SPINOR OPERATORS IN 3D LORENTZIAN GRAVITY

Giuseppe Sellaroli*

University of Waterloo

Third EFI winter conference on quantum gravity

* Work in collaboration with Florian Girelli

Representation theory of SU(1,1)

SU(1,1) is the double cover of the 3D Lorentz group $SO_0(2,1)$, with generators

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2J_0$$

and Casimir $Q = -J_0^2 + \frac{1}{2}(J_+J_- + J_-J_+)$.

They act on irreducible representations as

$$\begin{cases} J_0 |j\, m\rangle = m\, |j\, m\rangle \\ J_\pm |j\, m\rangle = C_\pm(j,m) |j\, m\pm 1\rangle \\ Q\, |j m\rangle = -j(j+1)\, |j\, m\rangle \end{cases}$$

with $C_\pm(j,m) = i\sqrt{j\mp m}\sqrt{j\pm m+1}$.

Representation theory of SU(1,1)

Representation classes:

- Continuous series $\mathcal{C}_j^{\varepsilon}$: $\varepsilon \in \{0, \frac{1}{2}\}$, $j \in \mathbb{C} \setminus \mathbb{Z}/2$, $m \in \varepsilon + \mathbb{Z}$
- Discrete series $D_j^{\pm}$: $j \in \{-\frac{1}{2}, 0, \frac{1}{2}, \dots\}$ $m \in \{\pm(j+1), \pm(j+2), \dots\}$
- Finite dimensional F_j: $j \in \{0, \frac{1}{2}, 1, \dots\}$ $m \in \{-j, \dots, j\}$

Only $D_j^{\pm}$ with $j \geq 0$ and $\mathcal{C}_j^{\varepsilon}$ with $j = -\frac{1}{2} + is$, $s > 0$ appear in the Plancherel decomposition.

Tensor operators

Tensor operators transform like vectors in a finite dimensional rep. F_γ :

$$[J_0, T^\gamma_\mu] = \mu T^\gamma_\mu \qquad [J_\pm, T^\gamma_\mu] = C_\pm(\gamma,\mu)\, T^\gamma_\mu \qquad |\mu| \leq \gamma$$

The algebra generators form a vector operator $V_0 = J_0, \quad V_{\pm 1} = \pm \frac{i}{\sqrt{2}} J_\pm$

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Two tensor operators can be used to build new operators:

$$T^\gamma_\mu = \sum_{\mu_1 \mu_2} \langle \gamma_1 \mu_1 \, \gamma_2 \mu_2 | \gamma \mu \rangle T^{\gamma_1}_{\mu_1} T^{\gamma_2}_{\mu_2}$$

Moreover, the Wigner-Eckart theorem holds: [GS, arXiv:1411.7467]

$$\langle j' m' | T^\gamma_\mu | j m \rangle = \langle j' \| T^\gamma \| j \rangle \, B(j' m' | \gamma \mu \, j m)$$

Spinor Operators

In the spinor formalism we try to find to spinor operators $T, \tilde{T}$ such that $V_\mu = -\frac{1}{\sqrt{2}} \sum_{\mu_1, \mu_2} \langle \frac{1}{2}\mu_1 \, \frac{1}{2}\mu_2 | 1 \mu \rangle T_{\mu_1} \tilde{T}_{\mu_2}$.

Explicitly $J_\pm = \pm i \, T_\pm \tilde{T}_\pm \qquad J_0 = -\frac{1}{2}(T_- \tilde{T}_+ + T_+ \tilde{T}_-)$.

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Using the Wigner-Eckart theorem one finds

$$\begin{cases} T_- |j m\rangle = \sqrt{j-m+1} \, |j+\tfrac{1}{2} \; m-\tfrac{1}{2}\rangle \\ T_+ |j m\rangle = \sqrt{j+m+1} \, |j+\tfrac{1}{2} \; m-\tfrac{1}{2}\rangle \\ \tilde{T}_- |j m\rangle = -\sqrt{j-m} \, |j+\tfrac{1}{2} \; m-\tfrac{1}{2}\rangle \\ \tilde{T}_+ |j m\rangle = \sqrt{j-m} \, |j+\tfrac{1}{2} \; m-\tfrac{1}{2}\rangle \end{cases}$$

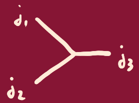
$$[T_+, \tilde{T}_-] = [\tilde{T}_+, T_-] = \mathbb{1}$$

$$\begin{cases} F_j : & T_\pm = \mp \tilde{T}_\mp^\dagger \\ D^+ : & T_\pm = -\tilde{T}_\mp^\dagger \\ D^+ : & T_\pm = -\tilde{T}_\mp^\dagger \\ \mathcal{C}^\varepsilon : & ? \end{cases}$$

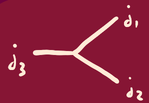
Intertwiners

LQG with SU(1,1) [Freidel, Livine, Rovelli: 2003] and spin networks for noncompact groups [Freidel, Livine 2003] has already been investigated.

We consider here spinor operators acting on single nodes, e.g.



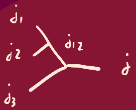
$$: \; |j_1 m_1\rangle \otimes |j_2 m_2\rangle \longmapsto \sum_{m_3} B(j_3 m_3 | j_1 m_1 \; j_2 m_2)\, |j_3 m_3\rangle$$



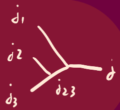
$$: \; |j_3 m_3\rangle \longmapsto \sum_{m_1, m_2} A(j_1 m_1 \; j_2 m_2 | j_3 m_3)\, |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

Intertwiners

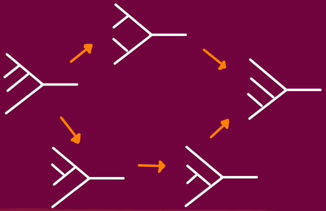
One can define **Racah coefficients** as



$$= \sum_{j_{23}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{bmatrix}$$



They satisfy the **Biedenharn-Elliott identity**



$$\sum_{j_{23}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{bmatrix} \begin{bmatrix} j_1 & j_{23} & j_{123} \\ j_4 & j & j_{234} \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ j_4 & j_{234} & j_{34} \end{bmatrix} = \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_{34} & j & j_{234} \end{bmatrix} \begin{bmatrix} j_{12} & j_3 & j_{123} \\ j_4 & j & j_{34} \end{bmatrix}$$

Spinor formalism

The spinor formalism is implemented by constructing scalar operators out of the $T, \tilde{T}$ operators, acting on different links.

$$E_{ab} = T_-^a \tilde{T}_+^b - T_+^a \tilde{T}_-^b \qquad F_{ab} = T_-^a T_+^b - T_+^a T_-^b \qquad \tilde{F}_{ab} = \tilde{T}_-^a \tilde{T}_+^b - \tilde{T}_+^a \tilde{T}_-^b$$

They form a closed algebra, e.g. $[E_{ab}, E_{cd}] = \delta_{cb} E_{ad} - \delta_{ad} E_{cb}$ ($u(N)$ algebra)

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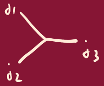
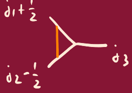
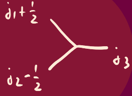
The usual observables $Q_{ab} = \vec{J}^{(a)} \cdot \vec{J}^{(b)}$ are recovered as

$$Q_{ab} = \frac{1}{4} E_{aa} E_{bb} - \frac{1}{2} E_{ab} E_{ba} + \left(\frac{1}{2} - \delta_{ab}\right) E_{aa} .$$

The action on intertwiners is given by

E_{12} 

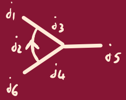
$\propto$ 

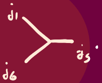
$= \begin{bmatrix} j_1 & 1/2 & j_1+1/2 \\ j_2-1/2 & j_3 & j_2 \end{bmatrix}$ 

and so on.

Spinor formalism

The $E, F, \tilde{F}$ operators encode the symmetries of the Racah coefficients.

Consider the intertwiner $\tau =$ 

$\equiv \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}$ 

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The operator $H = F_{23}\tilde{F}_{23} - \frac{i}{2j_4+1}\left(\tilde{F}_{34}\tilde{F}_{42} + E_{43}E_{42}\right)\tilde{F}_{23}$ annihilates it:

$H\tau \propto$ [Biedenharn - Elliot] 

$= 0$

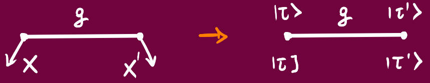
An analogous operator can be constructed in the SU(2) spinor formalism, encoding the quantization of a (proposed) Hamiltonian constraint [Bonzom, Livine 2011].

We will see next that, in the (classical) phase-space picture, this operator gives the flatness constraint in the spinor formalism.

Phase space picture

Poisson brackets on $su(1,1)$: $\{X_0, X_\pm\} = \mp i X_\pm \quad \{X_+, X_-\} = 2i X_0 \quad X_\pm = X_1 \pm i X_2$

We introduce two spinors $|\tau\rangle = \begin{pmatrix} \tilde{t}_- \\ \tilde{t}_+ \end{pmatrix} \in \mathbb{C}^2 \quad |\tau] = \begin{pmatrix} t_- \\ t_+ \end{pmatrix}$ satisfying $\{t_+, \tilde{t}_-\} = \{\tilde{t}_+, t_-\} = -i$. The idea is to replace the flux at each vertex of a graph with two spinors:



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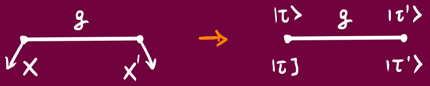


The spinors have information on both the flux and holonomy:

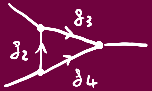
$$g = \frac{|\tau'\rangle[\tau| - |\tau'][\langle\tau|}{\sqrt{\langle\tau|\tau\rangle\langle\tau'|\tau'\rangle}} \in SU(1,1) \qquad g\,\frac{|\tau\rangle}{\sqrt{\langle\tau|\tau\rangle}} = -\frac{|\tau']}{\sqrt{\langle\tau'|\tau'\rangle}} \qquad g\,\frac{|\tau]}{\sqrt{\langle\tau|\tau\rangle}} = -\frac{|\tau'\rangle}{\sqrt{\langle\tau'|\tau'\rangle}}$$

$\langle\tau| = (t_+, -t_-) \qquad [\tau| = (\tilde{t}_+, -\tilde{t}_-)$

Phase space picture

A classical analogue of the $E, F, \tilde{F}$ operators is obtained as

$e_{ab} = -\langle \tau_a | \tau_b \rangle$ $\quad$ $f_{ab} = -\langle \tau_a | \tau_b]$ $\quad$ $\tilde{f}_{ab} = -[\tau_a | \tau_b \rangle$

Consider now the graph 

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The flatness constraint $\prod_{e \subset f} g_e = \mathbb{1}$ is implemented in the spinor formalism as $[\tau_2' | (\mathbb{1} - g_3^{-1} g_2^{-1} g_4) | \tau_3 \rangle = 0$ which can be shown to be the classical equivalent of the H operator.

Conclusions

- Results
  - Wigner-Eckart and Jordan-Schwinger for $su(1,1)$
  - Spinor formalism in Lorentzian gravity
  - Phase-space picture and Hamiltonian constraint
- Issues
  - unclear "classical" analogue for continuous class
- Next steps
  - generalization to $SL(2,\mathbb{C})$ [WIP]
  - generalization to $U_q(su(1,1))$

Thank you for listening!