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Complete quantization of vacuum spherically symmetric gravity

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In spite of their simplicity, spherically symmetric vacuum space-times are challenging to quantize.

One has a Hamiltonian and (one) diffeomorphism constraint and, like those in the full theory, they do not form a Lie algebra. Therefore a traditional Dirac quantization is problematic.

There has been some progress in the past:

Kuchař (PRD50, 3961 (1994)) through a series of canonical transformation isolated the single degree of freedom of the model (the ADM mass). The resulting quantization has waveforms Ψ(M), with M being a Dirac observable. There is no sense in which the singularity is “resolved”.

Campiglia, Gambini and JP (CQG24, 3649 (2007)) gauge fixed the diffeomorphism constraint and rescaled the Hamiltonian constraint to make it Abelian. The quantization ends up being equivalent to Kuchař’s.

Various authors (Modesto, Boehmer and Vandersloot, Ashtekar and Bojowald, Campiglia, Gambini, JP) studied the quantization of the interior of a black hole using the isometry to Kantowski-Sachs and treating it as a LQC. The singularity is resolved.

Gambini and JP (PRL101, 161301 (2008)) studied the semiclassical theory for the complete space-time of a black hole. The singularity is replaced by a region of high curvature that tunnels into another region of space-time.

Summary:

The main point today: one can **rescale the Hamiltonian without gauge fixing the diffeomorphism constraint**. The resulting constraint algebra is a Lie algebra.

$[D,D]=D, \quad [D,H]=H, \quad [H,H]=0$

The Dirac quantization using loop quantum gravity techniques can be completed in exact form, finding the space of physical states H_{phys}.

The metric can be represented as an operator corresponding to an evolving constant of the motion on H_{phys} and the singularity is resolved

We use the variables adapted to spherical symmetry developed by Bojowald and Swiderski (CQG23, 2129 (2006)). One ends up with two canonical pairs, E^x, E^φ, K_x, K_φ.

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\mathrm{sign}(E^x)\frac{(E^\varphi)^2}{\sqrt{|E^x|}} K_x \qquad K_{\theta\theta} = -\sqrt{|E^x|}\frac{A_\varphi}{2\gamma},$$

$$H_T = N\left[-\frac{E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} + \frac{((E^x)')^2}{8\sqrt{E^x}E^\varphi} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''E^x}{2E^\varphi}\right] + N_r\left[-(E^x)'K_x + E^\varphi(K_\varphi)'\right].$$

Rescaling the lapse and shift:

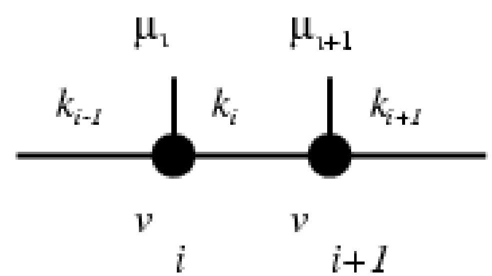
$$N_r^{\mathrm{old}} = N_r^{\mathrm{new}} - 2N^{\mathrm{old}}\frac{K_\varphi\sqrt{E^x}}{(E^x)'} \text{ and } N^{\mathrm{old}} = N^{\mathrm{new}}\frac{(E^x)'}{E^\varphi},$$

Yields the constraints with the Lie algebra structure:

$$H_T = \int dx \left[-N' \left(-\sqrt{E^x}\left(1 + K_\varphi^2\right) + \frac{\left((E^x)'\right)^2 \sqrt{E^x}}{4\left(E^\varphi\right)^2} + 2GM \right) + N_r \left[-(E^x)' K_x + E^\varphi (K_\varphi)' \right] \right]$$

To proceed to quantize we again follow Bojowald and Swiderski and define suitable one-dimensional "spin networks"

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \langle K_x, K_\varphi \big|$$



$$\big\rangle$$

$$= \prod_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} K_x(x) dx \right) \prod_{v_j \in g} \exp\left(\frac{i}{2} \mu_j \gamma K_\varphi(v_j) \right)$$

On such states the triads are well defined

$$\hat{E}^x(x)T_{g,\vec{k},\vec{\mu}}(K_x,K_\varphi) = \ell^2_{\rm Planck}k_i(x)T_{g,\vec{k},\vec{\mu}}(K_x,K_\varphi),$$

$$\hat{E}^\varphi(x)T_{g,\vec{k},\vec{\mu}}(K_x,K_\varphi) = \ell^2_{\rm Planck}\sum_{v_i\in g}\delta(x-x(v_i))\mu_i T_{g,\vec{k},\vec{\mu}}(K_x,K_\varphi),$$

And we proceed to polymerize and factor order the rescaled Hamiltonian constraint,

$$\hat{H}(N) = \int dx N(x)\left(2\left\{\sqrt{\sqrt{\hat{E}^x}\left(1+\frac{\sin^2\left(\rho\hat{K}_\varphi\right)}{\rho^2}\right)} - 2GM\right\}\hat{E}^\varphi - \sqrt[4]{\hat{E}^x}\left(\hat{E}^x\right)'\right),$$

And its action is well defined on the spin network states,

$$\hat{H}(N)T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \sum_{v_i \in g} N(v_i)\left(k_i \ell^2_{\text{Planck}}\right)^{\frac{1}{4}} \left[2\sqrt{1 + \frac{\sin^2\left(\rho K_\varphi(v_i)\right)}{\rho^2} - \frac{2GM}{\sqrt{k_i \ell^2_{\text{Planck}}}}} \ell^2_{\text{Planck}} \mu_i - \left(k_i - k_{i-1}\right)\ell^2_{\text{Planck}}\right] T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi).$$

And one can exactly solve it,

$$\Psi\left(K_\varphi, K_x, g, \vec{k}\right) = \Psi(M)\exp\left(f\left(K_\varphi, g, \vec{k}\right)\right)\Pi_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} K_x(x)dx\right),$$

$$f = \sum_{v_j \in g} -\tfrac{i}{2}\Delta K_j m_j F\left(\sin\left(\rho K_\varphi(v_j)\right), i m_j\right),$$

$$\text{with} \quad \Delta K_j = K_\varphi(v_j) - K_\varphi(v_{j-1}),$$

$$m_j = \left[\rho\sqrt{1 - 2GM/\sqrt{k_j}\ell_{\text{Planck}}}\right]^{-1}$$

$F(\phi, m) = \int_0^\phi \left(1 - m^2\sin^2 t\right)^{-1/2} dt$ the Jacobi elliptic function of the first kind.

The diffeomorphism constraint is solved by traditional group averaging. $|\vec{k}, \tilde{g}\rangle$.

The model has quantum observables without classical counterparts.

Since the basis of the physical space H_{phys} have a well defined number of vertices, one can construct a Dirac observable operator $\hat{V}$ with eigenvalue V, the number of vertices.

E^x is not well defined on H_{phys} as an operator, since it is not invariant under diffeomorphisms. However, since it must be a monotonous function of x, there is a portion of it that can be isolated as diffeo invariant.

One starts by noticing that the sequence $\vec{k}$ is well defined in H_{phys}

One defines a Dirac observable O(z) z in [0,1]

$$\hat{O}(z)|\vec{k},\tilde{g}\rangle_{\rm phys} = \ell^2_{\rm Planck} k_{{\rm Int}(V z)}|\vec{k},\tilde{g}\rangle_{\rm phys},$$

And in terms of it and an arbitrary function from the real line to [0,1] z(x) one can define an action for E^x in H_{phys},

$$\hat{E}^x(x)|\vec{k},\tilde{g}\rangle_{\rm phys} = \hat{O}(z(x))|\vec{k},\tilde{g}\rangle_{\rm phys}.$$

Recalling the form of the space-time metric, e.g., g_{tx},

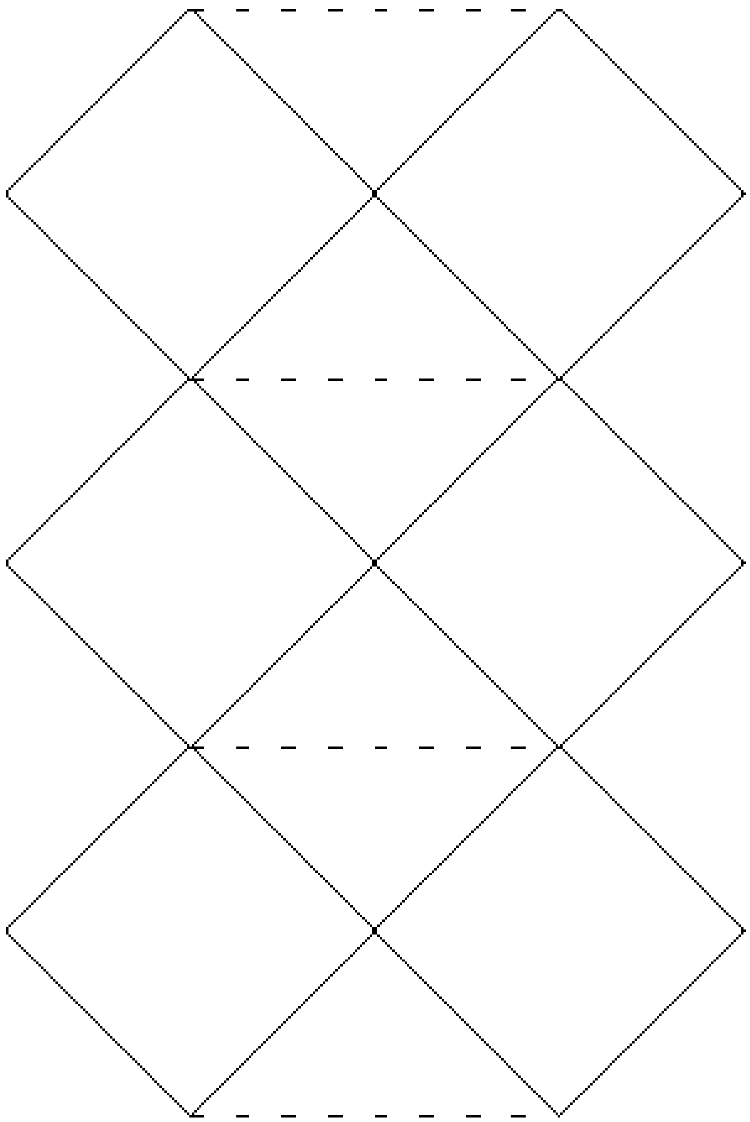
$$g_{tx} = g_{xx} N_r = -\frac{(E^x)' K_\varphi}{2\sqrt{E^x}\sqrt{1 + K_\varphi^2 - \frac{2GM}{\sqrt{E^x}}}},$$

One can straightforwardly write it as an evolving constant of motion acting on Hphys parameterized by the functional parameters Kφ and z(x).

In order to be a self-adjoint operator the radical should be Positive. This imposes limitations on the values of $\vec{k}$. The limitations imply that the metric is not singular where the classical singularity should be.

What kind of space-time emerges? It depends on the state.

If one wants a semi-classical space-time, one will have to choose $\Psi(M)$ peaked around some value of the mass, and one will need small jumps between k_i and k_{i+1}. The resulting geometry is distributional since Ex is only non-vanishing at vertices. One would be approximating a smooth function with Dirac deltas.



Some problems:

We do not know which of the quantum states possible is the on realized in nature. That would require a dynamical evolution.

Solutions of this sort are known to be unstable (Brady, Chambers, PRD51, 4177 (1995)).

Relevance to Firewalls?

Almheiri, Marolf, Polchinski, Sully, arXiv: 1207.3123 have proposed that black holes should be surrounded by "firewalls" to avoid certain contradictions involving evaporation. One way of seeing the problem stems from Marolf's observation (PRD79, 044710 (2009)) that for any theory whose Hamiltonian reduces to a boundary term and with an algebra of observables at infinity, the algebra evolves unitarily. So information must be preserved.

In our case in addition to the observable at infinity, we have observables in the bulk. So information at infinity could be lost either through the tunneling to another region in the interior of the black hole or to the observables in the bulk.

Although our analysis does not seem to extend to the inclusion of matter, the observables of the bulk still are present in that case.

Summary

- Rescaling the Hamiltonian constraint leads to a Lie algebra of constraints without the need to gauge fix.
- The Dirac quantization can be completed and the physical space of states found exactly.
- New quantum observables appear without classical counterpart.
- The metric can be realized on the space of physical states as an evolving constant of the motion.
- It is non-singular in the black hole interior and the space-time can be extended.
- It may open new possibilities for the “firewall” problem