Deparametrising GR with distances and angles

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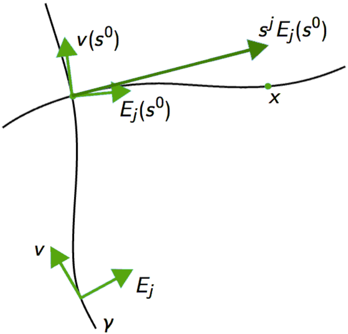
Plan of the talk

- Introduction - a perspective
- Setup - definitions
- Observables
- Variations - Poisson brackets
- Summary and final remarks

Fermi coordinates:



Introduction - last time...

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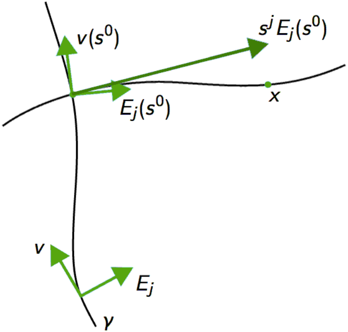

$$g^{\mu\nu}(x) \quad \rightsquigarrow \quad \mathcal{O}^{IJ}_{g^{\mu\nu};f}(s^0, s^j) := s^I_{,\mu}(x) s^J_{,\nu}(x) g^{\mu\nu}(x)|_{x \text{ s.t. } f(\mathbf{s^0}, \mathbf{s^j}) = x}$$

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where e_I is an ONB w.r.t. q and M_{IJ} is lower-triangular.

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and $(y^a) = (y^r, y^A)$ - "spherical" adapted coordinates such that

$$z^1 = y^r \sin y^\theta \cos y^\varphi \qquad z^2 = y^r \sin y^\theta \sin y^\varphi \qquad z^3 = y^r \cos y^\theta$$

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- The map $q \mapsto (y^a)$ is invariant w.r.t. diffeomorphisms $\psi \in \mathrm{Diff}_{\sigma_0}$ which preserve the observer, namely such that

$$\psi(\sigma_0) = \sigma_0 \qquad \psi'(\sigma_0) = M$$

Observables

Introduce observables:

$$
\begin{aligned}
Q_{ab}(r,\theta) : (q, p, \phi_\alpha, \pi^\alpha) &\mapsto q_{ab}(r,\theta) \\
P^{ab}(r,\theta) : (q, p, \phi_\alpha, \pi^\alpha) &\mapsto p^{ab}(r,\theta) \\
\Phi_\alpha(r,\theta) : (q, p, \phi_\alpha, \pi^\alpha) &\mapsto \phi_\alpha(r,\theta) \\
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- The above observables are Diff_{σ_0}-invariant.
- By construction
$$Q_{rr}(r,\theta) = 1 \qquad Q_{rA}(r,\theta) = 0$$
- Denote the interesting observables by $F(r,\theta)$ where
$$F \text{ runs through } \{Q_{AB}, P^{AB}, \Phi_\alpha, \Pi^\alpha\}$$

Variations of the observables

Given a point $\check{\gamma} = (\check{q}, \check{p}, \check{\phi}_\alpha, \check{\pi}^\alpha) \in \Gamma$ perform the following decomposition at a point $\gamma = (q, p, \phi_\alpha, \pi^\alpha) \in \Gamma$

$$F(r,\theta)|_\gamma = f(r,\theta)|_\gamma + G_{F(r,\theta)}\big|_\gamma$$

where the point at which f is evaluated is the one for which coordinates adapted to $\check{q}$ (and not q!) give the values (r, θ).

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Consider $\gamma = (\check{q} + \epsilon\delta q, \check{p} + \epsilon\delta p, \check{\phi}_\alpha + \epsilon\delta\phi_\alpha, \check{\pi}^\alpha + \epsilon\delta\pi^\alpha)$. The variations

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f(r, \theta)|_\gamma$$

are found imidiately. The variations

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G_{F(r,\theta)}\big|_\gamma$$

require more care.

Variations of $G_{F(r,\theta)}$

- Notice that

$$G_{F(r,\theta)}\big|_{\gamma} = 0$$

when the coordinates adapted to q and $\check{q}$ coincide (in a neighbourhood of the geodesic line connecting σ_0 and σ such that $(y^a(\sigma)) = (r, \theta)$). It implies also that

$$q_{rr} = 1 \qquad q_{rA} = 0, \text{ when expressed in the coordinates adapted to } \check{q}$$

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- Moreover, the observables we defined are Diff_{σ_0}-invariant, so

$$\{F(r,\theta),\ C(\vec{N})\} = 0$$

for $\vec{N}$ generating Diff_{σ_0}. Hence

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Combining the former two points we find that

$$\int d^3\sigma\ \frac{\delta G_{F(r,\theta)}}{\delta q_{ab}(\sigma)} \mathcal{L}_{\vec{N}} \check{q}_{ab}(\sigma) = -\mathcal{L}_{\vec{N}} f(r,\theta)$$

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Variations of $G_{F(r,\theta)}$ - part 2

It turns out that every variation δq can be cast into the form

$$\delta q = \mathcal{L}_{\vec{N}} \check{q} + \delta \tilde{q} \qquad (*)$$

for some $\vec{N}$ generating Diff_{σ_0}, where $\delta \tilde{q}$ is such that $\delta \tilde{q}_{rr} = 0 = \delta \tilde{q}_{rA}$.

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From $(*)$ one finds that

$$N^r{}_{,r} = \frac{1}{2}\delta q_{rr} \qquad N^A{}_{,r} = \check{q}^{AB}(\delta q_{rB} - N^r{}_{,B})$$

with initial conditions at $r = 0$ given by the requirement of belonging to $\mathrm{Diff}_{\sigma_{\mathbf{0}}}$.

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$$\vec{N} = \left[\frac{1}{2}\int_0^r dr' \delta q_{rr}(r',\theta)\right]\partial_r + \\ + \left[\int_0^r dr' \check{q}^{AB}(r',\theta)\left(\delta q_{rB}(r',\theta) - \frac{1}{2}\int_0^{r'} dr'' \partial_B \delta q_{rr}(r'',\theta)\right)\right]\partial_A + \\ + \left[\frac{1}{2}\overline{\delta q}_{IJ}(\sigma_0) z^J\left(\delta^{IK} - \frac{z^I z^K}{r^2}\right)\right]\partial_K$$

Poisson brackets

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They are canonical!

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The Poisson brackets of $P^{ra}(r,\theta)$'s can be found similarly. They turn out to be expressible in terms of the observables $F(r,\theta)$.

Summary and final remarks

- Reduced phase space for GR:
  The vanishing-of-the-vector-constraint condition ($q_{ab}\nabla_c P^{bc} = 0$) can be integrated radialy to give an expression of P^{ra} in terms of the observables $F(r,\theta)$. Hence we (almost) solved the vector constraint.

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- How do the diffeomorphisms not belonging to Diff_{σ_0} act on $F(r,\theta)$?
  The action of a general diffeomorphism on F's can be understood using formulas derived above. It turns out that a general generator can be decomposed into a field preserving F's (the part belonging to Diff_{σ_0}) and a part which preserves the "radial form of the metric". This additional part can be parametrised by 6 numbers:
  - 3 coming from the value of the field at σ_0 (the translational part)
  - 3 coming from the (necessarily antisymmetric) value of the first derivative of the field at σ_0 (the rotational part)

Thank you for your attention!

based on:

Duch, Kamiński, Lewandowski, Świeżewski
Observables for General Relativity related to geometry - to appear soon

Let

$$\psi_\epsilon : \sigma(\check{q}; r, \theta) \mapsto \sigma(\check{q} + \epsilon \delta q; r, \theta)$$

notice that

$$\delta q = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \psi_\epsilon^*(\check{q} + \epsilon \delta q) - \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \psi_\epsilon^*(\check{q}) = \delta \tilde{q} + \mathcal{L}_{\vec{N}} \check{q}$$

where

- $\delta \tilde{q}_{ra} = 0$ when expressed in coordinates adapted to $\check{q}$
- $\vec{N} = - \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \psi_\epsilon^*$