Radial Gauge

reduced phase space of General Relativity

Jędrzej Świeżewski

in collaboration with: Norbert Bodendorfer and Jerzy Lewandowski

Tux, 17.02.2015

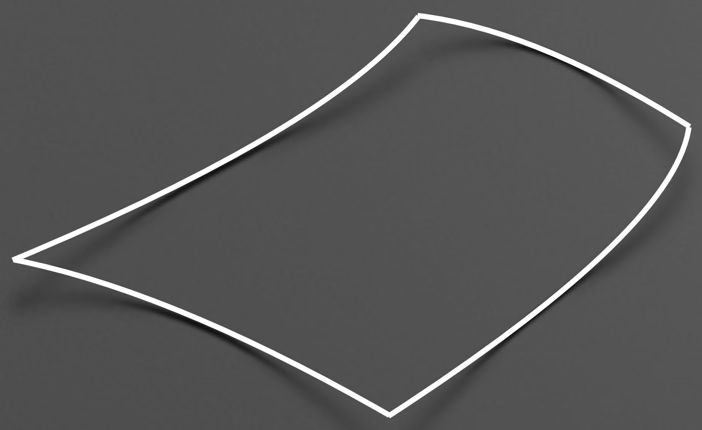
Radial Gauge

is a certain gauge for canonical General Relativity

useful for discussing quantum spherical symmetry

Preparations

spatial slice

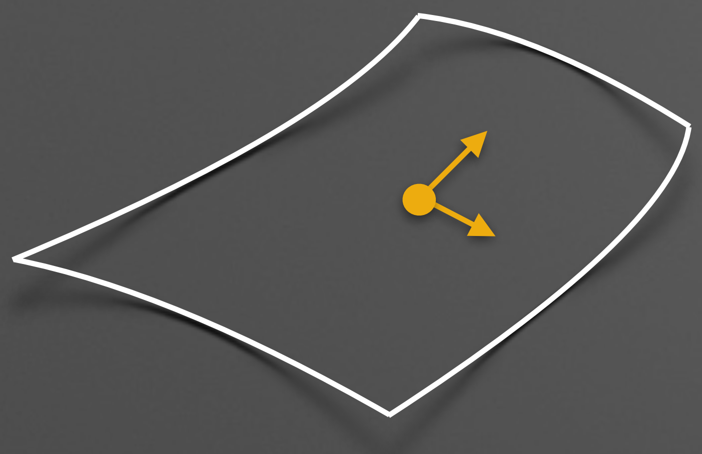


metric

q_{ab}

Preparations

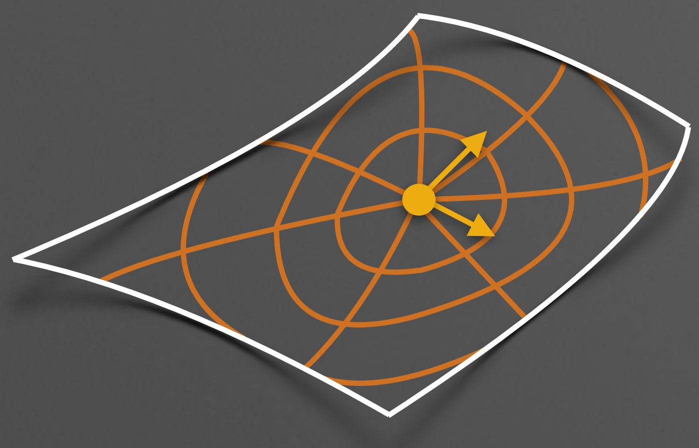
observer



q_{ab}

Preparations

observer



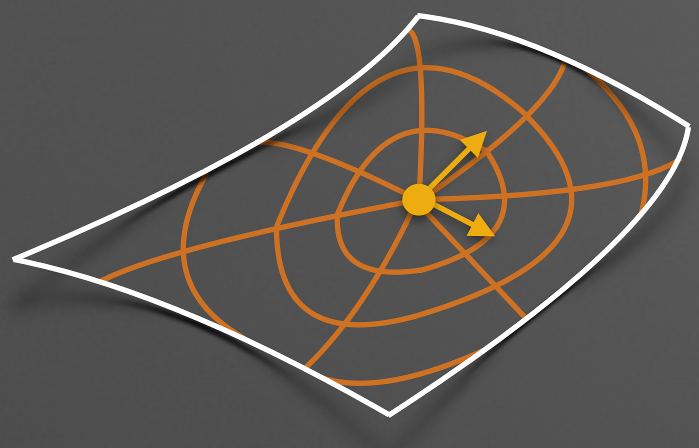
adapted coordinates

(r, θ)

q_{ab}

Preparations

observer



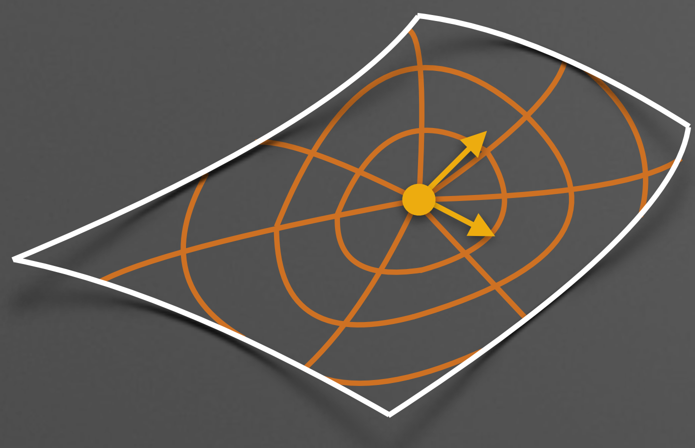
adapted coordinates
(r, θ)

observables

$$q_{ab} \rightsquigarrow Q_{ab}(r, \theta)$$

Preparations

observer



adapted coordinates
(r, θ)

observables

$$q_{ab} \rightsquigarrow Q_{ab}(r, \theta)$$

in adapted coordinates
metric has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & Q_{AB} & \end{bmatrix}$$

$$\{Q_{AB}(r, \theta),\ P^{CD}(r', \theta')\} = \delta\delta\delta$$

canonical subalgebra

appeared in: Duch, Kamiński, Lewandowski, JŚ JHEP05(2014)077

Gauge fixing

we want to fix the gauge: $q_{ra} = \delta_{ra}$

Gauge fixing

we want to fix the gauge: $q_{ra} = \delta_{ra}$

Dirac matrix $\left\{ q_{ra},\ C[\vec{N}] \right\} = \omega_{ra}$ is invertible (in $\mathrm{Diff}_{\,\mathrm{obs}}$)

Gauge fixing

we want to fix the gauge: $q_{ra} = \delta_{ra}$

Dirac matrix $\left\{q_{ra},\ C[\vec{N}]\right\} = \omega_{ra}$ is invertible (in $\mathrm{Diff}_{\mathrm{obs}}$)

$$\vec{N}(r,\theta) = \left[\frac{1}{2}\overline{\omega}_{KJ}(0)h^{JL}rn^{K}\right]\partial_L + \frac{1}{2}\left[\int_0^r dr'\ \omega_{rr}(r',\theta)\right]\partial_r +$$
$$+\left[\int_0^r dr'\ q^{BA}(r',\theta)\left(\omega_{rA}(r',\theta) - \frac{1}{2}\partial_A\left(\int_0^{r'} dr''\ \omega_{rr}(r'',\theta)\right)\right)\right]\partial_B$$

Gauge fixed dynamics

we need to find $\vec{N}_H$ such that $H[N] + C[\vec{N}_H]$ preserves our gauge

$$\left\{q_{ra},\ H[N] + C[\vec{N}_H]\right\} = 0$$

Gauge fixed dynamics

we need to find $\vec{N}_H$ such that $H[N]+C[\vec{N}_H]$ preserves our gauge

$$\{q_{ra},\ C[\vec{N}_H]\} = -\{q_{ra},\ H[N]\}$$

Gauge fixed dynamics

we need to find $\vec{N}_H$ such that $H[N]+C[\vec{N}_H]$ preserves our gauge

$$\{q_{ra},\ C[\vec{N}_H]\} = -\{q_{ra},\ H[N]\}$$

$$H[N]|_{\text{gauge-fix}} = \int N\left(\frac{1}{\sqrt{\det q}}G - \sqrt{\det q}\,{}^{(3)}R\right)$$

where

$$G = \frac{1}{2}(\widetilde{p}^r{}_r)^2 + 2q^{AB}\widetilde{p}^r{}_A\widetilde{p}^r{}_B - q_{AB}p^{AB}\widetilde{p}^r{}_r + (q_{AC}q_{BD} - \frac{1}{2}q_{AB}q_{CD})p^{AB}p^{CD}$$

$${}^{(3)}R = {}^{(2)}R - q^{AB}q_{AB,rr} - \frac{3}{4}q^{AB}{}_{,r}q_{AB,r} - \frac{1}{4}(q^{AB}q_{AB,r})^2$$

Jędrzej Świeżewski, University of Warsaw

Gauge fixed dynamics

we need to find $\vec{N}_H$ such that $H[N] + C[\vec{N}_H]$ preserves our gauge

$$\{q_{ra},\ C[\vec{N}_H]\} = -\{q_{ra},\ H[N]\}$$

$$H[N]|_{\text{gauge-fix}} = \int N\left(\frac{1}{\sqrt{\det q}}G - \sqrt{\det q}\,{}^{(3)}R\right)$$

where

$$G = \frac{1}{2}(\widetilde{p}^r{}_r)^2 + 2q^{AB}\widetilde{p}^r{}_A\widetilde{p}^r{}_B - q_{AB}p^{AB}\widetilde{p}^r{}_r + (q_{AC}q_{BD} - \frac{1}{2}q_{AB}q_{CD})p^{AB}p^{CD}$$

$${}^{(3)}R = {}^{(2)}R - q^{AB}q_{AB,rr} - \frac{3}{4}q^{AB}{}_{,r}q_{AB,r} - \frac{1}{4}(q^{AB}q_{AB,r})^2$$

$$\widetilde{p}^r{}_A(r,\theta) = \int_r^\infty dr'\mathcal{D}_Bp^B{}_A(r',\theta)$$

$$\widetilde{p}^r{}_r(r,\theta) = -\frac{1}{2}\int_r^\infty dr'\left(p^{AB}q_{AB,r}\right)(r',\theta) + \int_r^\infty dr'\mathcal{D}_A\left(q^{AB}(r',\theta)\int_{r'}^\infty dr''\left(\mathcal{D}_Cp^C{}_B(r'',\theta)\right)\right)$$

Example 1: Spherical symmetry

take metric in the form $\begin{bmatrix} \Lambda^2(r) & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{R^2(r)\eta_{AB}} \end{bmatrix}$

the constraints are

$$C[\vec{N}] = \int_0^\infty dr N^r \left(P_R R' - \Lambda P_\Lambda'\right)$$

$$H[N] = \int_0^\infty dr N \left(\frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}\right)$$

Example 1: Spherical symmetry

$$\begin{bmatrix} \Lambda^2(r) & 0 & 0 \\ 0 & R^2(r)\eta_{AB} & \\ 0 & & \end{bmatrix}$$

$$C[\vec{N}] = \int_0^\infty dr N^r \left(P_R R' - \Lambda P_\Lambda'\right)$$

$$H[N] = \int_0^\infty dr N \left(\frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}\right)$$

impose gauge $\Lambda = 1$

the Hamiltonian preserving the gauge is

$$H[N]|_{\text{gauge-fix}} = \int_0^\infty dr N \left(\frac{1}{2R^2}\left(-\int_r^\infty dr' \left(P_R R'\right)(r')\right)^2 + \frac{P_R}{R}\int_r^\infty dr' \left(P_R R'\right)(r') + RR'' + \frac{R'^2}{2} - \frac{1}{2}\right)$$

Example 1: Spherical symmetry

$$\begin{bmatrix} \Lambda^2(r) & 0 & 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \multicolumn{2}{c}{R^2(r)\eta_{AB}} \end{bmatrix}$$

$$C[\vec{N}] = \int_0^\infty dr N^r \left(P_R R' - \Lambda P_\Lambda'\right)$$

$$H[N] = \int_0^\infty dr N \left(\frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}\right)$$

impose gauge $\Lambda = 1$

the Hamiltonian preserving the gauge is

$$H[N]|_{\text{gauge-fix}} = \int_0^\infty dr N \left(\frac{1}{2R^2}\left(-\int_r^\infty dr' \left(P_R R'\right)(r')\right)^2 + \frac{P_R}{R}\int_r^\infty dr' \left(P_R R'\right)(r') + RR'' + \frac{R'^2}{2} - \frac{1}{2}\right)$$

it gives the following equations of motion

$$\frac{1}{N}\dot{R}(r) = -\frac{F(r)}{R(r)} + R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

$$\frac{1}{N}\dot{P}_R(r) = -R''(r) + \frac{P_R^2(r)}{R(r)} - 2\frac{P_R(r)F(r)}{R^2(r)} + \frac{F^2(r)}{R^3(r)} + P_R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

Example 1: Spherical symmetry

$$\begin{bmatrix} \Lambda^2(r) & 0 & 0 \\ 0 & \multicolumn{2}{c}{R^2(r)\eta_{AB}} \\ 0 & & \end{bmatrix}$$

$$C[\vec{N}] = \int_0^\infty dr N^r \left(P_R R' - \Lambda P'_\Lambda\right)$$

$$H[N] = \int_0^\infty dr N \left(\frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}\right)$$

impose gauge $\Lambda = 1$

the Hamiltonian preserving the gauge is

$$H[N]|_{\text{gauge-fix}} = \int_0^\infty dr N \left(\frac{1}{2R^2}\left(-\int_r^\infty dr' \left(P_R R'\right)(r')\right)^2 + \frac{P_R}{R}\int_r^\infty dr' \left(P_R R'\right)(r') + RR'' + \frac{R'^2}{2} - \frac{1}{2}\right)$$

it gives the following equations of motion

$$\frac{1}{N}\dot{R}(r) = -\frac{F(r)}{R(r)} + R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

$$\frac{1}{N}\dot{P}_R(r) = -R''(r) + \frac{P_R^2(r)}{R(r)} - 2\frac{P_R(r)F(r)}{R^2(r)} + \frac{F^2(r)}{R^3(r)} + P'_R(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

Jędrzej Świeżewski, University of Warsaw

$$\frac{1}{N}\dot{R}(r) = -\frac{F(r)}{R(r)} + R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

$$\frac{1}{N}\dot{P}_R(r) = -R''(r) + \frac{P_R^2(r)}{R(r)} - 2\frac{P_R(r)F(r)}{R^2(r)} + \frac{F^2(r)}{R^3(r)} + P_R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

Example 1a: Minkowski

setting $P_R(r) = 0$ constantly in time, we obtain

$$\dot{R}(r) = 0$$

$$0 = R''(r)$$

→

$$R(r) = r$$

$$(\ N^r = 0\ )$$

$$\frac{1}{N}\dot{R}(r) = -\frac{F(r)}{R(r)} + R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

$$\frac{1}{N}\dot{P}_R(r) = -R''(r) + \frac{P_R^2(r)}{R(r)} - 2\frac{P_R(r)F(r)}{R^2(r)} + \frac{F^2(r)}{R^3(r)} + P_R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

Example 1b: Schwarzschild

Schwarzschild metric in free-fall-coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2$$

N $\quad N^r$ $\quad \Lambda^2$ $\quad R^2$

Jędrzej Świeżewski, University of Warsaw

$$\frac{1}{N}\dot{R}(r) = -\frac{F(r)}{R(r)} + R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

$$\frac{1}{N}\dot{P}_R(r) = -R''(r) + \frac{P_R^2(r)}{R(r)} - 2\frac{P_R(r)F(r)}{R^2(r)} + \frac{F^2(r)}{R^3(r)} + P_R'(r)\int_0^r dr' \left(\frac{P_R(r')}{R(r')} - \frac{F(r')}{R^2(r')}\right)$$

Example 1b: Schwarzschild

Schwarzschild metric in free-fall-coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2$$

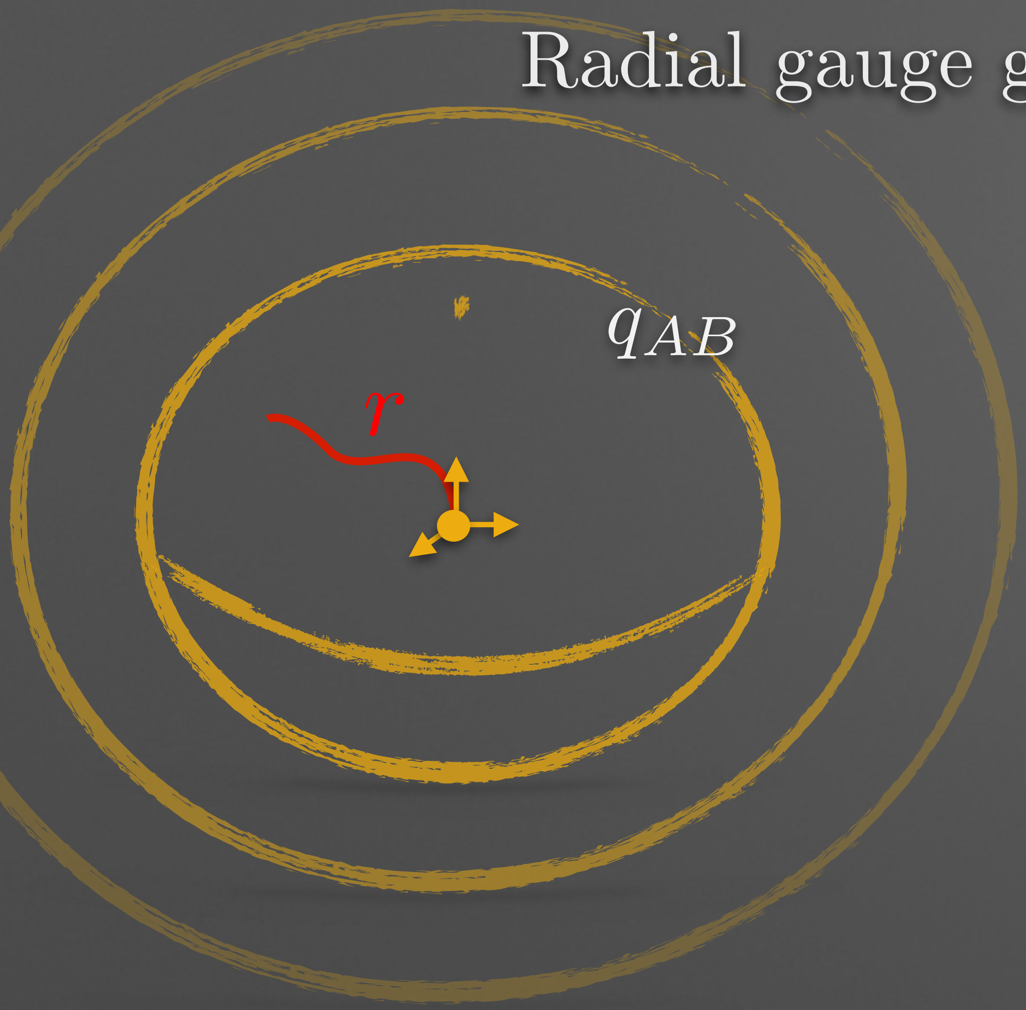
$$N \; N^r \; \Lambda^2 \; R^2 \quad \Rightarrow \quad P_\Lambda \; \& \; P_R$$

It turns out

$$N_H^r = N^r \qquad H[N] + C[\vec{N}_H] = 0 \qquad \dot{R} = 0 \qquad \dot{P}_R = 0$$

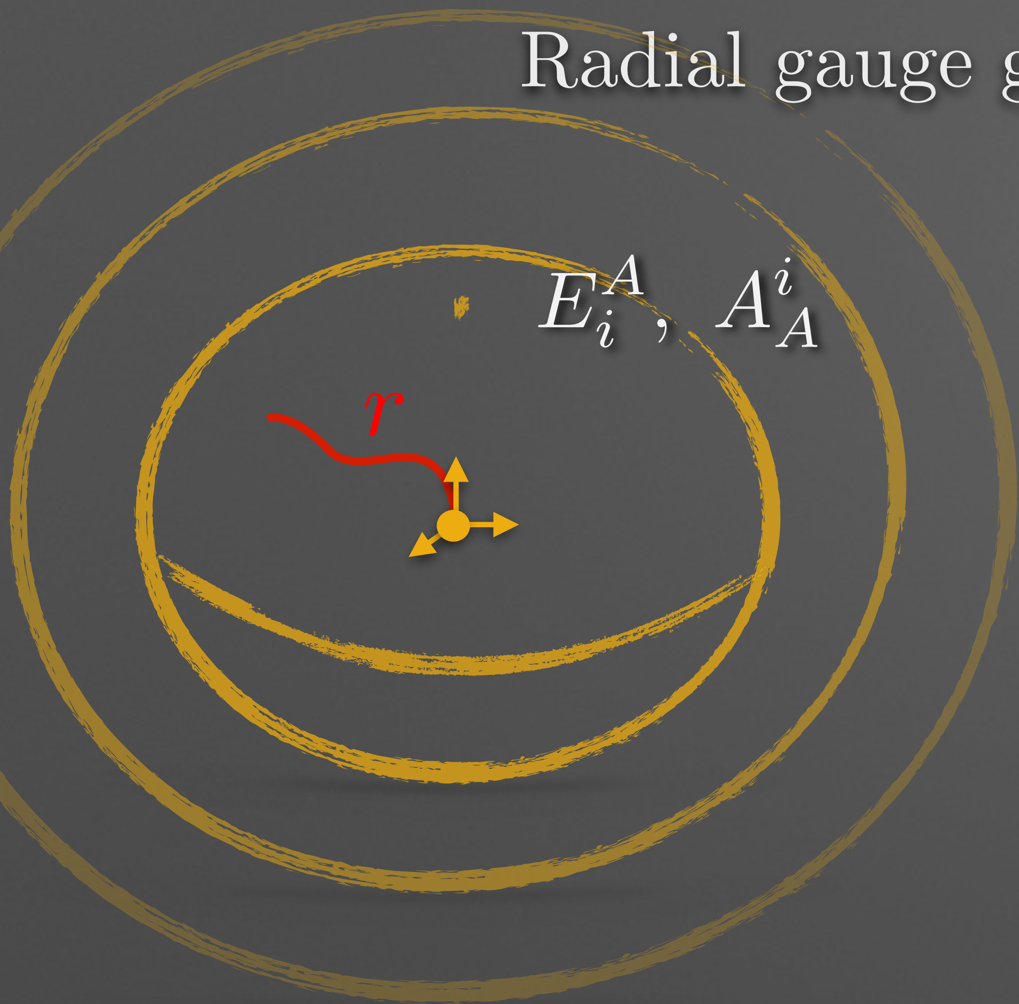
Quantisation

Radial gauge gives a reduced phase space of GR



Quantisation

Radial gauge gives a reduced phase space of GR



What can we do with it?

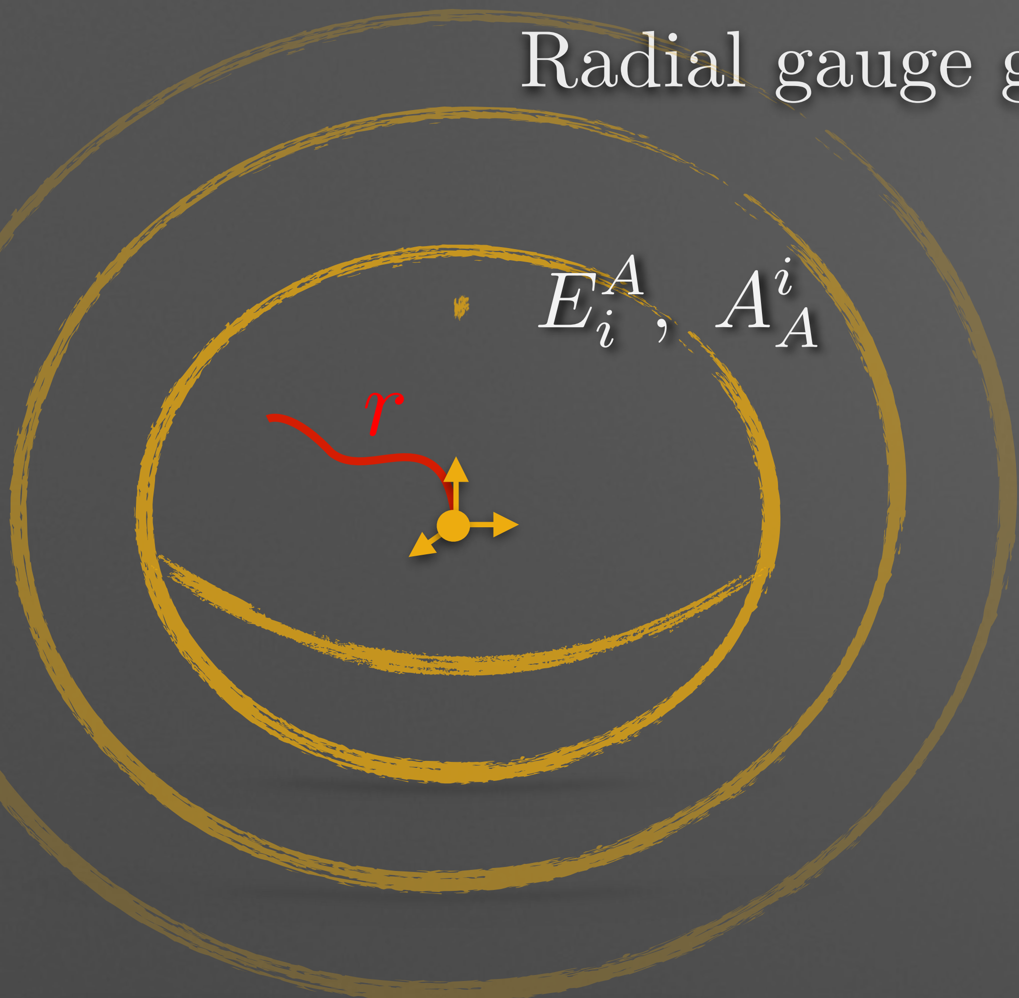
- quantum picture

$$h_e(A) = \mathcal{P}\exp\left(\int_e A_{Ai}\tau^i dx^A\right)$$

$$E_\lambda(S) = \int_S E_i^A \lambda^i \epsilon_{AB} dr dx^B$$

Quantisation

Radial gauge gives a reduced phase space of GR



What can we do with it?

- quantum picture

$$h_e(A) = \mathcal{P}\exp\left(\int_e A_{Ai}\tau^i dx^A\right)$$

$$E_\lambda(S) = \int_S E_i^A \lambda^i \epsilon_{AB} dr dx^B$$

Spherical Symmetry

- can be obtained imposing $\quad p^{rA} = 0$

$$\widetilde{p}^r{}_A(r,\theta) = \int_r^\infty dr' \mathcal{D}_B p^B{}_A(r',\theta)$$

- or by averaging w.r.t. rigid rotations around the centre

Jędrzej Świeżewski, University of Warsaw

Thank you for your attention

Bodendorfer, Lewandowski, JŚ

A quantum reduction to spherical symmetry in loop quantum gravity arXiv:1410.5609

Bodendorfer, Lewandowski, JŚ

General Relativty in the Radial Gauge I & II (to appear soon)

Extra slide

condition on canonical data in spherical symmetry

- $$\int_0^\infty dr P_R R' = 0$$

conditions on canonical data in general case

- $$\int_0^\infty dr \mathcal{D}_B p^B{}_A = 0$$
- $$-\frac{1}{2}\int_0^\infty dr p^{AB} q_{AB,r} + \int_0^\infty dr \mathcal{D}_A \left(q^{AB} \int_r^\infty dr' \mathcal{D}_C p^C{}_B \right) = 0$$
- $$\lim_{r\to 0} \frac{\left(p^{AB} q_{AB}\right)_{,rr} - \frac{1}{2}\left(p^{AB} q_{AB,r}\right)_{,r}}{\left(\sqrt{\det q}\right)_{,rr}} = 0$$