Large $j$ behaviour of dipole cosmology transition amplitudes

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Tux, 27th of February 2013
EFI winter conference on canonical and covariant loop quantum gravity

The project „International PhD Studies at the Faculty of Physics, University of Warsaw” is realized within the MPD programme of Foundation for Polish Science, cofinanced from European Union, Regional Development Fund
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Aim of the research and motivation

The aim

to investigate the properties of Lorentzian Polyhedra Propagator:

\[ T = \int_{SL(2,\mathbb{C})} d g Y^\dagger g Y \]

in large \( j \) limit
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- DC amplitude is proportional to \( \int dg \prod_{i=1}^{4} \langle \vec{n}_i | Y^\dagger gY | \vec{n}_i \rangle_j \)

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Motivation

- DC amplitude is proportional to $\int dg \prod_{i=1}^4 \langle \vec{n}_i | Y^\dagger g Y | \vec{n}_i \rangle_j$

- 2-vertex-and-1-edge DC:
  $\int dg dg' \prod_{i=1}^4 \langle \vec{n}_i | Y^\dagger g Y Y^\dagger g' Y | \vec{n}_i \rangle_j$ [ JP, in progress]
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- radiative corrections of "melonic" graph proportional to 
  \[ \log \Lambda \cdot T^2 \]  [ Riello: arXiv:1302.1781 (2013)]
The SPA method

We estimate the following integral for $\Lambda \gg 1$ by the value of integrand in the critical point $x_0$ of $f(x)$

$$\int dx \ g(x)e^{-\Lambda f(x)} = \sqrt{\frac{(2\pi)}{\Lambda |f''(x_0)|}} g(x_0)e^{-\Lambda f(x_0)}$$  \hspace{1cm} (1)
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(1)

for multidimensional integrals:

$$\int d^n x \ g(x) e^{-\Lambda f(x)} = \sqrt{\left(\frac{2\pi}{\Lambda}\right)^n \left(\left|\frac{\partial^2 f}{\partial x^2}\right|_{x_0}\right)^{-1/2}} g(x_0) e^{-\Lambda f(x_0)}$$

(2)
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Note!

$\nabla f(x)$ must vanish in $x_0$, i.e. $x_0$ must not be the extremum, where $\nabla f(x)$ is discontinues.
Not obvious decomposition of integrand

What if the function we integrate does not have an obvious decomposition into $g(x)e^{-\Lambda f(x)}$?
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Consider \( F(\Lambda) = \int d^x \Phi(x, \Lambda) \). Let’s assume, that the integrand \( \Phi(x, \Lambda) \) has appropriate asymptotic behaviour, but we don’t know its decomposition into \( f(x) \) and \( g(x) \). In such a case we need to investigate the function

\[
\phi(x) := \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln(\Phi(x, \Lambda))
\]  

(3)
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we will call it *the exponent part* of the integrand.

The SPA formula will be true for critical points and Hessian matrix of \( \phi(x) \).
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\[ f_m^{(j)}(\eta) \]
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Definition

Def: Lorentzian polyhedra propagator

Given a set of spins $j_1, \ldots, j_N$ we define an operator

$$T := \int_{SL(2,\mathbb{C})} dg \ [Y^\dagger g Y]^{(j_1 \otimes \cdots \otimes j_N)}$$

acting on $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$
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acting on $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$

We can consider its matrix elements in the $|m\rangle_j$ basis:

$$T_{m_1' \cdots m_N'}^{m_1 \cdots m_N} := \int_{SL(2, \mathbb{C})} \text{d}g \langle m_1, \ldots, m_N | Y^\dagger g Y | m_1', \ldots, m_N'\rangle_{j_1 \otimes \cdots \otimes j_N}$$

$$= \int_{SL(2, \mathbb{C})} \text{d}g \prod_{i=1}^N \langle m_i | Y^\dagger g Y | m_i'\rangle_{j_i}$$

$$= \int_{SL(2, \mathbb{C})} \text{d}g \prod_{i=1}^N D^{(\gamma_{j_i}, j_i)}(g)_{j_i, m_i}^{j_i', m_i'}$$
Domain and rank

It is easy to check, that $T$ acts non-trivially only on the invariant subspace of $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$:

$$T = \int_{SL(2,\mathbb{C})} dg \ Y^\dagger gY = \int_{\mathbb{R}^3 \times SU(2)} dk du \ Y^\dagger kuY$$

$$= \int_{\mathbb{R}^3} dk \ Y^\dagger kY \int_{SU(2)} u du = \hat{A} \cdot P_{Inv}$$
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Thus

$$
\mathbb{T} = P_{\text{Inv}} \cdot \hat{B} \cdot P_{\text{Inv}}
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Thus

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$$

So it’s enough to study the matrix elements between the $SU(2)$ invariants:

$$
\mathbb{T}_{\iota \iota'} = \int_{SL(2,\mathbb{C})} dg \langle \iota | Y^\dagger g Y | \iota' \rangle
$$
Symmetries of the integrand 1 - $SU(2)$

We need to integrate the function $\Phi_{\iota\iota'}(g, J) := \langle \iota | Y^\dagger g Y | \iota' \rangle$ on $SL(2, \mathbb{C})$, where $J = \max_{i=1,\ldots,N}(j_i)$. We anticipate that the critical point will be in $g = 1$. Let us study the behaviour of $\Phi$ close to $g = 1$. 

There are six-dimensional basis vector fields on $SL(2, \mathbb{C})$ given by the generators of rotations $J_i$ and generators of boosts $K_i$ ($i = 1, 2, 3$). It's straightforward to see, that $J_i \Phi_{\iota\iota'}(g) \equiv 0$ Indeed:

$J_i$ are $SU(2)$ generators, thus they commute with the $Y$ map, and $J_i | \iota \rangle = 0$, so $\langle \iota | Y^\dagger g J_i Y | \iota' \rangle = \langle \iota | Y^\dagger g Y J_i | \iota' \rangle = 0$
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Indeed: $J_i$ are $SU(2)$ generators, thus they commute with the $Y$ map, and $J_i | \iota \rangle = 0$, so

$$\langle \iota | Y^\dagger gJ_i Y | \iota' \rangle = \langle \iota | Y^\dagger gYJ_i | \iota' \rangle = 0$$
Symmetries of the integrand 2 - z-boost

Let’s now consider a z-boost $Y^\dagger e^{\eta K_3} Y$. Since $[K_3, J_3] = 0$, it’s convenient to consider it in the $|m\rangle_j$ basis.
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Let's now consider a $z$-boost $Y^\dagger e^{\eta K_3} Y$. Since $[K_3, J_3] = 0$, it's convenient to consider it in the $|m\rangle_j$ basis.

Let's define the function $f_{m}^{(j)}(\eta)$

$$\langle m | Y^\dagger e^{\eta K_3} Y | m' \rangle_j =: \delta_{m,m'} f_{m}^{(j)}(\eta)$$

(5)
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$$\langle m | Y^\dagger e^{\eta K_3} Y | m' \rangle_j =: \delta_{m,m'} f^{(j)}_m(\eta)$$

(5)

We decompose the invariant tensors in $|m\rangle_j$ basis

$$|\iota\rangle = \sum_{\{m_i\}} \iota_{m_1 \cdots m_N} |m_1, \ldots, m_N\rangle_{j_1 \otimes \cdots \otimes j_N}$$

and thus

$$\Phi_{\iota,\iota'}(e^{\eta K_3}) = \sum_{\{m_i\}} \iota_{m_1 \cdots m_N} \iota'_{m_1 \cdots m_N} \Phi^{m_1 \cdots m_N}_{m_1 \cdots m_N}(e^{\eta K_3})$$

where $\Phi^{m_1 \cdots m_N}_{m_1 \cdots m_N}(e^{\eta K_3}) = \prod_{i=1}^{N} f^{(j_i)}_{m_i}(\eta)$
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We decompose the invariant tensors in $|m\rangle_j$ basis

$$|\ell\rangle = \sum_{\{m_i\}} \ell_{m_1\cdots m_N} |m_1, \ldots, m_N\rangle_{j_1\otimes\cdots\otimes j_N}$$

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$$\Phi_{\ell,\ell'}(e^{\eta K_3}) = \sum_{\{m_i\}} \ell_{m_1\cdots m_N} \ell'_{m_1\cdots m_N} \Phi_{m_1\cdots m_N}(e^{\eta K_3})$$

where $\Phi_{m_1\cdots m_N}(e^{\eta K_3}) = \prod_{i=1}^{N} f_{m_i}^{(j_i)}(\eta)$

Note that

since $J_i |\ell\rangle = 0$, only the terms with $\sum_{i=0}^{N} m_i = 0$ counts.
Consider now a boost in arbitrary direction $\vec{n}$. 
Symmetries of the integrand 3 - boost direction

Consider now a boost in arbitrary direction \( \vec{n} \).

Since
\[
\exp(\eta \vec{n} \cdot \vec{K}) = \exp(u^{-1} \eta K_3 u) = u^{-1} \exp(\eta K_3) u
\]
for some \( u \in SU(2) \),
Consider now a boost in arbitrary direction $\vec{n}$.

Since

$$e^{\eta \vec{n} \cdot \vec{K}} = e^{u^{-1} \eta K_3 u} = u^{-1} e^{\eta K_3} u$$

for some $u \in SU(2)$, the value of $\Phi_{\nu
u'}(e^{\eta \vec{n} \cdot \vec{K}})$ is given by

$$\Phi_{\nu
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for some \( u \in SU(2) \), the value of \( \Phi_{\nu \nu'}(e^{\eta \vec{n} \cdot \vec{K}}) \) is given by
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\Phi_{\nu \nu'}(e^{\eta \vec{n} \cdot \vec{K}}) = \langle \nu | Y^\dagger u^{-1} e^{\eta K_3} u Y | \nu' \rangle = \langle \nu | u^{-1} Y^\dagger e^{\eta K_3} Y u | \nu' \rangle
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$$= \langle \ell | Y^\dagger e^{\eta K_3} Y | \ell' \rangle$$
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Consider now a boost in arbitrary direction $\vec{n}$.

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$$e^{\eta \vec{n} \cdot \vec{K}} = e^{u^{-1} \eta K_3 u} = u^{-1} e^{\eta K_3 u}$$

for some $u \in SU(2)$, the value of $\Phi_{u' u}(e^{\eta \vec{n} \cdot \vec{K}})$ is given by

$$\Phi_{u' u}(e^{\eta \vec{n} \cdot \vec{K}}) = \langle \ell | Y^{\dagger} u^{-1} e^{\eta K_3} u Y | \ell' \rangle = \langle \ell | u^{-1} Y^{\dagger} e^{\eta K_3} Y u | \ell' \rangle$$

$$= \langle \ell | Y^{\dagger} e^{\eta K_3} Y | \ell' \rangle = \Phi_{u' u}(e^{\eta K_3})$$
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for some $u \in SU(2)$, the value of $\Phi_{\mu\nu}'(e^{\eta \vec{n} \cdot \vec{K}})$ is given by

$$\Phi_{\mu\nu}'(e^{\eta \vec{n} \cdot \vec{K}}) = \langle \nu | Y^\dagger u^{-1} e^{\eta K_3} u Y | \nu' \rangle = \langle \nu | u^{-1} Y^\dagger e^{\eta K_3} Y u | \nu' \rangle = \langle \nu | Y^\dagger e^{\eta K_3} Y | \nu' \rangle = \Phi_{\mu\nu}'(e^{\eta K_3})$$

Thus the behaviour of $\Phi_{\mu\nu}'(e^{\eta K_3})$ and $f_m^{(j)}(\eta)$ is crucial in further calculation.
Strategy of integration 1: parametrisation of $SL(2, \mathbb{C})$

We need to integrate $\int_{SL(2, \mathbb{C})} dg \Phi_{\nu\nu'}(e^\eta(g)K_3)$. Since the integrand depend only on $\eta$, one may be tempted to use the decomposition $g = u_1^{-1} e^{\eta K_3} u_2$ and the measure

$$\int_{SU(2) \times SU(2) \times \mathbb{R}_+} du_1 du_2 \frac{\sinh^2 \eta}{4\pi} d\eta \Phi_{\nu\nu'}(e^{\eta K_3})$$
Strategy of integration 1: parametrisation of $SL(2, \mathbb{C})$

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$$\int_{SU(2) \times SU(2) \times \mathbb{R}_+} d\eta d\Phi_{\nu\nu'}(e^{\eta K_3})$$

however we anticipate that the maximum of $\Phi$ is $\eta = 0$, where the parametrisation breaks down.
Strategy of integration 1: parametrisation of $SL(2, \mathbb{C})$

We need to integrate $\int_{SL(2,\mathbb{C})} dg \; \Phi_{\nu\nu} (e^{\eta(g)K_3})$.
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$$\int_{SU(2) \times SU(2) \times \mathbb{R}_+} d\eta_1 d\eta_2 \frac{\sinh^2 \eta}{4\pi} d\eta \Phi_{\nu\nu} (e^{\eta K_3})$$

however we anticipate that the maximum of $\Phi$ is $\eta = 0$, where the parametrisation breaks down.

Thus let’s parametrise $SL(2, \mathbb{C})$ with $(u, \vec{x}) \in SU(2) \times \mathbb{R}^3$:

$$g(u, \vec{x}) := un_{\vec{x}}^{-1} e^{||\vec{x}||K_3} n_{\vec{x}}$$

where $n_{\vec{x}} \in SU(2)$ is such a rotation, that $n_{\vec{x}}^{-1}||\vec{x}||L_3 n_{\vec{x}} = \vec{x} \cdot \vec{L}$, i.e.

$$n_{\vec{x}} = \begin{pmatrix} \cos \frac{\theta(\vec{x})}{2} & -e^{i\phi(\vec{x})} \sin \frac{\theta(\vec{x})}{2} \\ e^{-i\phi(\vec{x})} \sin \frac{\theta(\vec{x})}{2} & \cos \frac{\theta(\vec{x})}{2} \end{pmatrix}$$
Strategy of integration 2: measure

\[ dg = \frac{\sinh^2 \eta}{4\pi} d\eta = d\mu d^3x \mu(\vec{x}) \]
Strategy of integration 2: measure

\[ dg = du_1 \, du_2 \, \frac{\sinh^2 \eta}{4\pi} \, d\eta = du \, d^3x \, \mu(\vec{x}) \]

\[ = du \left( \frac{1}{(4\pi)^2} \right) d\phi(\vec{x}) \sin \theta(\vec{x}) d\theta(\vec{x}) \sinh^2 \eta(\vec{x}) d\eta(\vec{x}) \]
Strategy of integration 2: measure

\[
dg = du_1 \, du_2 \, \frac{\sinh^2 \eta}{4\pi} \, d\eta = du \, d^3x \, \mu(x)
\]

\[
= du \, \frac{1}{(4\pi)^2} \, d\phi(x) \sin \theta(x) \, d\theta(x) \sinh^2 \eta(x) \, d\eta(x)
\]

Putting \( d^3x = |x|^2 \sin \theta \, d\theta \, d\phi \) and \( \eta = |x| \) we get

\[
\mu(x) = \left( \frac{\sinh \eta}{4\pi \eta} \right)^2
\]
Strategy of integration 2: measure

\[ dg = du_1 \, du_2 \, \frac{\sinh^2 \eta}{4\pi} \, d\eta = du \, d^3x \, \mu(\vec{x}) \]
\[ = du \, \frac{1}{(4\pi)^2} d\phi(\vec{x}) \sin \theta(\vec{x}) d\theta(\vec{x}) \sinh^2 \eta(\vec{x}) d\eta(\vec{x}) \]

Putting \( d^3x = |\vec{x}|^2 \sin \theta d\theta d\phi \) and \( \eta = |\vec{x}| \) we get

\[ \mu(\vec{x}) = \left( \frac{\sinh \eta}{4\pi \eta} \right)^2 \]

Note that

There are two properties of the measure important for the SPA method:

- \( \lim_{\vec{x} \to 0} \mu(\vec{x}) = \frac{1}{(4\pi)^2} \)

- \( \lim_{J \to \infty} \frac{1}{J} \ln [\mu(\vec{x})] = 0 \) - thus it does not effect the behaviour of the exponent part of the integrand.
Strategy of integration 3: SPA and Hessian

We expect the critical point to be $\vec{x}_0 = 0$. If the exponent part $\phi_{\mu\nu}'(\vec{x})$ of the integrand $\Phi_{\mu\nu}'(\vec{x})$ is smooth (and so is its derivative) in $\vec{x}_0$, the integral equals

$$T_{\mu\nu} = \int_{SL(2,\mathbb{C})} \mathrm{d}g \Phi_{\mu\nu}'(g) = \left(\frac{2\pi}{J}\right)^{3/2} \frac{1}{\sqrt{|H|}} \mu(0) \Phi_{\mu\nu}'(0) \left(1 + O\left(\frac{1}{J}\right)\right)$$

where $H$ is the Hessian matrix of $\phi_{\mu\nu}'(\vec{x}) := \lim_{J \to \infty} \frac{1}{J} \ln \Phi_{\mu\nu}'(\vec{x})$.
We expect the critical point to be $\vec{x}_0 = 0$. If the exponent part $\phi_{\mu'}(\vec{x})$ of the integrand $\Phi_{\mu'}(\vec{x})$ is smooth (and so is its derivative) in $\vec{x}_0$, the integral equals

$$T_{\mu'} = \int_{SL(2,\mathbb{C})} dg \frac{\Phi_{\mu'}(g)}{\sqrt{H}} = \left( \frac{2\pi}{J} \right)^{\frac{3}{2}} \frac{1}{|H|^{\frac{1}{2}}} \mu(0)\Phi_{\mu'}(0) \left( 1 + O\left( \frac{1}{J} \right) \right)$$

where $H$ is the Hessian matrix of $\phi_{\mu'}(\vec{x}) := \lim_{J \to \infty} \frac{1}{J} \ln \Phi_{\mu'}(\vec{x})$

**Hessian of $\phi$**

Since $\phi_{\mu'}$ is spherically symmetric (so it is a function of one variable $\eta$), we can express it’s Hessian for $\eta \to 0$ as

$$\det [H_{ij}]_{\eta=0} = \det \left[ \frac{1}{2} \left( \partial_i \partial_j \eta^2 \right) \frac{d^2 \phi}{d\eta^2} \right] = \det \left[ \delta_{ij} \frac{d^2 \phi}{d\eta^2} \right] = \left( \frac{d^2 \phi}{d\eta^2} \right)^3$$
The Lorentzian polyhedra propagator

Recall now, that

\[ \Phi_{\ell\ell'}(0) = \langle \ell | Y^\dagger e^{0\cdot K_3} Y | \ell' \rangle = \langle \ell | \ell' \rangle = \delta_{\ell\ell'} \]
Recall now, that

\[ \Phi_{\nu \nu'}(0) = \langle \nu | Y^\dagger e^{0 \cdot K_3} Y | \nu' \rangle = \langle \nu | \nu' \rangle = \delta_{\nu \nu'} \]

So neglecting the \( \frac{1}{j} \) terms the operator \( T \) is

\[ T_{\nu \nu'} = \left( \frac{2\pi}{J} \right)^{\frac{3}{2}} \left( \frac{d^2 \phi_{\nu \nu'}}{d\eta^2} \right)^{-\frac{3}{2}} \frac{1}{(4\pi)^2} \delta_{\nu \nu'} \]
Recall now, that

$$\Phi_{\mathbb{U},\mathbb{U}'}(0) = \langle \mathbb{U} | Y^\dagger e^{0 \cdot \mathbf{K}_3} Y | \mathbb{U}' \rangle = \langle \mathbb{U} | \mathbb{U}' \rangle = \delta_{\mathbb{U},\mathbb{U}'}$$

So neglecting the $\frac{1}{J}$ terms the operator $\mathbb{T}$ is

$$\mathbb{T}_{\mathbb{U},\mathbb{U}'} = \left( \frac{2\pi}{J} \right)^{\frac{3}{2}} \left( \frac{d^2 \Phi_{\mathbb{U},\mathbb{U}'}(\eta)}{d\eta^2} \right)^{-\frac{3}{2}} \frac{1}{(4\pi)^2} \delta_{\mathbb{U},\mathbb{U}'}$$

Now we will check smoothness of the exponent part of integrand, and calculate the Hessian.
Outline

Introduction
  Motivation
  Saddle Point Approximation
The lorentzian polyhedra propagator - preliminary analysis
  Definition and basic properties
  Symmetries of the integrand
  Strategy of integration
Integrand’s smoothness check
  \( f_m^{(j)}(\eta) \)
  \( \frac{1}{j} \ln\left(f_m^{(j)}(\eta)\right) \)
  \( \phi_m(\eta, J) \)
  Hessian
Results and summary
  Result
  Summary
Strategy

We are going to investigate the function

$$\phi_{\tilde{m}}(\eta, J) = \frac{1}{J} \ln \left( \prod_{i=1}^{N} f_{m_i}^{(j_i)}(\eta) \right) = \sum_{i=1}^{N} \frac{1}{J} \ln \left( f_{m_i}^{(j_i)}(\eta) \right)$$

in the limit $J \gg 1$ and $\eta \ll 1$. 
Strategy

We are going to investigate the function

$$\phi_m(\eta, J) = \frac{1}{J} \ln \left( \prod_{i=1}^{N} f_{m_i}^{(j)}(\eta) \right) = \sum_{i=1}^{N} \frac{1}{J} \ln \left( f_{m_i}^{(j)}(\eta) \right)$$

in the limit $J \gg 1$ and $\eta \ll 1$.

We will do it by finding a compact form of $f_{m_i}^{(j)}(\eta)$ and analysing its Taylor series.
The $f_m^{(j)}(\eta)$ function

$$f_m^{(j)}(\eta) = \langle m | Y^\dagger e^{\eta K_3} Y | m \rangle_j = D^{(\gamma j, j)}(e^{\eta K_3})_{j, m}$$
The $f^{(j)}_m(\eta)$ function

$$f^{(j)}_m(\eta) = \langle m | Y^\dagger e^{\eta K_3} Y | m \rangle_j = D(\gamma j, j) (e^{\eta K_3})_{j, m}^{j', m}$$

Thanks to $Y$ maps, which make the parameters $(p, k)$ of primary series dependent on $j$, and which pick the lowest spin subspace of $(p, k)$, these matrix elements has rather simple form
The $f_m^{(j)}(\eta)$ function

$$f_m^{(j)}(\eta) = \langle m | Y^\dagger e^{\eta K_3} Y | m \rangle_j = D^{(\gamma j, j)}(e^{\eta K_3})_{j, m}$$

Thanks to $Y$ maps, which make the parameters $(p, k)$ of primary series dependent on $j$, and which pick the lowest spin subspace of $(p, k)$, these matrix elements has rather simple form

$$f_m^{(j)}(\eta) = (2j + 1) \left( \begin{array}{c} 2j \\ j + m \end{array} \right) e^{-m\eta} e^{i\gamma j \eta} e^{-(j+1)\eta}$$

$$\int_0^1 dx \, x^{j+m}(1-x)^{j-m} \left( 1 - (1 - e^{-2\eta}) x \right)^{i\gamma j -(j+1)}$$

[ Ruhl (1970)]
The $f^{(j)}_{m}(\eta)$ function

$$f^{(j)}_{m}(\eta) = \langle m | Y^\dagger e^{\eta K_3} Y | m \rangle_j = D^{(\gamma j, j)}(e^{\eta K_3})j,m$$

Thanks to $Y$ maps, which make the parameters $(p, k)$ of primary series dependent on $j$, and which pick the lowest spin subspace of $(p, k)$, these matrix elements has rather simple form

$$f^{(j)}_{m}(\eta) = (2j + 1) \left(\begin{array}{c} 2j \\ j + m \end{array}\right) e^{-m \eta} e^{i \gamma j \eta} e^{-(j+1)\eta}$$

$$\int_0^1 dx x^{j+m}(1-x)^{j-m} (1 - (1 - e^{-2\eta}) x)^{i \gamma j - (j+1)}$$

[Ruhl (1970)]

(simple when compared to the general $SL(2, \mathbb{C})$ representation's matrix elements)
The $f_m^{(j)}(\eta)$ function in hypergeometric representation

$$f_m^{(j)}(\eta) = (2j + 1) \binom{2j}{j + m} e^{-m\eta} e^{i\gamma j \eta} e^{-(j+1)\eta}$$

$$\int_0^1 dx x^{j+m} (1 - x)^{j-m} (1 - (1 - e^{-2\eta}) x)^{i\gamma j - (j+1)}$$
The $f_m^{(j)}(\eta)$ function in hypergeometric representation

\[ f_m^{(j)}(\eta) = (2j + 1) \binom{2j}{j+m} e^{-m\eta} e^{i\gamma j \eta} e^{-(j+1)\eta} \]
\[ \int_0^1 dx x^{j+m} (1-x)^{j-m} (1 - (1 - e^{-2\eta}) x)^{i\gamma j -(j+1)} \]

Recalling the integral definition of the Hypergeometric Function of 2nd kind

\[ _2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \]
The \( f_m^{(j)}(\eta) \) function in hypergeometric representation

\[
f_m^{(j)}(\eta) = (2j + 1) \binom{2j}{j + m} e^{-m\eta} e^{i\gamma j \eta} e^{-(j+1)\eta} \int_0^1 dx \, x^{j+m} (1 - x)^{j-m} (1 - (1 - e^{-2\eta}) x)^{i\gamma j - (j+1)}
\]

Recalling the integral definition of the Hypergeometric Function of 2nd kind

\[
\_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt \, t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}
\]

we get

\[
f_m^{(j)}(\eta) = e^{-(j+m+1)\eta} e^{ij\gamma \eta} \_2F_1((j + m + 1), (j + 1 - i\gamma), (2j + 2); (1 - e^{-2\eta}))
\]
\[
\ln \left( f_m^{(j)}(\eta) \right)
\]

\[
\ln \left( f_m^{(j)}(\eta) \right) =
\]
\[ \ln \left( f_{m}^{(j)}(\eta) \right) \]

\[ \ln \left( f_{m}^{(j)}(\eta) \right) = -(j + m + 1)\eta \]
\[ \ln \left( f_m^{(j)}(\eta) \right) = -(j + m + 1)\eta + ij\gamma\eta \]
\[ \ln \left( f_m^{(j)}(\eta) \right) \]

\[ \ln \left( f_m^{(j)}(\eta) \right) = - (j + m + 1)\eta + ij\gamma\eta + \psi(\eta) \]
\[ \ln \left( f_m^{(j)}(\eta) \right) = -(j + m + 1)\eta + ij\gamma\eta + \psi(\eta) \]

\[ \psi(\eta) := \ln \left[ \, _2F_1 \left( j + m + 1, j + 1 - ij\gamma, 2j + 2; 1 - e^{-2\eta} \right) \, \right] \]
\[ \ln \left( f_m^{(j)}(\eta) \right) \]

\[ \ln \left( f_m^{(j)}(\eta) \right) = - (j + m + 1) \eta + ij \gamma \eta + \psi(\eta) \]

\[ \psi(\eta) := \ln \left[ _2F_1 \left( j + m + 1, j + 1 - ij \gamma, 2j + 2; 1 - e^{-2\eta} \right) \right] \]

Note that the fourth argument of \(_2F_1\) is small for \( \eta \) close to 0. Indeed, \( 1 - e^{-2\eta} = 2\eta + O(\eta^2) \).
\[
\ln \left( f_m^{(j)}(\eta) \right)
\]

\[
\ln \left( f_m^{(j)}(\eta) \right) = -(j + m + 1) \eta + ij \gamma \eta + \psi(\eta)
\]

\[
\psi(\eta) := \ln \left[ \, _2F_1 \left( j + m + 1, j + 1 - ij \gamma, 2j + 2; 1 - e^{-2\eta} \right) \, \right]
\]

Note that the fourth argument of \(_2F_1\) is small for \(\eta\) close to 0. Indeed, \(1 - e^{-2\eta} = 2\eta + O(\eta^2)\).

Let’s now recall the series definition of \(_2F_1\):

\[
_2F_1(a, b, c; z) := \sum_{k=0}^{\infty} \frac{a^k b^k}{c^k k!} z^k
\]

\[
\ln \left( f_m^{(j)}(\eta) \right) = - (j + m + 1) \eta + ij \gamma \eta + \psi(\eta)
\]

\[
\psi(\eta) := \ln \left[ \frac{2}{1 - e^{-2\eta}} \right]
\]

Note that the fourth argument of \( {}_2F_1 \) is small for \( \eta \) close to 0. Indeed, \( 1 - e^{-2\eta} = 2\eta + O(\eta^2) \).

Let's now recall the series definition of \( {}_2F_1 \):

\[
{}_2F_1(a, b, c; z) := \sum_{k=0}^{\infty} \frac{a^k b^k}{c^k k!} z^k
\]

For \( z \) close to 0 we Taylor expand it obtaining

\[
{}_2F_1(a, b, c; z) = 1 + \frac{ab}{c} z + \frac{a(a + 1)b(b + 1)}{2c(c + 1)} z^2 + O(z^3) = 1 + \epsilon
\]
\[
\ln \left( f_m^{(j)}(\eta) \right) = -(j + m + 1) \eta + i j \gamma \eta + \psi(\eta)
\]

\[
\psi(\eta) := \ln \left[ \, _2F_1 \left( j + m + 1, j + 1 - i j \gamma, 2j + 2; 1 - e^{-2\eta} \right) \, \right]
\]

Note that the fourth argument of \(_2F_1\) is small for \(\eta\) close to 0. Indeed, \(1 - e^{-2\eta} = 2\eta + O(\eta^2)\).

Let's now recall the series definition of \(_2F_1\):

\[
_2F_1(a, b, c; z) := \sum_{k=0}^{\infty} \frac{a^k b^k}{c^k k!} z^k
\]

For \(z\) close to 0 we Taylor expand it obtaining

\[
_2F_1(a, b, c; z) = 1 + \frac{ab}{c} z + \frac{a(a + 1)b(b + 1)}{2c(c + 1)} z^2 + O(z^3) = 1 + \epsilon
\]

Thus we can apply the Taylor expansion to \(\ln \left[ _2F_1(a, b, c; z) \right] \)
Taylor expansion of $\ln \left[ {\frac{\text{d}}{\text{d}z}} F_1 \left( a, b, c; z \right) \right]$

\[ \psi(\eta) := \ln \left[ {\frac{\text{d}}{\text{d}z}} F_1 \left( j + m + 1, j + 1 - ij\gamma, 2j + 2; 1 - e^{-2\eta} \right) \right] \]
Taylor expansion of $\ln \left[ \, _2F_1 (a, b, c; z) \right]$ 

$$
\psi(\eta) := \ln \left[ \, _2F_1 \left( j + m + 1, \ j + 1 - ij\gamma, \ 2j + 2; \ 1 - e^{-2\eta} \right) \right] 
$$

$$
= \frac{2(j + 1)(j + m + 1)}{2j + 1} \eta - \frac{2\gamma j(j + m + 1)}{2j + 1} \eta 
$$

$$
- \frac{(j - m)(j + m + 1) \left[(1 - \gamma^2)j + 1\right]}{(j + 1)(2j + 1)} \eta^2 
$$

$$
+ \frac{i \gamma j(j - m)(j + m + 1)}{(j + 1)(2j + 1)} \eta^2 + O(\eta^3) 
$$
Taylor expansion of $\ln \left[ _2F_1(a, b, c; z) \right]$:

Separating into $m$ dependent and $m$ independent part:

$$\psi(\eta) = \frac{2(j + 1)^2}{2j + 1}\eta - \frac{i}{2j + 1}2\gamma j(j + 1)\eta$$

$$- \frac{j^2 \left[(1 - \gamma^2)j + 1\right]}{(2j + 1)^2}\eta^2 + \frac{i\gamma j^2}{(2j + 1)}\eta^2 + O(\eta^3)$$

$$+ \frac{2(j + 1)}{2j + 1}\eta \cdot m - \frac{i}{2j + 1}2\gamma j\eta \cdot m$$

$$+ \frac{\left[(1 - \gamma^2)j + 1\right]j}{(j + 1)(2j + 1)^2}\eta^2 \cdot m - \frac{i\gamma j}{(j + 1)(2j + 1)}\eta^2 \cdot m$$

$$+ \frac{\left[(1 - \gamma^2)j + 1\right]j}{(j + 1)(2j + 1)^2}\eta^2 \cdot m^2 - \frac{i\gamma j}{(j + 1)(2j + 1)}\eta^2 \cdot m^2$$
Taylor expansion of $\ln \left[ _2F_1(a, b, c; z) \right]$

Extracting the leading order terms (in $j$):

$$\psi(\eta) = j\eta \left[ 1 + O(j^{-1}) \right] - i\gamma j\eta \left[ 1 + O(j^{-1}) \right]$$

$$- \frac{j(1 - \gamma^2)}{4} \eta^2 \left[ 1 + O(j^{-1}) \right] + i\frac{\gamma j}{2} \eta^2 \left[ 1 + O(j^{-1}) \right]$$

$$+ \eta \cdot m \left[ 1 + O(j^{-1}) \right] - i\gamma \eta \cdot m \left[ 1 + O(j^{-1}) \right]$$

$$+ \frac{1 - \gamma^2}{4j} \eta^2 \cdot m \left[ 1 + O(j^{-1}) \right] - i\frac{\gamma}{2j} \eta^2 \cdot m \left[ 1 + O(j^{-1}) \right]$$

$$+ \frac{1 - \gamma^2}{4j} \eta^2 \cdot m^2 \left[ 1 + O(j^{-1}) \right] - i\frac{\gamma}{2j} \eta^2 \cdot m^2 \left[ 1 + O(j^{-1}) \right]$$

The terms proportional to $m \eta^2$ are negligible with respect to terms $m^2 \eta^2$. 
Taylor expansion of $\ln \left[ \frac{\Gamma(a)}{\Gamma(b)} \right]$

Extracting the leading order terms (in $j$):

$$
\psi(\eta) = j\eta \left[ 1 + O\left( j^{-1} \right) \right] - i\gamma j \eta \left[ 1 + O\left( j^{-1} \right) \right]
$$

$$
- \frac{j(1 - \gamma^2)}{4} \eta^2 \left[ 1 + O\left( j^{-1} \right) \right] + i \frac{\gamma j}{2} \eta^2 \left[ 1 + O\left( j^{-1} \right) \right]
$$

$$
+ \eta \cdot m \left[ 1 + O\left( j^{-1} \right) \right] - i\gamma \eta \cdot m \left[ 1 + O\left( j^{-1} \right) \right]
$$

$$
+ \frac{1 - \gamma^2}{4j} \eta^2 \cdot m \left[ 1 + O\left( j^{-1} \right) \right] - i \frac{\gamma}{2j} \eta^2 \cdot m \left[ 1 + O\left( j^{-1} \right) \right]
$$

$$
+ \frac{1 - \gamma^2}{4j} \eta^2 \cdot m^2 \left[ 1 + O\left( j^{-1} \right) \right] - i \frac{\gamma}{2j} \eta^2 \cdot m^2 \left[ 1 + O\left( j^{-1} \right) \right]
$$

the terms proportional to $m\eta^2$ are negligible with respect to terms $m^2\eta^2$. 
Taylor expansion of $\ln \left[ \frac{2}{\Gamma(a, b, c; z)} \right]$

Extracting the leading order terms (in $j$):

$$
\psi(\eta) = j\eta \left[ 1 + O(j^{-1}) \right] - i\gamma j \eta \left[ 1 + O(j^{-1}) \right]
$$

$$
- \frac{j(1 - \gamma^2)}{4} \eta^2 \left[ 1 + O(j^{-1}) \right] + i\gamma j \eta^2 \left[ 1 + O(j^{-1}) \right]
$$

$$
+ \eta \cdot m \left[ 1 + O(j^{-1}) \right] - i\gamma \eta \cdot m \left[ 1 + O(j^{-1}) \right]
$$

$$
+ \frac{1 - \gamma^2}{4j} \eta^2 \cdot m^2 \left[ 1 + O(j^{-1}) \right] - i\gamma \eta^2 \cdot m^2 \left[ 1 + O(j^{-1}) \right]
$$

the terms proportional to $m\eta^2$ are negligible with respect to terms $m^2\eta^2$. 
Back to $\ln \left( f_m^{(j)}(\eta) \right)$

$$\ln \left( f_m^{(j)}(\eta) \right) = - (j + 1) \eta - m \eta + ij \gamma \eta + \psi(\eta)$$
Back to \( \ln \left( f_m^{(j)}(\eta) \right) \)

\[
\ln \left( f_m^{(j)}(\eta) \right) = -(j + 1)\eta - m\eta + ij\gamma\eta + \psi(\eta)
\]

\[
= -j\eta \left[ 1 + O(j^{-1}) \right] - m\eta + ij\gamma\eta \\
+ j\eta \left[ 1 + O(j^{-1}) \right] - i\gamma j\eta \left[ 1 + O(j^{-1}) \right] \\
+ \eta \cdot m \left[ 1 + O(j^{-1}) \right] - i\gamma \eta \cdot m \left[ 1 + O(j^{-1}) \right] \\
+ j\eta^2 \left[ -\frac{(1 - \gamma^2)}{4} + \frac{i\gamma}{2} \right] \left[ 1 + O(j^{-1}) \right] \\
+ \eta^2 \cdot m^2 \frac{1 - \gamma^2}{j} \left[ -\frac{(1 - \gamma^2)}{4} - \frac{i\gamma}{2} \right] \left[ 1 + O(j^{-1}) \right]
\]
Back to $\ln \left( f_m^{(j)}(\eta) \right)$

\[
\ln \left( f_m^{(j)}(\eta) \right) = -(j + 1)\eta - m\eta + ij\gamma\eta + \psi(\eta)
\]

\[
= -j\eta \left[ 1 + O\left( j^{-1} \right) \right] - m\eta + ij\gamma\eta
+ j\eta \left[ 1 + O\left( j^{-1} \right) \right] - i\gamma j\eta \left[ 1 + O\left( j^{-1} \right) \right]
+ \eta \cdot m \left[ 1 + O\left( j^{-1} \right) \right] - i\gamma \eta \cdot m \left[ 1 + O\left( j^{-1} \right) \right]
+ j\eta^2 \left[ -\frac{1 - \gamma^2}{4} + \frac{i\gamma}{2} \right] \left[ 1 + O\left( j^{-1} \right) \right]
+ \eta^2 \cdot \frac{m^2}{j} \left[ \frac{1 - \gamma^2}{4} - \frac{i\gamma}{2} \right] \left[ 1 + O\left( j^{-1} \right) \right]
\]

The linear terms cancel
Back to $\ln \left( f_m^{(j)}(\eta) \right)$

\[
\ln \left( f_m^{(j)}(\eta) \right) = -(j+1)\eta - m\eta + ij\gamma\eta + \psi(\eta)
\]

\[
= -m\eta + ij\gamma\eta - i\gamma j\eta \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ \eta \cdot m \left[ 1 + O \left( j^{-1} \right) \right] - i\gamma \eta \cdot m \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ j\eta^2 \left[ -\frac{1-\gamma^2}{4} + i\frac{\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ \eta^2 \cdot \frac{m^2}{j} \left[ \frac{1-\gamma^2}{4} - i\frac{\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ \eta O(1)
\]

The linear terms cancel
Back to $\ln \left( f_m^{(j)}(\eta) \right)$

$$
\ln \left( f_m^{(j)}(\eta) \right) = - (j + 1) \eta - m \eta + i j \gamma \eta + \psi(\eta)
$$

$$
= - m \eta
$$

$$
+ \eta \cdot m \left[ 1 + O \left( j^{-1} \right) \right] - i \gamma \eta \cdot m \left[ 1 + O \left( j^{-1} \right) \right]
$$

$$
+ j \eta^2 \left[ - \frac{(1 - \gamma^2)}{4} + i \frac{\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
$$

$$
+ \eta^2 \cdot \frac{m^2}{j} \left[ - \frac{(1 - \gamma^2)}{4} - i \frac{\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
$$

$$
+ \eta O \left( 1 \right) + i \gamma \eta O \left( 1 \right)
$$

The linear terms cancel
Back to $\ln \left( f_m^{(j)}(\eta) \right)$

\[
\ln \left( f_m^{(j)}(\eta) \right) = - (j + 1)\eta - m\eta + ij\gamma\eta + \psi(\eta)
\]

\[
= - i\gamma\eta \cdot m \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ j\eta^2 \left[ - \frac{(1 - \gamma^2)}{4} + \frac{i\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ \eta^2 \cdot \frac{m^2}{j} \left[ \frac{1 - \gamma^2}{4} - \frac{i\gamma}{2} \right] \left[ 1 + O \left( j^{-1} \right) \right]
\]

\[
+ \eta O \left( 1 \right) + i\gamma\eta O \left( 1 \right) + \frac{m}{j} \eta O \left( 1 \right)
\]

The linear terms cancel
Finally $\frac{1}{j} \ln \left( f_{m}^{(j)}(\eta) \right)$

Let's reorganise the quadratic terms:

$$\ln \left( f_{m}^{(j)}(\eta) \right) = \left( 1 + i\gamma + \frac{m}{j} \right) \eta O(1)$$

$$- i\gamma \eta \cdot m \left[ 1 + O(j^{-1}) \right]$$

$$+ j\eta^2 \left[ 1 - \frac{m^2}{j^2} \right] \left[ - \frac{(1 - \gamma^2)}{4} + i\frac{\gamma}{2} \right] \left[ 1 + O(j^{-1}) \right]$$
Finally $\frac{1}{j} \ln \left( f_m^{(j)}(\eta) \right)$

Lets reorganise the quadratic terms:

$$\ln \left( f_m^{(j)}(\eta) \right) = \left( 1 + i\gamma + \frac{m}{j} \right) \eta O(1)$$

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$$+ j\eta^2 \left[ 1 - \frac{m^2}{j^2} \right] \left[ - \frac{(1 - \gamma^2)}{4} + i\frac{\gamma}{2} \right] \left[ 1 + O(j^{-1}) \right]$$

And finally divide everything by $J$

$$\frac{1}{j} \ln \left( f_m^{(j)}(\eta) \right) = \eta O \left( \frac{1}{j} \right) - i\gamma \eta \cdot \frac{m}{j} \left( 1 + O \left( \frac{1}{j} \right) \right)$$

$$+ x\eta^2 \left[ 1 - \frac{m^2}{j^2} \right] \left[ - \frac{(1 - \gamma^2)}{4} + i\frac{\gamma}{2} \right] \left( 1 + O \left( \frac{1}{j} \right) \right)$$

where $x := \frac{i}{j} \in [0, 1]$
Finally $\frac{1}{j} \ln \left( f_m^{(j)}(\eta) \right)$

Let's reorganise the quadratic terms:

$$\ln \left( f_m^{(j)}(\eta) \right) = \left( 1 + i\gamma + \frac{m}{j} \right) \eta O(1)$$

$$- i\gamma \eta \cdot m \left[ 1 + O(j^{-1}) \right]$$

$$+ j\eta^2 \left[ 1 - \frac{m^2}{j^2} \right] \left[ - \frac{(1 - \gamma^2)}{4} + \frac{i\gamma}{2} \right] \left[ 1 + O(j^{-1}) \right]$$

And finally divide everything by $J$

$$\frac{1}{j} \ln \left( f_m^{(j)}(\eta) \right) = - i\gamma \eta \cdot \frac{m}{j}$$

$$+ x\eta^2 \left[ 1 - \frac{m^2}{j^2} \right] \left[ - \frac{(1 - \gamma^2)}{4} + \frac{i\gamma}{2} \right] + O \left( \frac{1}{j} \right)$$

where $x := \frac{i}{j} \in [0, 1]$
\[
\phi_m(\eta, J) = \sum_{i=1}^{N} \frac{1}{J} \ln \left( f_{m_i}(\eta) \right)
\]
\[ \phi \tilde{m}(\eta, J) = \sum_{i=1}^{N} \frac{1}{j} \ln \left( f_{m_i}^{(j)}(\eta) \right) \]

\[ = \mathcal{O} \left( \frac{1}{j} \right) - \eta \frac{i\gamma}{J} \sum_{i=1}^{N} m_i \]

\[ + \eta^2 \left[ - \frac{(1 - \gamma^2)}{4} + \frac{i\gamma}{2} \right] \sum_{i=1}^{N} x_i \left[ 1 - \frac{m_i^2}{j_i^2} \right] \]
Large $j$ behavior of dipole cosmology transition amplitudes

Jacek Puchta

Introduction

The lorentzian polyhedra propagator - preliminary analysis

Integrand’s smoothness check

\[ \phi_m(\eta, J) = \sum_{i=1}^{N} \frac{1}{j} \ln \left( f_{m_i}(\eta) \right) \]

\[ = O \left( \frac{1}{J} \right) - \eta \frac{i \gamma}{J} \sum_{i=1}^{N} m_i \]

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Recall, that we consider only these matrix elements, that are $SU(2)$ invariant, thus $\sum_{i=1}^{N} m_i = 0$, so the term linear in $\eta$ vanish
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Second derivative of $\phi$

It is immediate to read $\left( \frac{d^2 \phi}{d\eta^2} \right)_{\eta=0}$ from (6)

$$\left( \frac{d^2 \phi}{d\eta^2} \right)_{\eta=0} = \left[ - \frac{(1 - \gamma^2)}{2} + i\gamma \right] \sum_{i=1}^{N} x_i \left[ 1 - \frac{m_i^2}{j_i^2} \right]$$

however we need to handle with the term $\sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2}$.
\[
\sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2}
\]

Note, that so far we had \( \phi_\mathbf{m} \) in basis \( |m_1, \ldots, m_N\rangle_{j_1 \otimes \cdots \otimes j_N} \). In this basis

\[
\langle \mathbf{m} | \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} | \mathbf{m} \rangle_j = \frac{1}{J} \langle \mathbf{m} | \sum_{i=1}^{N} (J_i)^2 \frac{1}{2} \frac{J_z(i)^2}{J(i)^2} | \mathbf{m} \rangle_j
\]
\[ \sum_{i=1}^{N} \frac{x_i}{j_i^2} m_i^2 \]

Note, that so far we had \( \phi_\bar{m} \) in basis \(|m_1, \ldots, m_N\rangle_{j_1 \otimes \ldots \otimes j_N} \). In this basis

\[ \langle \bar{m} | \sum_{i=1}^{N} \frac{x_i}{j_i^2} m_i^2 | \bar{m} \rangle_{\bar{j}} = \frac{1}{J} \langle \bar{m} | \sum_{i=1}^{N} \left( J(i)^2 \right)^{1/2} \frac{J_{z,(i)^2}}{J(i)^2} | \bar{m} \rangle_{\bar{j}} \]

But we are interested in matrix elements between invariant tensors \(|\iota\rangle\), thus let’s consider

\[ \langle \iota | \sum_{i=1}^{N} \left( J(i)^2 \right)^{1/2} \frac{J_{z,(i)^2}}{J(i)^2} | \iota' \rangle \]
\[ \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} \]

Note, that so far we had \( \phi m \) in basis \(| m_1, \ldots, m_N \rangle_{j_1 \otimes \ldots \otimes j_N} \). In this basis

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\]
\[
\sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2}
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Note, that so far we had \( \phi_\vec{m} \) in basis \( |m_1, \ldots, m_N\rangle \otimes \cdots \otimes |j_N\rangle \). In this basis

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\]
\[ \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} \]

Note, that so far we had \( \phi \vec{m} \) in basis \( |m_1, \ldots, m_N\rangle_{j_1 \otimes \cdots \otimes j_N} \). In this basis

\[ \langle \vec{m} | \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} | \vec{m} \rangle_{\vec{j}} = \frac{1}{j} \langle \vec{m} | \sum_{i=1}^{N} \left( \frac{J(i)}{2} \right)^{1/2} \frac{J_z(i)}{j(i)}^2 | \vec{m} \rangle_{\vec{j}} \]

But we are interested in matrix elements between invariant tensors \( |l\rangle \), thus let’s consider

\[ \langle l | \sum_{i=1}^{N} \left( \frac{J(i)}{2} \right)^{1/2} \frac{J_z(i)}{j(i)}^2 | l' \rangle = \langle l | \sum_{i=1}^{N} \left( \frac{J(i)}{2} \right)^{1/2} \frac{J_x(i)}{j(i)}^2 | l' \rangle = \langle l | \sum_{i=1}^{N} \left( \frac{J(i)}{2} \right)^{1/2} \frac{J_y(i)}{j(i)}^2 | l' \rangle \]

\[ = \frac{1}{3} \langle l | \sum_{i=1}^{N} \left( \frac{J(i)}{2} \right)^{1/2} \frac{J_x(i)^2 + J_y(i)^2 + J_z(i)^2}{j(i)^2} | l' \rangle \]
\[
\sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2}
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Note, that so far we had \( \phi_m \) in basis \( |m_1, \ldots, m_N\rangle_{j_1 \otimes \cdots \otimes j_N} \). In this basis

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\langle \bar{m} | \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} | \bar{m} \rangle_{\bar{j}} = \frac{1}{J} \langle \bar{m} | \sum_{i=1}^{N} \left( J(i) \right)^2 \frac{1}{2} \frac{J_z(i)^2}{J(i)^2} | \bar{m} \rangle_{\bar{j}}
\]

But we are interested in matrix elements between invariant tensors \( |\eta\rangle \), thus let's consider

\[
\langle \eta | \sum_{i=1}^{N} \left( J(i) \right)^2 \frac{1}{2} \frac{J_x(i)^2}{J(i)^2} | \eta' \rangle = \langle \eta | \sum_{i=1}^{N} \left( J(i) \right)^2 \frac{1}{2} \frac{J_y(i)^2}{J(i)^2} | \eta' \rangle = \langle \eta | \sum_{i=1}^{N} \left( J(i) \right)^2 \frac{1}{2} \frac{J_z(i)^2}{J(i)^2} | \eta' \rangle
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It is immediate to read \( \left( \frac{d^2 \phi}{d\eta^2} \right)_{\eta=0} \) from (6)

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however we need to handle with the term \( \sum_{i=1}^{N} x_i \frac{m_i^2}{j_i^2} \), which appear to be equal

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so at the end of the day we have

\[
\left( \frac{d^2 \phi}{d\eta^2} \right)_{\eta=0} = \frac{2}{3} \left[ - \frac{1 - \gamma^2}{2} + i\gamma \right] \sum_{i=1}^{N} x_i
\]
Outline

Introduction
  Motivation
  Saddle Point Approximation
The lorentzian polyhedra propagator - preliminary analysis
  Definition and basic properties
  Symmetries of the integrand
  Strategy of integration
Integrand’s smoothness check
  $f^{(j)}_m(\eta)$
  $\frac{1}{j} \ln \left( f^{(j)}_m(\eta) \right)$
  $\phi_{\bar{m}}(\eta, J)$
  Hessian
Results and summary
  Result
  Summary
The final form of the Lorentzian polyhedra propagator

Given two basis elements \( \iota, \iota' \in \text{Inv} (H_{j_1} \otimes \cdots \otimes H_{j_N}) \) the Lorentzian polyhedra propagator’s matrix element is

\[
T_{\iota \iota'} = \frac{1}{(4\pi)^2} \left( \frac{6\pi}{w_\gamma \sum_{i=1}^{N} j_i} \right)^\frac{3}{2} \delta_{\iota \iota'}
\]

for \( w_\gamma = -(1 - \gamma^2) + 2i\gamma \)
Summary

The integrand $\langle \iota | Y^\dagger g Y | \iota' \rangle$ has been studied.
- Smoothness of the exponent part has been proven - thus the SPA is applicable.
- The direct formula of the Hessian in the critical point has been found.
- The leading order of the operator $T$ has been studied.
  - It splits into direct sum of $T|j_1 \otimes \cdots \otimes j_N$.
  - On each space $\text{Inv}(\otimes H_{j_i})$ it is proportional to the identity with a factor dependent on total area of polyhedron $A = \sum j_i$.

Further directions:
- Subleading order
- Applications to concrete examples
- Boundary conditions...
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Result

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Thank you for your attention!