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Compact Phase Space

for Loop Quantum Gravity

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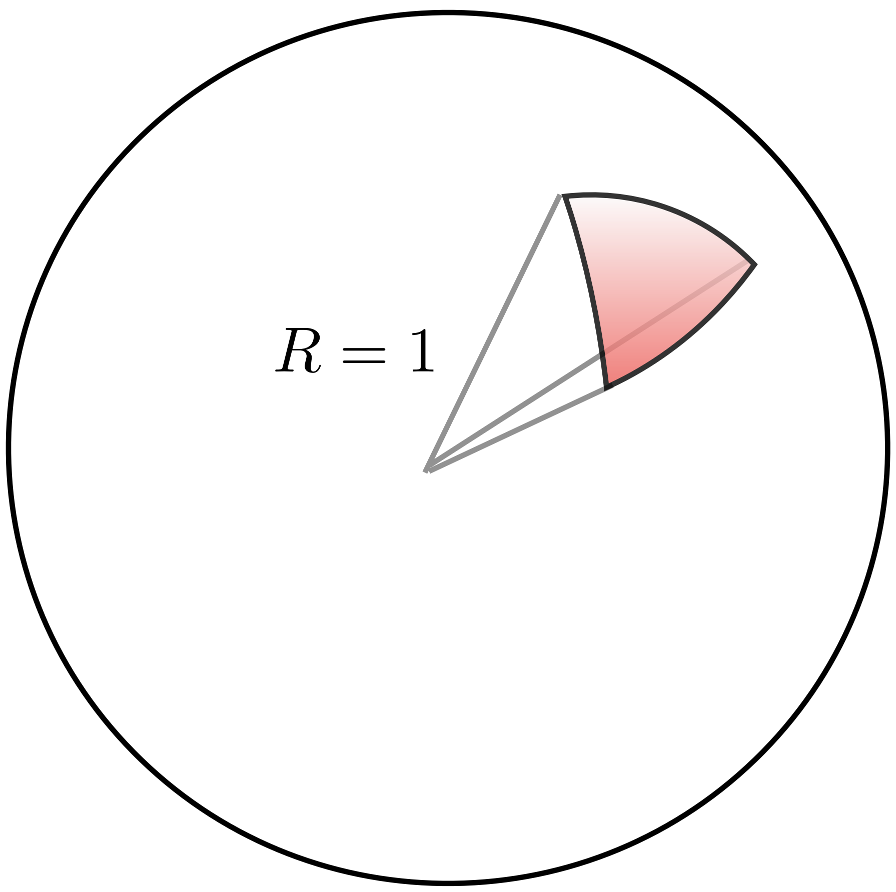
- Fact: our universe has a positive cosmological constant
- Does this affect the kinematics of LQG? (Dupuis-Girelli '13)
- Replacing flat cells with uniformly curved cells (Bahr-Dittrich '09)
- Classical kinematics: $\Gamma \equiv su(2) \times SU(2) \quad \forall \ell$
- Idea: replace the algebra with the group —> finiteness (Haggard-Han-Kaminski-Riello '14)
- Classically: compact phase space —> finite Liouville volume
- Quantum: finite # of Planck cells, finite # orthogonal states —> finite dim Hilbert space

$$\alpha_\ell = L_\ell / R$$



- $\boldsymbol{k}_\ell$: rotation associated to the curved arc ℓ
- $\boldsymbol{h}_\ell$: holonomy of the 3d connection

$$k_\ell, h_\ell \in SO(3) \sim SU(2)$$

- $SU(2) \times SU(2)$ local isometry group, or Chern-Simon gauge group

(Meusburger-Schroers '08)

- small triangles: $k_\ell = e^{\vec{J}_\ell \cdot \vec{\tau}} \sim 1\!\!1 + J_\ell = 1 + \vec{J}_\ell \cdot \vec{\tau}$

- Standard LQG phase space:

$$SU(2) \times SU(2) \underset{R\to\infty}{\longrightarrow} su(2) \times SU(2) = T^* SU(2)$$

$$(k, h) \underset{R\to\infty}{\mapsto} (J, h)$$

U(1)xU(1)

in the limit in which
the radius of one of the two circles can
be considered large we want to recover the
symplectic form of the cotangent space
$\omega = d\alpha \wedge d\beta$

- $h = e^{i\alpha}, k = e^{i\beta} \in \mathbb{C}$
- Symplectic 2-form: $\omega = -h^{-1}dh \wedge k^{-1}dk$
- Poisson brakets: $\{k, h\} = hk$

QUANTIZATION

- Hilbert space: $|n\rangle \qquad n = 1, ..., N = \dim \mathcal{H}$
- Operators: $k|n\rangle = e^{i\frac{2\pi}{N}n}|n\rangle$

  $h|n\rangle = |n+1\rangle \qquad$ cyclic: $h|N\rangle = |1\rangle$

discrete spectrum

- Commutator: $[h, k] = \left(e^{i\frac{2\pi}{N}} - 1\right) hk$

  $[\hat{a}, \hat{b}] = i\hbar\widehat{\{a, b\}}$

$\hbar = \frac{2\pi}{N}$ ⟶ $\frac{\text{[Planck length]}}{\text{[cosmological constant]}}$

limit: arc ≪ R
where
$\theta = Tr[kh^{-1}dh]$

- $(k = e^J, h) \in SU(2) \times SU(2)$
- Symplectic 2-form: $\omega = Tr[dk \wedge h^{-1}dh - kh^{-1}dh \wedge h^{-1}dh]$
- Poisson brakets? quasi Poisson-Lie groups

- QUANTIZATION
  - Hilbert space: $L_2[SU(2)] \sim \oplus_{j=0}^{\infty}(\mathcal{H}_j \otimes \mathcal{H}_j)$
  - Operators: $\langle U|jmn\rangle = D^j_{mn}(U)$
    - $h\psi(U) = U\psi(U)$ $\quad h_{AB}|jmn\rangle = \begin{pmatrix} \frac{1}{2} & j & j' \\ A & m & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & j & j' \\ B & n & n' \end{pmatrix}|j'm'n'\rangle$
    - $J^i\psi(U) = L^i\psi(U)$ $\quad J^i|jmn\rangle = \tau^{(j)}_{mk}|jkn\rangle$

SU(2)xSU(2)

limit: arc ≪ R
where
$\theta = Tr[kh^{-1}dh]$

- $(k = e^J, h) \in SU(2) \times SU(2)$
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- **QUANTIZATION** QUANTUM GROUPS
  - Hilbert space: $L_2[SU(2)] \sim \oplus_{j=0}^{j_{max}} (\mathcal{H}_j \otimes \mathcal{H}_j)$
  - Operators: $\langle U|jmn\rangle = D^j_{mn}(U)$
    - $h\psi(U) = U\psi(U)$ $\quad h_{AB}|jmn\rangle = \begin{pmatrix} \frac{1}{2} & j & j' \\ A & m & m' \end{pmatrix}_q \begin{pmatrix} \frac{1}{2} & j & j' \\ B & n & n' \end{pmatrix}_q |j'm'n'\rangle$
    - $J^i\psi(U) = L^i\psi(U)$ $\quad J^i|jmn\rangle = \tau^{(j)}_{mk}|jkn\rangle$

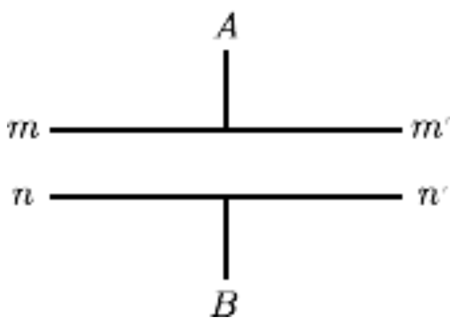
$q^r = -1 \qquad j_{max} = \frac{r-2}{2}$

$$h_{AB}|jmn\rangle = \begin{pmatrix} \frac{1}{2} & j & j' \\ A & m & m' \end{pmatrix}_q \begin{pmatrix} \frac{1}{2} & j & j' \\ B & n & n' \end{pmatrix}_q |j'm'n'\rangle$$ does not commute any more

$\nexists$ *h reps*

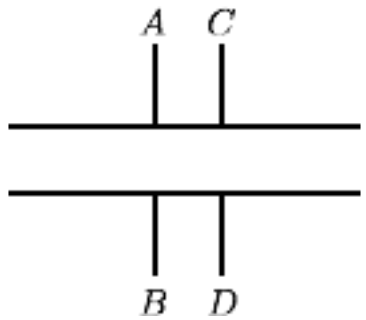
- Wigner symbols as trivalent nodes

$(h_{AB})^{mn}_{m'n'} \quad =$



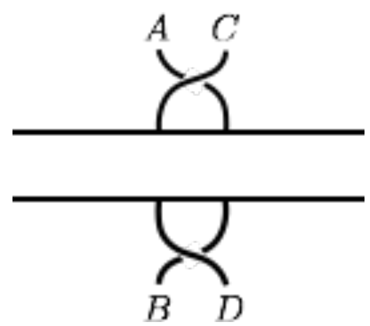
- Acting with two operators:

$h_{AB}h_{CD} \quad =$



- Inverse order:

$h_{CD}h_{AB} \quad =$



crossing operators

$$h_{AB}h_{CD} = R^{A'C'}_{AC} R^{B'D'}_{BD} \; h_{C'D'}h_{A'B'}$$

Expanding in $\hbar$ so that $R \sim 1 + r$:

$$\{h_{AB}, h_{CD}\} = r^{B'D'}_{BD} h_{CD'}h_{AB'} + r^{A'C'}_{AC} h_{C'D}h_{A'B}$$

- We have introduced a modification of LQG kinematics
  - compact phase space
  - allows to introduce a positive cosmological constant
- finite dimensional Hilbert space dim determined by the ratio between the two constants:
  - quantization (physically: Planck constant scale)
  - simplex curvature / deformation of Poisson algebra (physically: cosmological constant)
- Hilbert space reduces to usual LQG one for triangles small compared to curvature radius
- A q-deformation of the dynamics:
  - renders quantum gravity finite (Turaev-Viro '92, Han '10)
  - amount to introduce the cosmological constant (Mizoguchi-Tada '91, Han '10) $q = e^{i\sqrt{\Lambda}\hbar G}$
- Compactness: discretization of the intrinsic and extrinsic geometry
- Time discreteness: $K_{ab} \sim dq_{ab}/dt$ where $q_{ab}(\Delta t) \sim q_{ab}(0) + dq_{ab}/dt\ \Delta t$
  minimum proper time Planckian, full discrete spectrum depends on cosmological constant
- Euclidean 2+1 $\rightarrow$ to do: Lorentzian 3+1 !!!