Large j behaviour of dipole cosmology transition amplitudes

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Tux, 27th of Februart 2013
EFI winter conference on canonical and covariant loop quantum gravity

The project „International PhD Studies at the Faculty of Physics, University of Warsaw" is realized within the MPD programme of Foundation for Polish Science, cofinanced from European Union, Regional Development Fund

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Outline

Aim of the research and motivation

The aim

to investigate the properties of Lorentzian Polyhedra Propagator:

$$\mathbb{T} = \int_{SL(2,\mathbb{C})} \mathrm{d}g Y^{\dagger} g Y$$

in large j limit

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- DC amplitude is proportional to $\int \mathrm{d}g \prod_{i=1}^{4} \langle \vec{n}_i | \, Y^{\dagger} g Y \, | \vec{n}_i \rangle_j$

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- 2-vertex-and-1-edge DC:
  $\int \mathrm{d}g \mathrm{d}g' \prod_{i=1}^{4} \langle \vec{n}_i | \, Y^\dagger g Y \;\; Y^\dagger g' Y \, | \vec{n}_i \rangle_j$ [JP, in progress]

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- radiative corrections of "melonic" graph proportional to $\log \Lambda \cdot \mathbb{T}^2$ [Riello: arXiv:1302.1781 (2013)]

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The SPA method

We estimate the following integral for $\Lambda \gg 1$ by the value of integrand in the critical point x_0 of $f(x)$

$$\int \mathrm{d}x \; g(x) e^{-\Lambda f(x)} = \sqrt{\frac{(2\pi)}{\Lambda |f''(x_0)|}} g(x_0) e^{-\Lambda f(x_0)} \tag{1}$$

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Note!

$\nabla f(x)$ must vanish in x_0, i.e. x_0 must not be the extremum, where $\nabla f(x)$ is discontinues.

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$$\phi(x) := \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln(\Phi(x, \Lambda)) \tag{3}$$

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we will call it *the exponent part* of the integrand.
The SPA formula will be true for critical points and Hessian matrix of $\phi(x)$.

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Definition

Def: Lorentzian polyhedra propagator

Given a set of spins $j_1, \ldots, j_N$ we define an operator

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We can consider its matrix elements in the $|m\rangle_j$ basis:

$$\begin{aligned}
\mathbb{T}_{m_1 \cdots m_N}^{m_1' \cdots m_N'} &:= \int_{SL(2,\mathbb{C})} \mathrm{d}g \, \langle m_1, \ldots, m_N | \, Y^\dagger g Y \, | m_1', \ldots, m_N' \rangle_{j_1 \otimes \cdots \otimes j_N} \\
&= \int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{i=1}^{N} \langle m_i | \, Y^\dagger g Y \, | m_i' \rangle_{j_i} \\
&= \int_{SL(2,\mathbb{C})} \mathrm{d}g \prod_{i=1}^{N} D^{(\gamma j_i, j_i)}(g)_{j_i, m_i}^{j_i, m_i'}
\end{aligned}$$

Domain and rank

It is easy to check, that $\mathbb{T}$ acts non-trivially only on the invariant subspace of $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$:

$$\begin{aligned}\mathbb{T} &= \int_{SL(2,\mathbb{C})} \mathrm{d}g \; Y^\dagger g Y = \int_{\mathbb{R}^3 \times SU(2)} \mathrm{d}k \mathrm{d}u \; Y^\dagger k u Y \\ &= \int_{\mathbb{R}^3} \mathrm{d}k \; Y^\dagger k Y \int_{SU(2)} u \mathrm{d}u = \hat{A} \cdot P_{\mathrm{Inv}}\end{aligned}$$

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Thus

$$\mathbb{T} = P_{\mathrm{Inv}}\cdot\hat{B}\cdot P_{\mathrm{Inv}}$$

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Thus

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So it's enough to study the matrix elements between the $SU(2)$ invariants:

$$\mathbb{T}_{\iota\iota'} = \int_{SL(2,\mathbb{C})} \mathrm{d}g \, \langle \iota | \, Y^\dagger g Y \, | \iota' \rangle$$

Symmetries of the integrand 1 - $SU(2)$

We need to integrate the function $\Phi_{\iota\iota'}(g, J) := \langle\iota| Y^\dagger g Y |\iota'\rangle$ on $SL(2,\mathbb{C})$, where $J = \max_{i=1,\ldots,N}(j_i)$. We anticipate that the critical point will be in $g = \mathbf{1}$. Let us study the behaviour of Φ close to $g = \mathbf{1}$.

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There are six-dimensional basis vector fields on $SL(2,\mathbb{C})$ given by the generators of rotations J_i and generators of boosts K_i $(i = 1,2,3)$.

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It's straightforward to see, that $J_i \Phi_{\iota\iota'}(g) \equiv 0$
Indeed: J_i are $SU(2)$ generators, thus they commute with the Y map, and $J_i |\iota\rangle = 0$, so

$$\langle\iota| Y^\dagger g J_i Y |\iota'\rangle = \langle\iota| Y^\dagger g Y J_i |\iota'\rangle = 0$$

Symmetries of the integrand 2 - z-boost

Let's now consider a z-boost $Y^{\dagger}e^{\eta K_3}Y$. Since $[K_3, J_3] = 0$, it's convenient to consider it in the $|m\rangle_j$ basis.

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Let's define the function $f_m^{(j)}(\eta)$

$$\langle m| Y^\dagger e^{\eta K_3} Y |m'\rangle_j =: \delta_{m,m'} f_m^{(j)}(\eta) \tag{5}$$

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$$|\iota\rangle = \sum_{\{m_i\}} \iota_{m_1 \cdots m_N} |m_1, \ldots, m_N\rangle_{j_1 \otimes \cdots \otimes j_N}$$

and thus

$$\Phi_{\iota,\iota'}(e^{\eta K_3}) = \sum_{\{m_i\}} \iota^{m_1 \cdots m_N} \iota'_{m_1 \cdots m_N} \Phi^{m_1 \cdots m_N}_{m_1 \cdots m_N}(e^{\eta K_3})$$

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Note that

since $J_i |\iota\rangle = 0$, only the terms with $\sum_{i=0}^N m_i = 0$ counts.

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Thus the behaviour of $\Phi_{\iota\iota'}(e^{\eta K_3})$ and $f_m^{(j)}(\eta)$ is crucial in further calculation.

Strategy of integration 1: parametrisation of $SL(2,\mathbb{C})$

We need to integrate $\int_{SL(2,\mathbb{C})} \mathrm{d}g\ \Phi_{\iota\iota'}(e^{\eta(g)K_3})$.
Since the integrand depend only on η, one may be tempted to use the decomposition $g = u_1^{-1} e^{\eta K_3} u_2$ and the measure

$$\int_{SU(2)\times SU(2)\times\mathbb{R}_+} \mathrm{d}u_1 \mathrm{d}u_2 \frac{\sinh^2\eta}{4\pi} \mathrm{d}\eta \Phi_{\iota\iota'}(e^{\eta K_3})$$

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Thus let's parametrise $SL(2,\mathbb{C})$ with $(u,\vec{x}) \in SU(2) \times \mathbb{R}^3$:

$$g(u,\vec{x}) := u {n_{\vec{x}}}^{-1} e^{|\vec{x}|K_3} n_{\vec{x}}$$

where $n_{\vec{x}} \in SU(2)$ is such a rotation, that ${n_{\vec{x}}}^{-1}|\vec{x}|L_3 n_{\vec{x}} = \vec{x}\cdot\vec{L}$, i.e.

$$n_{\vec{x}} = \begin{pmatrix} \cos\frac{\theta(\vec{x})}{2} & -e^{i\phi(\vec{x})}\sin\frac{\theta(\vec{x})}{2} \\ e^{-i\phi(\vec{x})}\sin\frac{\theta(\vec{x})}{2} & \cos\frac{\theta(\vec{x})}{2} \end{pmatrix}$$

Strategy of integration 2: measure

$$\mathrm{d}g \quad = \quad \mathrm{d}u_1 \, \mathrm{d}u_2 \, \frac{\sinh^2 \eta}{4\pi} \mathrm{d}\eta = \mathrm{d}u \, \mathrm{d}^3 x \, \mu(\vec{x})$$

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Putting $\mathrm{d}^3x = |\vec{x}|^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ and $\eta = |\vec{x}|$ we get

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Note that

There are two properties of the measure important for the SPA method:

- $\lim_{\vec{x}\to 0}\mu(\vec{x}) = \frac{1}{(4\pi)^2}$
- $\lim_{J\to\infty}\frac{1}{J}\ln\left[\mu(\vec{x})\right] = 0$ - thus it does not effect the behaviour of the exponent part of the integrand.

Strategy of integration 3: SPA and Hessian

We expect the critical point to be $\vec{x}_0 = 0$. If the exponent part $\phi_{\iota\iota'}(\vec{x})$ of the integrand $\Phi_{\iota\iota'}(\vec{x})$ is smooth (and so is its derivative) in $\vec{x}_0$, the integral equals

$$\mathbb{T}_{\iota\iota'} = \int_{SL(2,\mathbb{C})} \mathrm{d}g\, \Phi_{\iota\iota'}(g) = \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \frac{1}{\sqrt{|H|}} \mu(0)\Phi_{\iota\iota'}(0) \left(1 + O\left(\frac{1}{J}\right)\right)$$

where H is the Hessian matrix of $\phi_{\iota\iota'}(\vec{x}) := \lim_{J\to\infty} \frac{1}{J} \ln \Phi_{\iota\iota'}(\vec{x})$

Strategy of integration 3: SPA and Hessian

We expect the critical point to be $\vec{x}_0 = 0$. If the exponent part $\phi_{\iota\iota'}(\vec{x})$ of the integrand $\Phi_{\iota\iota'}(\vec{x})$ is smooth (and so is its derivative) in $\vec{x}_0$, the integral equals

$$\mathbb{T}_{\iota\iota'} = \int_{SL(2,\mathbb{C})} \mathrm{d}g\, \Phi_{\iota\iota'}(g) = \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \frac{1}{\sqrt{|H|}} \mu(0)\Phi_{\iota\iota'}(0) \left(1 + O\left(\frac{1}{J}\right)\right)$$

where H is the Hessian matrix of $\phi_{\iota\iota'}(\vec{x}) := \lim_{J\to\infty} \frac{1}{J} \ln \Phi_{\iota\iota'}(\vec{x})$

Hessian of ϕ

Since $\phi_{\iota\iota'}$ is spherically symmetric (so it is a function of one variable η), we can express it's Hessian for $\eta \to 0$ as

$$\det\left[H_{ij}\right]_{\eta=0} = \det\left[\frac{1}{2}\left(\partial_i\partial_j\eta^2\right)\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}\right] = \det\left[\delta_{ij}\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}\right] = \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}\right)^3$$

The Lorentzian polyhedra propagator

Recall now, that

$$\Phi_{\iota\iota'}(0) = \langle\iota| \, Y^{\dagger} e^{0\cdot K_3} Y \, |\iota'\rangle = \langle\iota| \, |\iota'\rangle = \delta_{\iota\iota'}$$

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So neglecting the $\frac{1}{J}$ terms the operator $\mathbb{T}$ is

$$\mathbb{T}_{\iota\iota'} = \left(\frac{2\pi}{J}\right)^{\frac{3}{2}} \left(\frac{\mathrm{d}^2\phi_{\iota\iota'}}{\mathrm{d}\eta^2}\right)^{-\frac{3}{2}} \frac{1}{(4\pi)^2} \quad \delta_{\iota\iota'}$$

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Now we will check smoothness of the exponent part of integrand, and calculate the Hessian.

Outline

Strategy

We are going to investigate the function

$$\phi_{\vec{m}}(\eta, J) = \frac{1}{J} \ln \left(\prod_{i=1}^{N} f_{m_i}^{(j_i)}(\eta) \right) = \sum_{i=1}^{N} \frac{1}{J} \ln \left(f_{m_i}^{(j_i)}(\eta) \right)$$

in the limit $J \gg 1$ and $\eta \ll 1$.

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in the limit $J \gg 1$ and $\eta \ll 1$.

We will do it by finding a compact form of $f_m^{(j)}(\eta)$ and analysing its Taylor series.

The $f_m^{(j)}(\eta)$ function

$$f_m^{(j)}(\eta) = \langle m | \, Y^\dagger e^{\eta K_3} Y \, | m \rangle_j = D^{(\gamma j, j)}(e^{\eta K_3})^{j,m}_{j,m}$$

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$$f_m^{(j)}(\eta) = (2j+1)\binom{2j}{j+m} e^{-m\eta} e^{i\gamma j\eta} e^{-(j+1)\eta} \int_0^1 \mathrm{d}x \, x^{j+m}(1-x)^{j-m} \left(1 - \left(1 - e^{-2\eta}\right) x\right)^{i\gamma j-(j+1)}$$

[Ruhl (1970)]

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(simple when compared to the general $SL(2, \mathbb{C})$ representation's matrix elements)

The $f_m^{(j)}(\eta)$ function in hypergeometric representation

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Recalling the integral definition of the Hypergeometric Function of 2nd kind

$$ {}_2F_1(a,b,c;\,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}$$

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we get

$$f_m^{(j)}(\eta) = e^{-(j+m+1)\eta} e^{ij\gamma\eta}\, {}_2F_1\left((j+m+1),\ (j+1-ij\gamma),\ (2j+2);\ (1-e^{-2\eta})\right)$$

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Let's now recall the series definition of ${}_2F_1$:

$$ {}_2F_1(a,b,c;z) := \sum_{k=0}^{\infty} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} z^k $$

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For z close to 0 we Taylor expand it obtaining

$${}_2F_1(a,b,c;\,z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2c(c+1)}z^2 + O(z^3) = 1 + \epsilon$$

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Thus we can apply the Taylor expansion to $\ln\left[\,{}_2F_1(a,b,c;\,z)\right]$

Taylor expansion of $\ln\left[\,{}_2F_1(a,b,c;\,z)\right]$

$$\psi(\eta) \;:=\; \ln\left[\,{}_2F_1\left(j+m+1,\ j+1-ij\gamma,\ 2j+2;\ 1-e^{-2\eta}\right)\right]$$

Taylor expansion of $\ln\left[\,_2F_1(a,b,c;\,z)\right]$

$$\psi(\eta) \;:=\; \ln\left[\,_2F_1\left(j+m+1,\; j+1-ij\gamma,\; 2j+2;\; 1-e^{-2\eta}\right)\right]$$

$$= \frac{2(j+1)(j+m+1)}{2j+1}\eta \;-\; i\frac{2\gamma j(j+m+1)}{2j+1}\eta$$

$$- \frac{(j-m)(j+m+1)\left[(1-\gamma^2)j+1\right]j}{(j+1)(2j+1)^2}\eta^2$$

$$+\, i\frac{\gamma j(j-m)(j+m+1)}{(j+1)(2j+1)}\eta^2 \;+\; O(\eta^3)$$

Taylor expansion of $\ln\left[\,{}_2F_1(a,b,c;\,z)\right]$

Separating into m dependent and m independent part:

$$
\begin{aligned}
\psi(\eta) \;=\;& \frac{2(j+1)^2}{2j+1}\eta \;-\; i\frac{2\gamma j(j+1)}{2j+1}\eta \\
&- \frac{j^2\left[(1-\gamma^2)j+1\right]}{(2j+1)^2}\eta^2 \;+\; i\frac{\gamma j^2}{(2j+1)}\eta^2 \;+\; O(\eta^3) \\
&+ \frac{2(j+1)}{2j+1}\eta\cdot m \;-\; i\frac{2\gamma j}{2j+1}\eta\cdot m \\
&+ \frac{\left[(1-\gamma^2)j+1\right]j}{(j+1)(2j+1)^2}\eta^2\cdot m \;-\; i\frac{\gamma j}{(j+1)(2j+1)}\eta^2\cdot m \\
&+ \frac{\left[(1-\gamma^2)j+1\right]j}{(j+1)(2j+1)^2}\eta^2\cdot m^2 \;-\; i\frac{\gamma j}{(j+1)(2j+1)}\eta^2\cdot m^2
\end{aligned}
$$

Taylor expansion of $\ln\left[\,{}_2F_1(a,b,c;\,z)\right]$

Extracting the leading order terms (in j):

$$
\begin{aligned}
\psi(\eta) \;=\;& j\eta\left[1+O\left(j^{-1}\right)\right] \;-\; i\gamma j\eta\left[1+O\left(j^{-1}\right)\right] \\
& - \frac{j(1-\gamma^2)}{4}\eta^2\left[1+O\left(j^{-1}\right)\right] \;+\; i\frac{\gamma j}{2}\eta^2\left[1+O\left(j^{-1}\right)\right] \\
& + \eta\cdot m\left[1+O\left(j^{-1}\right)\right] \;-\; i\gamma\eta\cdot m\left[1+O\left(j^{-1}\right)\right] \\
& + \frac{1-\gamma^2}{4j}\eta^2\cdot m\left[1+O\left(j^{-1}\right)\right] \;-\; i\frac{\gamma}{2j}\eta^2\cdot m\left[1+O\left(j^{-1}\right)\right] \\
& + \frac{1-\gamma^2}{4j}\eta^2\cdot m^2\left[1+O\left(j^{-1}\right)\right] \;-\; i\frac{\gamma}{2j}\eta^2\cdot m^2\left[1+O\left(j^{-1}\right)\right]
\end{aligned}
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Taylor expansion of $\ln\left[{}_2F_1(a,b,c;\,z)\right]$

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$$\begin{aligned}\psi(\eta) \;=\;& j\eta\left[1+O\left(j^{-1}\right)\right] - i\gamma j\eta\left[1+O\left(j^{-1}\right)\right] \\ & - \frac{j(1-\gamma^2)}{4}\eta^2\left[1+O\left(j^{-1}\right)\right] + i\frac{\gamma j}{2}\eta^2\left[1+O\left(j^{-1}\right)\right] \\ & + \eta\cdot m\left[1+O\left(j^{-1}\right)\right] - i\gamma\eta\cdot m\left[1+O\left(j^{-1}\right)\right] \\ & + \frac{1-\gamma^2}{4j}\eta^2\cdot m\left[1+O\left(j^{-1}\right)\right] - i\frac{\gamma}{2j}\eta^2\cdot m\left[1+O\left(j^{-1}\right)\right] \\ & + \frac{1-\gamma^2}{4j}\eta^2\cdot m^2\left[1+O\left(j^{-1}\right)\right] - i\frac{\gamma}{2j}\eta^2\cdot m^2\left[1+O\left(j^{-1}\right)\right]\end{aligned}$$

the terms proportional to $m\eta^2$ are negligible with respect to terms $m^2\eta^2$.

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Back to $\ln\left(f_m^{(j)}(\eta)\right)$

$$\ln\left(f_m^{(j)}(\eta)\right) \quad = \quad -(j+1)\eta - \ m\eta + \ ij\gamma\eta + \ \psi(\eta)$$

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$$\begin{aligned}
\ln\left(f_m^{(j)}(\eta)\right) &= -(j+1)\eta - m\eta + ij\gamma\eta + \psi(\eta) \\
&= -j\eta\left[1+O\left(j^{-1}\right)\right] - m\eta + ij\gamma\eta \\
&\quad + j\eta\left[1+O\left(j^{-1}\right)\right] - i\gamma j\eta\left[1+O\left(j^{-1}\right)\right] \\
&\quad + \eta\cdot m\left[1+O\left(j^{-1}\right)\right] - i\gamma\eta\cdot m\left[1+O\left(j^{-1}\right)\right] \\
&\quad + j\eta^2\left[-\frac{(1-\gamma^2)}{4} + i\frac{\gamma}{2}\right]\left[1+O\left(j^{-1}\right)\right] \\
&\quad + \eta^2\cdot\frac{m^2}{j}\left[\frac{1-\gamma^2}{4} - i\frac{\gamma}{2}\right]\left[1+O\left(j^{-1}\right)\right]
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The linear terms cancel

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\ln\left(f_m^{(j)}(\eta)\right) &= -(j+1)\eta - m\eta + ij\gamma\eta + \psi(\eta) \\
&= - m\eta + ij\gamma\eta \\
&\quad - i\gamma j\eta\left[1+O\left(j^{-1}\right)\right] \\
&\quad +\eta\cdot m\left[1+O\left(j^{-1}\right)\right] - i\gamma\eta\cdot m\left[1+O\left(j^{-1}\right)\right] \\
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Back to $\ln\left(f_m^{(j)}(\eta)\right)$

$$\ln\left(f_m^{(j)}(\eta)\right) = -(j+1)\eta - m\eta + ij\gamma\eta + \psi(\eta)$$

$$= - i\gamma\eta \cdot m\left[1 + O\left(j^{-1}\right)\right] + j\eta^2\left[-\frac{(1-\gamma^2)}{4} + i\frac{\gamma}{2}\right]\left[1 + O\left(j^{-1}\right)\right] + \eta^2 \cdot \frac{m^2}{j}\left[\frac{1-\gamma^2}{4} - i\frac{\gamma}{2}\right]\left[1 + O\left(j^{-1}\right)\right] + \eta O(1) + i\gamma\eta O(1) + \frac{m}{j}\eta O(1)$$

The linear terms cancel

Finally $\frac{1}{J}\ln\left(f_m^{(j)}(\eta)\right)$

Lets reorganise the quadratic terms:

$$
\begin{aligned}
\ln\left(f_m^{(j)}(\eta)\right) &= \left(1 + i\gamma + \frac{m}{j}\right)\eta O(1) \\
&- i\gamma\eta \cdot m\left[1 + O\left(j^{-1}\right)\right] \\
&+ j\eta^2\left[1 - \frac{m^2}{j^2}\right]\left[-\frac{(1-\gamma^2)}{4} + i\frac{\gamma}{2}\right]\left[1 + O\left(j^{-1}\right)\right]
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\end{aligned}
$$

And finally divide everything by J

$$
\begin{aligned}
\frac{1}{J}\ln\left(f_m^{(j)}(\eta)\right) &= \eta O\left(\frac{1}{J}\right)\ -\ i\gamma\eta\cdot\frac{m}{J}\left(1+O\left(\frac{1}{J}\right)\right) \\
&+ x\eta^2\left[1-\frac{m^2}{j^2}\right]\left[-\ \frac{(1-\gamma^2)}{4}+\ i\frac{\gamma}{2}\right]\left(1+O\left(\frac{1}{J}\right)\right)
\end{aligned}
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where $x := \frac{j}{J}\in[0,1]$

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\begin{aligned}
\frac{1}{J}\ln\left(f_m^{(j)}(\eta)\right) = & - i\gamma\eta \cdot \frac{m}{J} \\
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$\phi_{\vec{m}}(\eta, J)$

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$\phi_{\vec{m}}(\eta, J)$

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so at the end of the day we have

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Outline

The final form of the Lorentzian polyhedra propagator

Given two basis elements $\iota, \iota' \in \mathrm{Inv}\left(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}\right)$ the Lorentzian polyhedra propagator's matrix element is

$$\mathbb{T}_{\iota\iota'} = \frac{1}{(4\pi)^2} \left(\frac{6\pi}{w_\gamma \sum_{i=1}^{N} j_i} \right)^{\frac{3}{2}} \quad \delta_{\iota\iota'}$$

for $w_\gamma = -(1-\gamma^2) + 2i\gamma$

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INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

FNP
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Thank you for your attention!

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