial_a = \sin\theta \Big(E^x(x)\tau_3\partial_x + [E^1(x)\tau_1 + E^2(x)\tau_2]\partial_\theta \Big) \\
&\quad + [E^1(x)\tau_2 - E^2(x)\tau_1]\partial_\phi,
\end{aligned}
$$

where τ_i are the generators of $SU(2)$ (i.e. $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$ with ϵ_{ijk} the totally antisymmetric tensor). Setting $\gamma = 1$, the Poisson algebra is given by

$$
\begin{aligned}
\{A_x(x), E^x(x')\} &= 2G\delta(x - x'), \\
\{A_i(x), E^j(x')\} &= G\delta_i^j \delta(x - x'), \quad i,j = 1,2,
\end{aligned}
$$

Classical system: polar coordinates

2) One first introduces polar coordinates, i.e.,

$$E^1 = E^\varphi \cos(\alpha+\beta), \qquad E^2 = E^\varphi \sin(\alpha+\beta),$$
$$A_1 = A_\varphi \cos\beta, \qquad A_2 = A_\varphi \sin\beta,$$

and completes the canonical transformation defining

$$\eta = \alpha+\beta, \quad P^\eta = A_\varphi E^\varphi \sin\alpha = 2A_1E^2 - 2A_2E^1,$$
$$\bar{A}_\varphi = 2A_\varphi \cos\alpha.$$

Finally, the transformation

$$\bar{A}_x = A_x + \eta', \quad \bar{P}^\eta = P^\eta + (E^x)',$$

allows one to simplify the treatment of the pure gauge canonical pair η and $\bar{P}^\eta$. In the following we will set the second class condition $\eta = 0$ (gauge fixing the Gauss constraint $P^\eta = 0$).

Classical constraints

3) Within this gauge fixing, $2K_x = \bar{A}_x$ and $2K_\varphi = \bar{A}_\varphi$. The Hamiltonian is a linear combination of the constraints

$$H := \frac{((E^x)')^2}{8\sqrt{E^x}E^\varphi} - \frac{E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''}{2E^\varphi}, \quad H_r := E^\varphi K_\varphi' - (E^x)'K_x \,.$$

fulfilling the algebra

$$\{H_r(N_r), H_r(\tilde{N}_r)\} = H_r(N_r\tilde{N}_r' - N_r'\tilde{N}_r), \; \{H(N), H_r(N_r)\} = H(N_r N'),$$
$$\{H(N), H(\tilde{N})\} = H_r\left(\frac{E^x}{(E^\varphi)^2}\left[N\tilde{N}' - N'\tilde{N}\right]\right).$$

New constraint algebra

3) In order to write the scalar constraint as a total derivative, we ”gauge” and scale it as

$$H_{new} := \frac{(E^x)'}{E^\varphi} H_{old} - 2\frac{\sqrt{E^x}}{E^\varphi} K_\varphi H_r = \left[\sqrt{E^x}\left(1 - \frac{[(E^x)']^2}{4(E^\varphi)^2} + K_\varphi^2\right)\right]'$$

Now, smearing with the lapse integrating by parts and scaling with E^φ (together with appropriate boundary conditions)

$$H(N) = \int dx N\left(\sqrt{E^x} E^\varphi \left(1 + K_\varphi^2\right) - 2GME^\varphi - \frac{[(E^x)']^2\sqrt{E^x}}{4E^\varphi}\right),$$

The new constraint algebra is

$$\{H_r(N_r), H_r(\tilde{N}_r)\} = H_r(N_r\tilde{N}_r' - N_r'\tilde{N}_r),\ \{H(N), H_r(N_r)\} = H(N_rN'),$$
$$\{H(N), H(\tilde{N})\} = 0.$$

Kinematical Hilbert space

1) Spin networks

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \prod_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} dx\, K_x(x)\right) \prod_{v_j \in g} \exp\left(\frac{i}{2} \mu_j K_\varphi(v_j)\right),$$

$k_j \in \mathbb{Z}$ is the valence associated with the edge e_j, and $\mu_j \in \mathbb{R}$ the valence associated with the vertex v_j

2) Kinematical Hilbert space

$$\mathcal{H}^B_{\text{kin}} = \mathcal{H}^m_{\text{kin}} \otimes \left[\bigotimes_{j=1}^{V} \ell^2_j \otimes L^2_j(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})\right],$$

which is endowed with the inner product

$$\langle g, \vec{k}, \vec{\mu}, M | g', \vec{k}', \vec{\mu}', M' \rangle = \delta(M - M') \delta_{\vec{k},\vec{k}'} \delta_{\vec{\mu},\vec{\mu}'} \delta_{g,g'} \;.$$

Kinematical Hilbert space

3) Operator representation: mass and triads

$$\hat{M}|g,\vec{k},\vec{\mu},M\rangle = M|g,\vec{k},\vec{\mu},M\rangle,$$
$$\hat{E}^x(x)|g,\vec{k},\vec{\mu},M\rangle = \ell_{\rm Pl}^2 k_j|g,\vec{k},\vec{\mu},M\rangle,$$
$$\hat{E}^\varphi(x)|g,\vec{k},\vec{\mu},M\rangle = \ell_{\rm Pl}^2 \sum_{v_j\in g}\delta\big(x-x_j\big)\mu_j|g,\vec{k},\vec{\mu},M\rangle,$$

4) Holonomies (of K_φ) of length ρ

$$N^\varphi_{\pm n\rho}(x)|g,\vec{k},\vec{\mu},M\rangle = |g,\vec{k},\vec{\mu}'_{\pm n\rho},M\rangle, \quad n\in\mathbb{N},$$

here $\vec{\mu}'_{\pm n\rho}$ either has just the same components than $\vec{\mu}$ up to $\mu_j \to \mu_j \pm n\rho$ if x coincides with a vertex of the graph located at x_j, or $\vec{\mu}'_{\pm n\rho}$ will be $\vec{\mu}$ with a new component $\{\ldots,\mu_j,\pm n\rho,\mu_{j+1},\ldots\}$ with $x_j < x < x_{j+1}$.

Representation of the scalar constraint

The scalar constraint will be promoted to

$$\hat{H}(N) = \int dx N(x) \sqrt{\hat{E}^x}$$
$$\times \left(\hat{\Theta} \sqrt{\hat{E}^x} + \hat{E}^\varphi \sqrt{\hat{E}^x} - \frac{1}{4} \widehat{\left[\frac{1}{\hat{E}^\varphi} \right]} \left[(\hat{E}^x)' \right]^2 \sqrt{\hat{E}^x} - 2G\hat{M}\hat{E}^\varphi \right),$$

1) The operator $\hat{\Theta}(x)$ acting on the kinematical states

$$\hat{\Theta}(x)|g, \vec{k}, \vec{\mu}, M\rangle = \sum_{v_j \in g} \delta(x - x(v_j)) \hat{\Omega}^2_\varphi(v_j)|g, \vec{k}, \vec{\mu}, M\rangle,$$

$$\hat{\Omega}_\varphi(v_j) = \frac{1}{4i\rho} |\hat{E}^\varphi|^{1/4} \left[\widehat{\text{sgn}(E^\varphi)} \left(\hat{N}^\varphi_{2\rho} - \hat{N}^\varphi_{-2\rho} \right) \right.$$
$$\left. + \left(\hat{N}^\varphi_{2\rho} - \hat{N}^\varphi_{-2\rho} \right) \widehat{\text{sgn}(E^\varphi)} \right] |\hat{E}^\varphi|^{1/4} \Big|_{v_j},$$

Representation of the scalar constraint

2) Besides

$$|\hat{E}^\varphi|^{1/4}(v_j)|g,\vec{k},\vec{\mu},M\rangle = \ell_{\rm Pl}^{1/2}|\mu_j|^{1/4}|g,\vec{k},\vec{\mu},M\rangle,$$

$$\widehat{\mathrm{sgn}\big(E^\varphi(v_j)\big)}|g,\vec{k},\vec{\mu},M\rangle = \mathrm{sgn}(\mu_j)|g,\vec{k},\vec{\mu},M\rangle,$$

$$\widehat{\left[\frac{1}{\hat{E}^\varphi}\right]}|g,\vec{k},\vec{\mu},M\rangle =$$

$$= \sum_{v_j\in g}\delta(x-x(v_j))\frac{1}{\ell_{\rm Pl}^2\rho^2}(|\mu_j+\rho|^{1/2}-|\mu_j-\rho|^{1/2})^2|g,\vec{k},\vec{\mu},M\rangle,$$

$$(\hat{E}^x)'(v_j)|g,\vec{k},\vec{\mu},M\rangle = \ell_{\rm Pl}^2(k_j-k_{j-1})|g,\vec{k},\vec{\mu},M\rangle.$$

Representation of the scalar constraint

3) The action of the constraint on spin networks

$$\hat{H}(N)|g,\vec{k},\vec{\mu},M\rangle = \sum_{v_j\in g} N(x_j)(\ell_{\rm Pl}^3 k_j)\Big[f_0(\mu_j,k_j,M)|g,\vec{k},\vec{\mu},M\rangle$$
$$-f_+(\mu_j)|g,\vec{k},\vec{\mu}_{+4\rho_j},M\rangle - f_-(\mu_j)|g,\vec{k},\vec{\mu}_{-4\rho_j},M\rangle\Big],$$

If $s_\pm(\mu_j) = {\rm sgn}(\mu_j) + {\rm sgn}(\mu_j \pm 2\rho)$, then

$$f_\pm(\mu_j) = \frac{1}{16\rho^2}|\mu_j|^{1/4}|\mu_j \pm 2\rho|^{1/2}|\mu_j \pm 4\rho|^{1/4}s_\pm(\mu_j)s_\pm(\mu_j \pm 2\rho),$$
$$f_0(\mu_j,k_j,k_{j-1},M) = \frac{1}{16\rho^2}\Big[(|\mu_j||\mu_j+2\rho|)^{1/2}s_+(\mu_j)s_-(\mu_j+2\rho)$$
$$+(|\mu_j||\mu_j-2\rho|)^{1/2}s_-(\mu_j)s_+(\mu_j-2\rho)\Big] + \mu_j\left(1-\frac{2GM}{\ell_{\rm Pl}|k_j|^{1/2}}\right)$$
$$-\frac{{\rm sgn}(\mu_j)}{\rho^2}(k_j-k_{j-1})^2(|\mu_j+\rho|^{1/2}-|\mu_j-\rho|^{1/2})^2,$$

Invariant domain and singularity resolution

1) Invariant domain: i) the constraint does not create new vertices or edges, ii) it preserves the sequences of k_j, and iii) at each vertex, it is a difference operator mixing different μ_j's such that $\mu_j = \epsilon_j \pm 4\rho \mathrm{n}_j$, with $\mathrm{n}_j \in \mathbb{N}$ and $\epsilon_j \in (0, 4\rho]$.

2) Singularity resolution: i) the scalar constraint leaves invariant the subspace of spin networks with non-vanishing k_j and μ_j. ii) Additionally, spin networks with $k_j = 0$ and/or $\mu_j = 0$ can be ruled out by requiring selfadjointness to some metric components (locally).

Solutions to the constraint

1) Solutions: $(\Psi_g| = \int_0^\infty dM \sum_{\vec{k}} \sum_{\vec{\mu}} \langle g, \vec{k}, \vec{\mu}, M | \psi(M) \chi(\vec{k}) \phi(\vec{k}; \vec{\mu}; M)$.

   They are annihilated by the constraint $\sum_{v_j \in g} (\Psi_g | N_j \hat{C}_j^\dagger = 0$, factorizing as $\phi(\vec{k}, \vec{\mu}, M) = \prod_{j=1}^V \phi_j(\mu_j)$, $\phi_j(\mu_j) = \phi_j(k_j, k_{j-1}, \mu_j, M)$.

2) Difference equation: each function $\phi_j(\mu_j)$ satisfies

$$- F_+(\mu_j)\phi_j(\mu_j - 4\rho) - F_-(\mu_j)\phi_j(\mu_j + 4\rho) + f_0(k_j, k_{j-1}, \mu_j, M)\phi_j(\mu_j) = 0.$$

   where $f_\pm(\mu_j) = F_\pm(\mu_j \pm 4\rho)$ vanish on the intervals $[0, \mp 2\rho]$.

3) Asymptotically ($\mu_j \to \infty$) the difference eq. is approximated by

$$- 4\mu_j \partial^2_{\mu_j} \phi - 4\partial_{\mu_j} \phi - \frac{(k_j - k_{j-1})^2 - 1/4}{\mu_j} \phi + \underbrace{\left(1 - \frac{2GM}{\ell_{\rm Pl} |k_j|^{1/2}}\right)}_{=\tilde{\omega}} \mu_j \phi = 0.$$

Solutions to the constraint

3) The sign of $\tilde{\omega}$ (exterior or interior of the black hole) influences the asymptotic behavior of ϕ_j:

a) If $\tilde{\omega} < 0$ the constraint $\hat{C}_j^{\text{in}} = \hat{\mathcal{C}}_j^{\text{in}} + \left(1 - \frac{2GM}{\ell_{\text{Pl}}|k_j|^{1/2}}\right)$ adopts this form on the representation $\hat{C}_j^{\text{in}} = \hat{\mu}_j^{-1/2} \hat{C}_j \hat{\mu}_j^{-1/2}$. If $\tilde{\omega} < 0$. It can be diagonalized as $\omega_j + \left(1 - \frac{2GM}{\ell_{\text{Pl}}|k_j|^{1/2}}\right) = 0$, with

$$\hat{\mathcal{C}}_j^{\text{in}} |\phi_{\omega_j}^{\text{in}}\rangle = \omega_j |\phi_{\omega_j}^{\text{out}}\rangle, \quad \langle \phi_{\omega_j}^{\text{in}} | \phi_{\omega_j'}^{\text{in}} \rangle = \delta\left(\sqrt{\omega_j} - \sqrt{\omega_j'}\right),$$

and ω_j belonging to the positive real line (nondegenerated).

Solutions to the constraint

b) If $\tilde{\omega} > 0$, the constraint $\hat{C}_j^{\text{out}} = \hat{C}_j^{\text{out}} - (k_j - k_{j-1})^2$ adopts this form on the representation

$$\hat{C}_j^{\text{out}} = \widehat{\left[\frac{1}{\mu_j}\right]}^{-1/2} \hat{C}_j \widehat{\left[\frac{1}{\mu_j}\right]}^{-1/2}, \quad \hat{b}(\mu_j) = \frac{1}{\rho}(|\hat{\mu}_j + \rho|^{1/2} - |\hat{\mu}_j - \rho|^{1/2}).$$

The constraint equation reads $\lambda_n(M, k_j, \epsilon_j) - \Delta k_j^2 = 0$, with

$$\hat{C}_j^{\text{out}} |\phi_{\lambda_j}^{\text{out}}\rangle = \lambda_j |\phi_{\lambda_j}^{\text{out}}\rangle, \quad \langle \phi_{\lambda_n(\epsilon_j)}^{\text{out}} | \phi_{\lambda_{n'}(\epsilon_j)}^{\text{out}} \rangle = \delta_{nn'}.$$

$\lambda_n(M, k_j, \epsilon_j)$ is a sequence ($n \in \mathbb{N}$) of positive real numbers depending continuously on ϵ_j. Therefore, we expect that the positive real line would be completely covered (future research).

Physical Hilbert space

1) Group averaging: $(\Psi_g^C| = \int d\alpha_1 \cdots d\alpha_V \ \exp\left\{\sum_{j=1}^{V} i\alpha_j \hat{C}_j^\dagger\right\}(\Psi_g|$.

Equivalently, on the representation of τ (canonically conjugate of M)

$$\Psi_g^C(\vec{k},\vec{\mu};\tau) = \frac{2G}{\ell_{\rm Pl}\sqrt{k_j}} \int_0^\infty d\omega_j \psi(\omega_j)\chi(\vec{k})\phi^{\rm in}_{\vec{\omega}(\omega_j)}(\vec{\mu})e^{iM(\omega_j)\tau}$$

$$+ \sum_{\vec{\lambda}_n(\omega_j)} \psi(\vec{\lambda}_n(\omega_j))\chi(\vec{k})\phi^{\rm out}_{\vec{\lambda}_n(\omega_j)}(\vec{\mu})e^{iM(\vec{\lambda}_n(\omega_j))\tau},$$

2) Normalization $\|\Psi_g^C(\tau_0)\|^2 = \sum_{\vec{k}}\sum_{\vec{\mu}} |\Psi_g^C(\vec{k},\vec{\mu};\tau_0)|^2 < \infty$, i.e., the inner product is $\langle g,\vec{k},\vec{\mu}|g',\vec{k}',\vec{\mu}'\rangle = \delta_{\vec{k},\vec{k}'}\delta_{\vec{\mu},\vec{\mu}'}\delta_{g,g'}$.

Physical Hilbert space

3) Standard group averaging with the diffeomorphism constraint: rigging map

$$\eta : \mathrm{Cyl} \to \mathrm{Cyl}^*_{\mathrm{Diff}},$$

It induces the inner product $\langle \eta(\Psi) | \eta(\Phi) \rangle = \langle \eta(\Psi) | \Phi \rangle$, and yields the Hilbert space

$$\mathcal{H}_{\mathrm{Diff}} = \oplus_{[g]} \mathcal{H}_{[g],\mathrm{Diff}},$$

4) Observables: the model is characterized by the mass M (boundary), and on the bulk by the number of vertices V and the new observable

$$\hat{O}(z)\Psi_{\mathrm{phys}} = \ell_{\mathrm{Pl}}^2 k_{\mathrm{Int}(Vz)} \Psi_{\mathrm{phys}}, \quad z(x) : [0, x] \to [0, 1]$$

with $z(x)$ any monotonic function.

Conclusions and outlook

1) We quantize an spherically symmetric spacetime:
   a) We consider Ashtekar variables and a suitable modification of the classical constraint algebra.
   b) We adopt a loop representation together with the Dirac quantization scheme.
2) We find explicitly the solutions, construct the physical Hilbert space and provide the observables (some of them without classical analog).
3) Meticulous analytical and numerical study of the spectrum of some geometrical operators, as well as semiclassical geometries.
4) Study of the effects of the discrete geometry on Hawking radiation and extension to other classical models: CGHS (vacuum), grav. collapse (coupled matter), Gowdy, etc.