Continuous spinors for discretised gravity

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Introduction

This talk studies three-dimensional Euclidean gravity as a consitency check for the spinorial representation of loop gravity.

Motivation

- Spinors simplify the symplectic structure. Instead of $T^*SU(2)$, we can then use $\mathbb{C}^2 \times \mathbb{C}^2$.
- What about the dynamics of the theory?

Results

- The discretised Palatini action turns into a line integral over the one-skeleton.
- All fields are continuous, but have support only on the edges of the discretisation.
- The resulting path intgeral gives the Ponzano–Regge model.

Outline of the talk

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References:

*E. Freidel and S. Speziale, From twistors to twisted geometries, Phys. Rev. D 82 (2010), arXiv:1001.2748.
*E. Livine and Johannes Tambornino, Spinor Representation for Loop Quantum Gravity, J. Math. Phys. 53 (2012), arXiv:1105.3385.
*Maite Dupuis, Simone Speziale, Johannes Tambornino, Spinors and Twistors in Loop Gravity and Spin Foams , PoS QGQGS2011 (2011), arXiv:1201.2120.

A one-dimensional action for Euclidean gravity in three dimensions

Euclidean gravity in three dimensions

We are using first-order variables, the action thus becomes:

$$S_M[e,A] = \frac{\hbar}{2\ell_{\rm P}} \int_M \epsilon_{ijk} e^i \wedge F^{jk}[A] = -\frac{\hbar}{\ell_{\rm P}} \int_M e_i \wedge F^i[A]. \tag{1}$$

The equations of motion are:

$$\text{torsionless condition:} \quad T^i = De^i = 0, \tag{2a}$$

$$\text{flatness constraint:} \quad F^i = \mathrm{d}A^i + \frac{1}{2}\epsilon^i{}_{lm} A^l \wedge A^m = 0. \tag{2b}$$

The torsionless condition turns the $SU(2)$ connection A^i into the Levi-Civita connection, hence the metric $ds^2 = e^i \otimes e_i$ is locally flat. Performing a 3+1 split $M = \Sigma \times \mathbb{R}$ we obtain the symplectic structure:

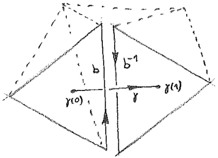
$$\{e^i{}_a(p), A^j{}_b(q)\} = \frac{\ell_{\rm P}}{\hbar} \delta^{ij} \underset{\sim}{\eta}_{ab} \tilde{\delta}_\Sigma(p,q). \tag{3}$$

Notation:

- $i,j,k,\dots = 1,2,3$ are internal indices, $a,b,c,\dots$ abstract indices on Σ.
- e^i is the triad, while $A^i{}_j = \epsilon^i{}_{lj} A^l$ denotes the $\mathfrak{so}(3)$ (respectively $\mathfrak{su}(2)$) connection, and D is the corresponding exterior covariant derivative.

Holonomy flux variables

We now discretise the continuum theory such that we still have a phase space. We introduce a simplicial decomposition of M, and assign holonmies and fluxes to links $\gamma_1, \gamma_2, \dots$ and dual bones $b_1, b_2, \dots$:



$$h^A{}_B[b] = \mathrm{Pexp}\Big(-\int_\gamma A\Big)^A{}_B \in SU(2), \qquad \text{(4a)}$$

$$\ell^A{}_B[b] = \int_{b \ni p} \big(h_p^{-1} e_p h_p\big)^A{}_B \in \mathfrak{su}(2). \qquad \text{(4b)}$$

The Poisson brackets of the continuum theory induce on each link the commutation relations of $T^*SU(2)$, e.g.:

$$\{\ell_i, \ell_j\} = \frac{\ell_\mathrm{P}}{\hbar}\epsilon_{ij}{}^m \ell_m, \quad \{\ell_i, h\} = \frac{\ell_\mathrm{P}}{\hbar} h \tau_i. \qquad \text{(5)}$$

Notation:

- $A, B, \dots$ are spinor indices, we move them by the two-dimensional ϵ-tensor.
- $\ell \equiv \ell^A{}_B = \ell^i \tau^A{}_{Bi}$, where τ_i are the Pauli matrices divided by $2\mathrm{i}$.

Spinors for loop gravity

Two orthogonal spinors diagonalise the flux in the frame of the initial point:

$$\ell[b] = \frac{\ell_{\mathrm{P}}}{4\mathrm{i}}\Big(|z\rangle\langle z| - |z][z|\Big). \quad \ell_{AB}[b] = \frac{\ell_{\mathrm{P}}}{2\mathrm{i}} z_{(A} z^{\dagger}_{B)}. \tag{6}$$

The holonomy maps them into the final point:

$$h|z\rangle = |\underset{\sim}{z}\rangle, \quad |z] = |\underset{\sim}{z}]. \tag{7}$$

The two spinors have the same norm:

$$C = \|\underset{\sim}{z}\|^2 - \|z\|^2 = \langle \underset{\sim}{z}|\underset{\sim}{z}\rangle - \langle z|z\rangle = 0. \tag{8}$$

This is the length–matching constraint.
The following scheme gives the relation between $|z\rangle$ and $|z]$:

$$\begin{aligned} [z| &= z_A = \epsilon_{BA} z^B, \qquad & |z\rangle &= z^A = \epsilon^{AB} z_B, \\ |z] &= z_{\dagger}^{A} = \delta^{A\bar{A}} \bar{z}_{\bar{A}}, \qquad & \langle z| &= z^{\dagger}_{A} = \delta_{A\bar{A}} \bar{z}^{\bar{A}}. \end{aligned} \tag{9}$$

Two invarinat tensors move the indices up and down: ϵ_{AB} is the anti-symmetric ϵ-tensor, while $\delta_{A\bar{A}}$ denotes the Hermitian metric.

Symplectic reduction

The spinors carry a natural symplectic structure:

$$\{z^\dagger_A, z^B\} = \frac{\mathrm{i}}{\hbar}\delta^B_A, \quad \{\underset{\sim}{z}^\dagger_A, \underset{\sim}{z}^B\} = -\frac{\mathrm{i}}{\hbar}\delta^B_A. \tag{10}$$

We can now reverse the logic:

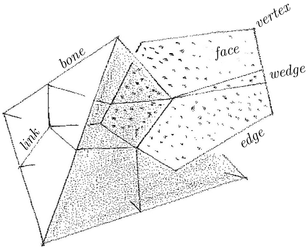
- Take the spinors z^A and $\underset{\sim}{z}^A$ as fundamental, and use them to parametrise holonomies and fluxes.
- The symplectic structure (10) induce commutations relations for holomies and fluxes. On the constraint hypersurface $C = 0$, we get the usual holonomy-flux algebra.
- The spinors are not unique, the constraint $C = 0$ generates a $U(1)$ symmetry.
- The parametrisation becomes singular once $\ell_i = 0$.
- Taking the symplectic quotient, we obtain $T^*SU(2) \ni (h, \ell)$ exempt of the hypersuface T_o of vanishing flux $\ell = 0$:

$$T^*SU(2) - T_o = \left(\mathbb{C}^2 \times \mathbb{C}^2\right)//_C. \tag{11}$$

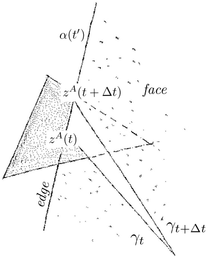
We now look at the thee-dimensional discretisation and study the dynamics of the theory.

The first step is to discretise the action as a sum over wedges:

$$S_M[e, A] = -\frac{\hbar}{\ell_{\mathrm{P}}} \int_M e_i \wedge F^i$$
$$\approx -\frac{\hbar}{\ell_{\mathrm{P}}} \sum_{w:\mathrm{wedges}} \int_{b_w} e_i \int_w F^i. \quad (12)$$



- Pick a spinfoam *face* f, and introduce the loop $\alpha(t)$ that runs around the boundary ∂f.
- Consider a family of paths (links) $\{\gamma_t\}_{t\in[0,1]}$ connecting $\alpha(t)$ with the center $c = \gamma_t(1)$ of the spinfoam face.



The covariant t-derivative of the h_{γ_t}-holonomy gives the curvature in the spinfoam face:

$$h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)} = \int_0^1 \mathrm{d}s \, h^{-1}_{\gamma_t(s)} F_{\gamma_t(s)}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\gamma_t(s), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(s)\Big) h_{\gamma_t(s)}. \tag{13}$$

A wedge w_o corresponds to some t-interval $[t_o, t_o+\Delta t]$. Each wedge contributes through its wedge-action S_w to the total action:

$$S_{w_o} \approx -\frac{2\hbar\Delta t}{\ell_{\mathrm{P}}} \, \ell_{AB}[b_{w_o}]\Big(h^{-1}_{\gamma_{t_o}(1)} \frac{D}{\mathrm{d}t} h_{\gamma_{t_o}(1)}\Big)^{AB}. \tag{14}$$

Where $\ell[b_{w_o}]$ is the flux through the bone dual to the wedge.

Continuum limit on a spinfoam face

- Next, we introduce spinors: $z^A(t)$ belongs to the boundary, while $w^A(t)$ sits at the center of the spinfoam face.
- The limit $\Delta t \to 0$ turns the sum over wedges into an intgeral.

We add the length-matching constraint $C = \|w\|^2 - \|z\|^2$, and end up with the following action:

$$S_f[z, w, \varphi, A] = -\mathrm{i}\hbar \int_0^1 \mathrm{d}t \Big(z^\dagger_A \frac{D}{\mathrm{d}t} z^A - w^\dagger_A \dot{w}^A - \mathrm{i}\,\varphi\,\big(\|z\|^2 - \|w\|^2\big)\Big). \qquad (15)$$

A one-dimensional action for gravity in three dimensions

The total action is the sum over all faces:

$$\begin{aligned} S_M[{}^{f_1}z, {}^{f_2}z, \dots; {}^{f_1}w, {}^{f_2}w, \dots; \varphi_{f_1}, \varphi_{f_2}, \dots; A_{e_1}, A_{e_2}, \dots] = \\ \equiv S_M[\underline{z}, \underline{w}, \underline{\varphi}, \underline{A}] = -\mathrm{i}\hbar \sum_f \oint_{\partial f} \Big({}^f z^\dagger_A D\, {}^f z^A - {}^f w^\dagger_A \mathrm{d}\,{}^f w^A + \\ - \mathrm{d}t\,\mathrm{i}\,\varphi_f\,\big(\|{}^f z\|^2 - \|{}^f w\|^2\big)\Big). \end{aligned} \qquad (16)$$

1. Variation of the spinors:

$$\frac{D}{\mathrm{d}t} z^A = \mathrm{i}\varphi z^A, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} w^A = \mathrm{i}\varphi w^A. \tag{17}$$

The spinors are periodic in t, implying:

$$\exp\Big(-\int_{\partial f} \mathrm{d}t A(t) \Big) = \mathbb{1}, \quad \mathrm{e}^{\mathrm{i}\int_{\partial f} \mathrm{d}t \varphi(t)} = 1. \tag{18}$$

This is the discretisation of the curvature constraint $F^i = 0$.

2. Variation of the φ-multiplier:

$$C = \|w(t)\|^2 - \|z(t)\|^2 = 0, \tag{19}$$

reducing the spinors to holonomy-flux variables.

3. Variation of the $SU(2)$ connection:

$$G_i := \frac{\hbar}{\ell_{\mathrm{P}}} \sum_{f=1}^{3} \ell_i [b_f]_t = \mathrm{i}\hbar \sum_{f=1}^{3} \tau^{AB}{}_i {}^f z^\dagger_A(t) {}^f z_B(t) = 0. \tag{20}$$

This is the discretisation of the torsionless condition $T^i = 0$.

Gauge symmetries

1. One-dimensional diffeomorphisms: The action is invariant under reparametrisations of the path.
2. $U(1)$ transformations:

$$\tilde{\varphi}(t) = \varphi(t) + \dot{\lambda}(t), \tag{21a}$$

$$\tilde{z}^A(t) = \mathrm{e}^{-\mathrm{i}\lambda(t)} z^A(t), \tag{21b}$$

$$\tilde{w}^A(t) = \mathrm{e}^{-\mathrm{i}\lambda(t)} w^A(t). \tag{21c}$$

3. $SU(2)$ transformations:

$$\tilde{A}(t) = g^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t) + g^{-1}(t)A(t)g(t), \tag{22a}$$

$$\tilde{z}(t) = g^{-1}(t)z(t), \tag{22b}$$

$$\tilde{w}(t) = w(t). \tag{22c}$$

Nota bene: There is no obvious representation of the shift symmetry.

Path intgeral quantisation

Bargmann quantisation of the harmonic oscialltor

Representation of the canonical commutation relations $\{z^\dagger_A, z^B\} = \frac{\mathrm{i}}{\hbar}\delta^B_A$ in the space of $\mathbb{C}^2$-analytic functions $f \in \mathcal{H}$:

$$\frac{\partial}{\partial \bar{z}^{\bar{A}}} f(z) = 0, \quad \big(\hat{z}^A f\big)(z) = z^A f(z), \quad \big(\hat{z}^\dagger_A f\big)(z) = \frac{\partial}{\partial z^A} f(z). \qquad (23)$$

The inner product is:

$$\langle f, f' \rangle = \frac{1}{\pi^2}\int^2_{\mathbb{C}} d^4 z \mathrm{e}^{-\delta_{A\bar{A}} z^A \bar{z}^{\bar{A}}} \overline{f(z)} f'(z). \qquad (24)$$

There is an orthonormal basis:

$$\langle z | j, m \rangle = \frac{1}{\sqrt{(j-m)!(j+m)!}} \big(z^0\big)^{j-m} \big(z^1\big)^{j+m}. \qquad (25)$$

With $j = 0, \frac{1}{2}, 1, \dots, m = -j, \dots, j$.

- This gives the Hilbertspace for a single bone.
- The constraints glue these individual Hilbert spaces together.
- The result agrees with the LQG Hilbert space on a fixed graph.

*E. Freidel and S. Speziale, From twistors to twisted geometries, Phys. Rev. D 82 (2010), arXiv:1001.2748.
*E. Livine and Johannes Tambornino, Spinor Representation for Loop Quantum Gravity, J. Math. Phys. 53 (2012), arXiv:1105.3385.
*Maite Dupuis, Simone Speziale, Johannes Tambornino, Spinors and Twistors in Loop Gravity and Spin Foams, PoS QGQGS2011 (2011), arXiv:1201.2120.
*Enrique F Borja, Laurent Freidel, Iñaki Garay, and Etera Livine, U(N) tools for loop quantum gravity: the return of the spinor, Class. Quantum Grav. 28 (2011), arXiv:1010.5451.

Definition of the path integral

We define the vacuum-to-vacuum amplitude $\langle\Omega|\Omega\rangle = Z_M$ for the discretised (closed) manifold M as the path integral over the continuous spinor action:

$$Z_M = \int_{\substack{\text{all spinors be} \\ \text{periodic in } \partial f}} \prod_{f:\text{faces}} \mathcal{D}[{}^f z]\mathcal{D}[{}^f w]\mathcal{D}[\varphi_f]\Delta^{\psi}_{\mathrm{FP}}[\varphi_f]\delta\big(\psi[\varphi_f]\big)$$

$$\times \prod_{e:\text{edges}} \mathcal{D}[A_e]\Delta^{\Psi}_{\mathrm{FP}}[A]\delta\big(\Psi[A]\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_M[\underline{z},\underline{w},\underline{\varphi},\underline{A}]}. \qquad (26)$$

Notation:

- Each face contributes to $S_M = \sum_{f:\text{faces}} S_f$ through: $S_f = -\mathrm{i}\hbar \int_{\partial f} \left(z^{\dagger}_A D z^A - w^{\dagger}_A \mathrm{d} w^A - \mathrm{i}\,\varphi\, \mathrm{d}t(\|z\|^2 - \|w\|^2)\right).$
- $\mathcal{D}$ denotes formal Lebesgue measures, e.g. $\mathcal{D}[z] = \prod_t \frac{d^4 z(t)}{\pi^2}$.
- There is a gauge fixing for the gauge potentials φ and A:
  - $\psi[\varphi]$ and $\Psi[A]$ are the gauge fixing conditions.
  - $\Delta^{\psi}_{\mathrm{FP}}[\varphi]$ and $\Delta^{\Psi}_{\mathrm{FP}}[A]$ are the corresponding Faddeev–Popov determinants.

Integral over the spinors

The integral over the spinors factorises into products over faces. This defines the face amplitude:

$$Z_f[A,\varphi] := \int\limits_{z^A(0)=z^A(1),\, w^A(0)=w^A(1)} \mathcal{D}[z]\mathcal{D}[w]\mathrm{e}^{\int_{\partial f}\mathrm{d}t\left(z^\dagger_A \frac{D}{\mathrm{d}t}z^A - w^\dagger_A\frac{\mathrm{d}}{\mathrm{d}t}w^A - \mathrm{i}\,\varphi(\|z\|^2-\|w\|^2)\right)}. \tag{27}$$

The Gaußian integral turns into the trace over Hilbert space:

$$Z_f[A,\varphi] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\Big[\mathrm{Pexp}\Big(-\int_{\partial f}\mathrm{d}t\big(A^i(t)\tau^{AB}{}_i\hat{z}_A\hat{z}^\dagger_B + \tag{28}$$

$$+\,\mathrm{i}\,\varphi(t)\,(:\!\|\hat{z}\|^2\!: - :\!\|\hat{w}\|^2\!:)\big)\Big)\Big]. \tag{29}$$

In terms of the canonical basis:

$$Z_f[A,\varphi] = \sum_{2j=0}^{\infty}\sum_{m=-j}^{j}\sum_{2l=0}^{\infty}(2l+1)\langle j,m|\mathrm{Pexp}\Big(\mathrm{i}\int_{\partial f}\mathrm{d}t A^i(t)\hat{L}_i\Big)|j,m\rangle$$
$$\times\,\mathrm{e}^{-\mathrm{i}\int_{\partial f}\mathrm{d}t\varphi(t)(2j-2l)}. \tag{30}$$

Notation:

- Generators of angular momentum: $\hat{L}_i = \mathrm{i}\tau^{AB}{}_i\hat{z}_A\hat{z}^\dagger_B$
- The spinor's norm: $:\!\|\hat{z}\|^2\!: = \frac{1}{2}\left(\hat{z}^A\hat{z}^\dagger_A + \hat{z}^\dagger_A\hat{z}^A\right) = z^A\frac{\partial}{\partial z^A} + 1,$

Also the φ-integration factorises into a product over spinfoam faces. We choose the gauge fixing condition:

$$\psi[\varphi](t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = 0. \tag{31}$$

Infinitesimal gauge transformations are $\varphi^\lambda(t) = \varphi(t) + \dot{\lambda}(t)$, and $\Delta^\psi_{\mathrm{FP}}[\varphi]$ is the determinant of the operator:

$$\hat{m}[\lambda] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\psi[\varphi^{\varepsilon\lambda}](t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda(t). \tag{32}$$

All eigenvalues are φ-independent, and $\Delta^\psi_{\mathrm{FP}}[\varphi]$ just gives an irrelevant overall constant.

$$\begin{aligned} Z_f[A] &= \int \mathcal{D}[\varphi]\Delta^\psi_{\mathrm{FP}}[\varphi]\delta\big(\delta\psi[\varphi]\big)Z_f[\varphi, A] = \\ &= \sum_{2j=0}^{\infty}\sum_{m=-j}^{j}(2j+1)\big\langle j, m\big|\mathrm{Pexp}\Big(\mathrm{i}\int_{\partial f}\mathrm{d}t A^i(t)L_i\Big)\big|j, m\big\rangle = \\ &= \delta_{SU(2)}\Big(\mathrm{Pexp}\Big(-\int_{\partial f}\mathrm{d}t A^i(t)\tau_i\Big)\Big). \end{aligned} \tag{33}$$

Integral over the $SU(2)$ gauge potential

The most difficult step is the integral over the gauge potential A.
We choose the following gauge fixing on the edges:

$$\forall e(t): \quad \Psi[A](t) = \frac{\mathrm{d}}{\mathrm{d}t} A_e(t) = 0. \tag{34}$$

- We can achieve this gauge globally, one every edge of the discretisation.
- It is only a partial gauge fixing: For a single edge we can map A_e to any other constant $\tilde{A}_e \in \mathfrak{su}(2)$.

The infinitesimal gauge transformations are $\delta_\Lambda A = \frac{D}{\mathrm{d}t}\Lambda = \dot{\Lambda} + [A, \Lambda]$. This yields the Faddeev–Popov operator:

$$\hat{M}^i{}_j \Lambda^j(t) = \delta_\lambda \Psi[A](t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Lambda^i(t) + \epsilon^i{}_{lm} A^l \frac{\mathrm{d}}{\mathrm{d}t}\Lambda^m(t). \tag{35}$$

We diagonalise this operator, under the condition that the gauge elements $\Lambda(t)$ are everywhere continuous. This yields the Faddeev–Popov determinant:

$$\Delta^\Psi_{\mathrm{FP}}[A] \propto \det \hat{M} \propto \prod_{e:\mathrm{edges}} \prod_{n \in \mathbb{Z}-\{0\}} \left(1 - \frac{|A_e|}{2\pi n}\right)^2 = \prod_{e:\mathrm{edges}} \frac{4\sin^2 \frac{|A_e|}{2}}{|A_e|^2}. \tag{36}$$

*E. Bianchi, Loop Quantum Gravity à la Aharonov-Bohm, Gen. Relativ. Gravit. 46, (2014), arXiv:0907.4388

Final result

The path integral over the connection turns into an ordinary integral:

$$\int \prod_{e:\text{edges}} \mathcal{D}[A_e]\Delta^{\Psi}_{\text{FP}}[A]\delta\big(\Psi[A]\big)\cdots \propto \prod_{e:\text{edges}} \int_{\mathbb{R}^3} d^3A_e \frac{4\sin^2\frac{|A_e|}{2}}{|A_e|^2}\dots \quad (37)$$

$$\propto \prod_{e:\text{edges}} \int_{SU(2)} d\mu_{\text{Haar}}(U_e)\dots \quad (38)$$

The Ponzano–Regge as a path integral in spinor space

$$Z_M = \int \prod_{f:\text{faces}} \mathcal{D}[{}^f z]\mathcal{D}[{}^f w]\mathcal{D}[\varphi_f]\Delta^{\psi}_{\text{FP}}[\varphi_f]\delta\big(\psi[\varphi_f]\big)$$

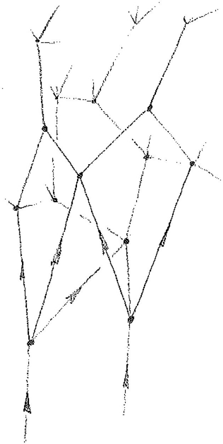
$$\times \prod_{e:\text{edges}} \mathcal{D}[A_e]\Delta^{\Psi}_{\text{FP}}[A]\delta\big(\Psi[A]\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_M[\underline{z},\underline{w},\underline{\varphi},\underline{A}]} =$$

$$= \prod_{e:\text{edges}} \int_{SU(2)} d\mu_{\text{Haar}}(U_e) \prod_{f:\text{faces}} \delta_{SU(2)}\Big(\mathrm{P}\prod_{e\in\partial f} U_e\Big). \quad (39)$$

where $\mathrm{P}\prod$ denotes the path ordered product.

Conclusion

Summary



- There is a one-dimensional action for simplicial gravity in three dimensions.
- All fields are continuous, but have support only on the one-dimensional edges of the discretisation.
- Once we have an action, we can define a canonical path intgeral.
- The resulting amplitudes reproduce the Ponzano–Regge model.

Spinors are useful for the following reasons: (i) They are canonical Darboux coordinates for loop gravity. (ii) Dynamics on a fixed discretisation of spacetime simplifies. (iii) The action turns into a bilinear in the spinors, this greatly simplifies the evaluation of the path integral.

Thanks for the attention!

In progress:

- WW., Continuous action for simplicial quantum gravity in three dimensions, (2014).
- Marc Geiller and WW, Semi-discrete actions for gravity, (2014).

See also:

- L. Freidel and S. Speziale, From twistors to twisted geometries; Phys. Rev. D 82 (2010), `arXiv:1001.2748`.
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