Spinor operators in 3D Lorentzian gravity

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* Work in collaboration with Florian Girelli
Representation theory of $SU(1,1)$

$SU(1,1)$ is the double coser of the 3D Lorentz group $SO_0(2,1)$, with generators

$$[J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = -2 J_0$$

and Casimir

$$Q = -J_0^2 + \frac{1}{2} (J_+ J_- + J_- J_+).$$

They act on irreducible representations as

$$
\begin{align*}
J_0 |j m\rangle &= m |j m\rangle \\
J_\pm |j m\rangle &= C_{\pm}(j, m) |j m\pm 1\rangle \\
Q |j m\rangle &= -j(j+1) |j m\rangle
\end{align*}
$$

with

$$C_{\pm}(j, m) = i\sqrt{j \mp m} \sqrt{j \mp m + 1}. $$
Representation theory of SU(1,1)

Representation classes:

- Continuous series $C^\varepsilon_j$: $\varepsilon \in \{0, \frac{1}{2}\}$, $j \in \mathbb{C} \setminus \mathbb{Z}/2$, $m \in \mathbb{Z} + \mathbb{Z}$

- Discrete series $D^+_j$: $j \in \{-\frac{1}{2}, 0, \frac{1}{2}, \ldots\}$, $m \in \{\pm (j+1), \pm (j+2), \ldots\}$

- Finite dimensional $F_j$: $j \in \{0, \frac{1}{2}, 1, \ldots\}$, $m \in \{-j, \ldots, j\}$

Only $D^+_j$ with $j \geq 0$ and $C^\varepsilon_j$ with $j = -\frac{1}{2} + is$, $s > 0$ appear in the Plancherel decomposition.
Tensor operators transform line sectors in a finite dimensional rep. $F_8$:

\[ [J_0, T^\gamma_\mu] = \mu T^\gamma_\mu \quad [J_\pm, T^\gamma_\mu] = C_\pm (\gamma, \mu) T^\gamma_\mu \quad 1 \mu \leq \gamma \]

The algebra generators form a vector operator $V_0 = J_0, \quad V_{\pm} = \pm \frac{i}{\sqrt{2}} J_{\pm}$
Tensor operators

Tensor operators transform like sectors in a finite dimensional rep. $F_8$:

$$[J_0, T_\mu^\gamma] = \mu T^\gamma_\mu$$  $$[J_{\pm}, T_\mu^\gamma] = C_{\pm}(\gamma, \mu) T^\gamma_\mu$$  \(\mu, 1 \leq \gamma\)

The algebra generators form a vector operator $V_0 = J_0$,  $V_{\pm} = \pm \frac{i}{\sqrt{2}} J_{\pm}$

Two tensor operators can be used to build new operators:

$$T_\mu^\gamma = \sum_{\mu_1, \mu_2} \langle \gamma, \mu_1, \gamma, \mu_2 | \gamma, \mu \rangle T^\gamma_{\mu_1} T^\gamma_{\mu_2}$$

Moreover, the Wigner-Eckart theorem holds: [65, arxiv:1411.7467]

$$\langle \delta', m' | T_\mu^\gamma | \delta m \rangle = \langle \delta | T^\gamma | \delta \rangle B(\delta', m' | \gamma, \mu | \delta m)$$
In the spinor formalism we try to find the spinor operators $\tau, \bar{\tau}$ such that

$$\nabla_{\mu} = -\frac{i}{\sqrt{2}} \sum_{\mu_1, \mu_2} \langle \frac{1}{2} \mu, \frac{1}{2} \mu_2 | \mu \rangle \, T_{\mu_1} \bar{T}_{\mu_2}.$$

Explicitly

$$J_{\pm} = \pm i \, T_{\pm} \bar{T}_{\pm} \quad J_0 = -\frac{i}{2} (T_- \bar{T}_+ + T_+ \bar{T}_-).$$
Spinor Operators

In the spinor formalism we try to find the spinor operators $T, \tilde{T}$ such that

$$V_\mu = -\frac{i}{\sqrt{2}} \sum_{\mu, \mu_2} \langle \frac{1}{2} \mu, \frac{1}{2} \mu_2 | 1 \mu \rangle T_\mu, \tilde{T}_{\mu_2}.$$ 

Explicitly

$$J_\pm = \pm i T_\pm \tilde{T}_\pm \quad J_0 = -\frac{i}{2} (T_- \tilde{T}_+ + T_+ \tilde{T}_-).$$

Using the Wigner-Eckart theorem one finds

$$\begin{align*}
T_- |j m\rangle &= \sqrt{j - m + 1} |j + \frac{1}{2} m - \frac{1}{2}\rangle \\
T_+ |j m\rangle &= \sqrt{j + m + 1} |j + \frac{1}{2} m - \frac{1}{2}\rangle \\
\tilde{T}_- |j m\rangle &= -\sqrt{j - m} |j + \frac{1}{2} m - \frac{1}{2}\rangle \\
\tilde{T}_+ |j m\rangle &= \sqrt{j - m} |j + \frac{1}{2} m - \frac{1}{2}\rangle
\end{align*}$$

$$[T_+, \tilde{T}_-] = [\tilde{T}_+, T_-] = 0$$

$$\begin{align*}
F_\pm: \quad & T_\pm = \mp \tilde{T}_\pm^t \\
D^+: \quad & T_\pm = -\tilde{T}_\pm^t \\
D^+: \quad & T_\pm = -\tilde{T}_\pm^t \\
C^\epsilon: \quad & ?
\end{align*}$$
Intertwiners

LQG with $SU(1,1)$ [Freidel, Livine, Rosell: 2003] and spin networks for noncompact groups [Freidel, Livine 2003] has already been investigated.

We consider here spinor operators acting on single nodes, e.g.

\[ \gamma \begin{array}{c} \downarrow d_3 \quad \downarrow d_2 \end{array} : |d_1, m, \otimes |d_2 m_2 \rangle \rightarrow \sum_{m_3} B(d_3, m_3 |d_1, m, d_2 m_2 ) |d_3 m_3 \rangle \]

\[ \gamma \begin{array}{c} \downarrow d_3 \quad \downarrow d_2 \end{array} : |d_3 m_3 \rangle \rightarrow \sum_{m, m_2} A(d, m, d_2 m_2 |d_3 m_3 ) |d, m, \otimes |d_2 m_2 \rangle \]
Intertwiners

One can define Racah coefficients as

$$d_1 d_2 \otimes d_2 d_3 = \sum_{d_3} \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_3 & d_{23} & d_{123} \end{bmatrix} \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_3 & d_{23} & d_{123} \end{bmatrix}$$

They satisfy the Biedenharn-Elliott identity

$$\sum_{d_23} \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_2 & d_3 & d_{23} \end{bmatrix} \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_4 & d_{23} & d_{123} \end{bmatrix} \begin{bmatrix} d_2 & d_3 & d_{23} \\ d_4 & d_{234} & d_{34} \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_3 & d_{123} & d_{123} \end{bmatrix} \begin{bmatrix} d_1 & d_2 & d_{12} \\ d_3 & d_{23} & d_{123} \end{bmatrix}$$
Spinor formalism

The spinor formalism is implemented by constructing scalar operators out of the $T, \tilde{T}$ operators, acting on different lines.

\[
E_{ab} = T_a T_b - T_b T_a \\
F_{ab} = T_a T_b - T_b T_a \\
\tilde{F}_{ab} = T_a T_b - T_b T_a
\]

They form a closed algebra, e.g. \([E_{ab}, E_{cd}] = Sc_b E_{ad} - Sd_a E_{cb}\) \((u(N)\text{ algebra})\)
Spinor formalism

The spinor formalism is implemented by constructing scalar operators out of the $T, \tilde{T}$ operators, acting on different links.

$$ E_{ab} = T^a_+ \tilde{T}^b_+ - T^a_- \tilde{T}^b_- \quad F_{ab} = T^a_+ T^b_+ - T^a_- T^b_- \quad \tilde{F}_{ab} = \tilde{T}^a_+ \tilde{T}^b_+ - \tilde{T}^a_- \tilde{T}^b_- $$

They form a closed algebra, e.g. $[E_{ab}, E_{cd}] = Scb E_{ad} - Sad E_{cb}$ (u(N) algebra)

The usual observables $Q_{ab} = \bar{J}^{(a)}(c) \cdot J^{(b)}(d)$ are recovered as

$$ Q_{ab} = \frac{1}{4} E_{ac} E_{bb} - \frac{1}{2} E_{ab} E_{ba} + (\frac{1}{2} - \delta_{ab}) E_{aa}. $$

The action on intertwiners is given by

$$ E_{12} \underset{d_3}{\underset{d_2}{\otimes}} \underset{d_1}{\underset{d_{1+\frac{1}{2}}}{\otimes}} = \left[ \begin{array}{ccc} d_1 & \frac{1}{2} & d_{1+\frac{1}{2}} \\ d_{2-\frac{1}{2}} & d_3 & d_2 \\ d_2^{-\frac{1}{2}} & \end{array} \right]. $$

and so on.
Spinor formalism

The $E,F,\tilde{F}$ operators encode the symmetries of the Racah coefficients.

Consider the intertwiner $\tau = \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_6 & d_5 & d_4 \end{pmatrix}$.

The operator $H = F_{23} \tilde{F}_{23} - \frac{i}{2 \delta_{d_6} + 1} (\tilde{F}_{34} F_{42} + E_{d_3} E_{d_2}) \tilde{F}_{23}$ annihilates it:

\[ H \tau = \alpha \left[ \text{Biedenharn-Elliott} \right] \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 0 \]

An analogous operator can be constructed in the $SU(2)$ spinor formalism, encoding the quantization of a (proposed) Hamiltonian constraint [Bonzom, Livine 2011]. We will see next that, in the (classical) phase-space picture, this operator gives the flatness constraint in the spinor formalism.
Phase space picture

Poisson brackets on $\text{su}(1,1)$:

$$\{X_0, X_\pm\} = \mp i X_\pm \quad \{X_+, X_-\} = 2i X_0 \quad X_\pm = X_0 \pm i X_1$$

We introduce two spinors $1\zeta = (\zeta^-, \zeta^+) \in \mathbb{C}^2$ $1\zeta' = (\zeta'^-, \zeta'^+)$ satisfying

$$\{\zeta^+, \zeta^-\} = \{\zeta'^+, \zeta'^-\} = -i$$

The idea is to replace the flux at each vertex of a graph with two spinors:

$$\zeta \xrightarrow{g} \zeta'$$

$$\zeta \xrightarrow{1\zeta} \zeta'$$
Phase space picture

Poisson brackets on $su(1,1)$:
\[
\{X_0, X_\pm\} = \pm i X_\pm, \quad \{X_+, X_-\} = 2i X_0, \quad X_\pm \equiv X_1 \pm i X_2
\]

We introduce two spinors $1\tilde{\zeta} = \begin{pmatrix} \tilde{\zeta}_- \\ \tilde{\zeta}_+ \end{pmatrix} \in \mathbb{C}^2$, $1\tilde{\zeta}' = \begin{pmatrix} t_- \\ t_+ \end{pmatrix}$ satisfying $\{t_+, \tilde{\zeta}_-\} = \{\tilde{\zeta}_+, t_-\} = -i$. The idea is to replace the flux at each vertex of a graph with two spinors:

The spinors have information on both the flux and holonomy:

\[
g : 1\tilde{\zeta}\left[1\bar{\zeta} - 1\bar{\zeta}'\right]<1\bar{\zeta}\bar{\zeta}> \in SU(1,1)
\]

\[
g \frac{1\tilde{\zeta}}{\sqrt{<1\bar{\zeta}2\bar{\zeta}> <1\bar{\zeta}'2\bar{\zeta}'>}} = -\frac{1\bar{\zeta}'}{\sqrt{<1\bar{\zeta}2\bar{\zeta}> <1\bar{\zeta}'2\bar{\zeta}'>}}
\]

\[
g \frac{1\bar{\zeta}}{\sqrt{<1\bar{\zeta}2\bar{\zeta}> <1\bar{\zeta}'2\bar{\zeta}'>}} = -\frac{1\bar{\zeta}}{\sqrt{<1\bar{\zeta}2\bar{\zeta}> <1\bar{\zeta}'2\bar{\zeta}'>}}
\]

$1\tilde{\zeta} = (t_+, -t_-)$, $1\bar{\zeta} = (\tilde{\zeta}_+, -\tilde{\zeta}_-)$
A classical analogue of the $E,F,\tilde{F}$ operators is obtained as

$$e_{ab} = - \langle \tau_1 \tau_2 \rangle$$
$$f_{ab} = - \langle \tau_2 \tau_3 \rangle$$
$$\tilde{f}_{ab} = - \langle \tau_3 \tau_4 \rangle$$

Consider now the graph

The flatness constraint $\Pi_{ecf} g_e = 1$ is implemented in the spinor formalism as

$$\left[ \tau_2 \left( 1 - \delta_3 \delta_2 \delta_4 \right) \right] \tau_3 = 0$$

which can be shown to be the classical equivalent of the $H$ operator.
Conclusions

• Results
  • Wigner–Eckart and Jordan–Schwinger for SU(1,1)
  • Spinor formalism in Lorentzian gravity
  • Phase-space picture and Hamiltonian constraint

• Issues
  • unclear "classical" analogue for continuous class

• Next steps
  • generalization to SL(2,C) (WIP)
  • generalization to $U_q(su(1,1))$
Thank you for listening!