Continuous spinors for discretised gravity
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This talk studies three-dimensional Euclidean gravity as a consistency check for the spinorial representation of loop gravity.

Motivation

- Spinors simplify the symplectic structure. Instead of $T^* SU(2)$, we can then use $\mathbb{C}^2 \times \mathbb{C}^2$.
- What about the dynamics of the theory?

Results

- The discretised Palatini action turns into a line integral over the one-skeleton.
- All fields are continuous, but have support only on the edges of the discretisation.
- The resulting path integral gives the Ponzano–Regge model.
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A one-dimensional action for Euclidean gravity in three dimensions
Euclidean gravity in three dimensions

We are using first-order variables, the action thus becomes:

\[ S_M[e, A] = \frac{\hbar}{2\ell_P} \int_M \epsilon_{ijk} e^i \wedge F^{jk}[A] = -\frac{\hbar}{\ell_P} \int_M e_i \wedge F^i[A]. \]  

(1)

The equations of motion are:

torsionless condition: \[ T^i = De^i = 0, \]  
flatness constraint: \[ F^i = dA^i + \frac{1}{2} \epsilon^{ilm} A^l \wedge A^m = 0. \]  

(2b)

The torsionless condition turns the SU(2) connection \( A^i \) into the Levi-Civita connection, hence the metric \( ds^2 = e^i \otimes e_i \) is locally flat. Performing a 3+1 split \( M = \Sigma \times \mathbb{R} \) we obtain the symplectic structure:

\[ \{ e^i_{\ a}(p), A^j_{\ b}(q) \} = \frac{\ell_P}{\hbar} \delta^{ij} \eta_{ab} \tilde{\delta} \Sigma(p, q). \]  

(3)

Notation:

- \( i, j, k, \cdots = 1, 2, 3 \) are internal indices, \( a, b, c, \ldots \) abstract indices on \( \Sigma \).
- \( e^i \) is the triad, while \( A^i_{\ j} = e^i_{\ l} A^l \) denotes the \( so(3) \) (respectively \( su(2) \)) connection, and \( D \) is the corresponding exterior covariant derivative.
We now discretise the continuum theory such that we still have a phase space. We introduce a simplicial decomposition of $M$, and assign holonomies and fluxes to links $\gamma_1, \gamma_2, \ldots$ and dual bones $b_1, b_2, \ldots$:

\[
h^{A}_{\ B}[b] = \text{Pexp} \left( - \int_{\gamma} A \right)^{A}_{\ B} \in SU(2), \quad (4a)
\]

\[
\ell^{A}_{\ B}[b] = \int_{b \ni p} (h^{-1}_p e_p h_p)^{A}_{\ B} \in \mathfrak{su}(2). \quad (4b)
\]

The Poisson brackets of the continuum theory induce on each link the commutation relations of $T^*SU(2)$, e.g.:

\[
\{\ell_i, \ell_j\} = \ell_P \frac{\epsilon_{ij}}{\hbar} \ell_m, \quad \{\ell_i, h\} = \ell_P \frac{h}{\hbar} h\tau_i. \quad (5)
\]

Notation:
- $A, B, \ldots$ are spinor indices, we move them by the two-dimensional $\epsilon$-tensor.
- $\ell \equiv \ell^{A}_{\ B} = \ell^i \tau^A_{\ B i}$, where $\tau_i$ are the Pauli matrices divided by $2i$. 

Holonomy flux variables
Two orthogonal spinors diagonalise the flux in the frame of the initial point:

\[
\ell[b] = \frac{\ell_P}{4i} \left( |z\rangle\langle z| - |\bar{z}\rangle\langle \bar{z}| \right), \quad \ell_{AB}[b] = \frac{\ell_P}{2i} \bar{z}_A z_B^\dagger.
\]  

(6)

The holonomy maps them into the final point:

\[
h|z\rangle = |\bar{z}\rangle, \quad |\bar{z}\rangle = |\bar{z}\rangle.
\]

(7)

The two spinors have the same norm:

\[
C = \|\bar{z}\|^2 - \|z\|^2 = \langle \bar{z}|\bar{z}\rangle - \langle z|z\rangle = 0.
\]

(8)

This is the length-matching constraint.

The following scheme gives the relation between $|z\rangle$ and $|\bar{z}\rangle$:

\[
|\bar{z}\rangle = z_A = \epsilon_{BA} z^B, \quad |z\rangle = z^A = \epsilon^{AB} z_B,
\]

\[
|\bar{z}\rangle = z_A^\dagger = \delta^{A\bar{A}} \bar{z}_{\bar{A}}, \quad \langle z| = z_A^\dagger = \delta^A A \bar{z}_{\bar{A}}.
\]

(9)

Two invariant tensors move the indices up and down: $\epsilon_{AB}$ is the anti-symmetric $\epsilon$-tensor, while $\delta^A A$ denotes the Hermitian metric.
The spinors carry a natural symplectic structure:

\[
\{z_A^\dagger, z^B\} = \frac{i}{\hbar} \delta_A^B, \quad \{\tilde{z}_A^\dagger, \tilde{z}^B\} = -\frac{i}{\hbar} \delta_A^B. \quad \text{(10)}
\]

We can now reverse the logic:

- Take the spinors \(z^A\) and \(\tilde{z}^A\) as fundamental, and use them to parametrise holonomies and fluxes.
- The symplectic structure (10) induce commutations relations for holonomies and fluxes. On the constraint hypersurface \(C = 0\), we get the usual holonomy-flux algebra.
- The spinors are not unique, the constraint \(C = 0\) generates a \(U(1)\) symmetry.
- The parametrisation becomes singular once \(\ell_i = 0\).
- Taking the symplectic quotient, we obtain \(T^*SU(2) \ni (\hbar, \ell)\) exempt of the hypersurface \(T_o\) of vanishing flux \(\ell = 0\):

\[
T^*SU(2) - T_o = (\mathbb{C}^2 \times \mathbb{C}^2) \parallel C. \quad \text{(11)}
\]
We now look at the thee-dimensional discretisation and study the dynamics of the theory.

The first step is to discretise the action as a sum over wedges:

\[
S_M[e, A] = -\frac{\hbar}{\ell_P} \int_M e_i \wedge F^i \\
\approx -\frac{\hbar}{\ell_P} \sum_{w: \text{wedges}} \int_{b_w} e_i \int_w F^i. \quad (12)
\]
The action on a wedge

- Pick a spinfoam face \( f \), and introduce the loop \( \alpha(t) \) that runs around the boundary \( \partial f \).
- Consider a family of paths (links) \( \{ \gamma_t \}_{t \in [0,1]} \) connecting \( \alpha(t) \) with the center \( c = \gamma_t(1) \) of the spinfoam face.

The covariant \( t \)-derivative of the \( h_{\gamma_t} \)-holonomy gives the curvature in the spinfoam face:

\[
\begin{align*}
    h_{\gamma_t(1)}^{-1} \frac{D}{dt} h_{\gamma_t(1)} &= \int_0^1 ds \ h_{\gamma_t(s)}^{-1} F_{\gamma_t(s)} \left( \frac{d}{ds} \gamma_t(s), \frac{d}{dt} \gamma_t(s) \right) h_{\gamma_t(s)}.
\end{align*}
\]

A wedge \( w_o \) corresponds to some \( t \)-interval \([t_o, t_o + \Delta t]\). Each wedge contributes through its wedge-action \( S_w \) to the total action:

\[
S_{w_o} \approx - \frac{2\hbar \Delta t}{\ell_P} \ell_{AB}[b_{w_o}] \left( h_{\gamma_{t_o}(1)}^{-1} \frac{D}{dt} h_{\gamma_{t_o}(1)} \right)^{AB}.
\]

Where \( \ell[b_{w_o}] \) is the flux through the bone dual to the wedge.
Continuum limit on a spinfoam face

- Next, we introduce spinors: \( z^A(t) \) belongs to the boundary, while \( w^A(t) \) sits at the center of the spinfoam face.
- The limit \( \Delta t \to 0 \) turns the sum over wedges into an integral.

We add the length-matching constraint \( C = \|w\|^2 - \|z\|^2 \), and end up with the following action:

\[
S_f[z, w, \varphi, A] = -i\hbar \int_0^1 dt \left( z_A^A \frac{D}{dt} z^A - w_A^A \dot{w}^A - i \varphi (\|z\|^2 - \|w\|^2) \right). \quad (15)
\]

A one-dimensional action for gravity in three dimensions

The total action is the sum over all faces:

\[
S_M[f_1 z, f_2 z, \ldots; f_1 w, f_2 w, \ldots; \varphi_{f_1}, \varphi_{f_2}, \ldots; A_{e_1}, A_{e_2}, \ldots] = \\
\equiv S_M[z, w, \varphi, A] = -i\hbar \sum_f \oint_{\partial f} \left( f_A^z D f_A^z z^A - f_A^w d f_A^w w^A + \\
- dt i \varphi_f (\|f_z\|^2 - \|f_w\|^2) \right). \quad (16)
\]
Equations of motion

1. Variation of the spinors:

\[ \frac{D}{dt} z^A = i\varphi z^A, \quad \text{and} \quad \frac{d}{dt} w^A = i\varphi w^A. \]  \hspace{1cm} (17)

The spinors are periodic in \( t \), implying:

\[ \exp\left(-\int_{\partial_f} dt A(t)\right) = 1, \quad e^{i\int_{\partial_f} dt \varphi(t)} = 1. \]  \hspace{1cm} (18)

*This is the discretisation of the curvature constraint \( F^i = 0. \)*

2. Variation of the \( \varphi \)-multiplier:

\[ C = \|w(t)\|^2 - \|z(t)\|^2 = 0, \]  \hspace{1cm} (19)

*reducing the spinors to holonomy-flux variables.*

3. Variation of the \( SU(2) \) connection:

\[ G_i := \frac{\hbar}{\ell_P} \sum_{f=1}^3 \ell_i [b_f]_t = i\hbar \sum_{f=1}^3 \tau_i^{AB} f z^\dagger_A(t) f z_B(t) = 0. \]  \hspace{1cm} (20)

*This is the discretisation of the torsionless condition \( T^i = 0. \).*
One-dimensional diffeomorphisms: The action is invariant under reparametrisations of the path.

**2** $U(1)$ transformations:

\[
\begin{align*}
\tilde{\varphi}(t) &= \varphi(t) + \dot{\lambda}(t), \\
\tilde{z}^A(t) &= e^{-i\lambda(t)} z^A(t), \\
\tilde{w}^A(t) &= e^{-i\lambda(t)} w^A(t).
\end{align*}
\]

**3** $SU(2)$ transformations:

\[
\begin{align*}
\tilde{A}(t) &= g^{-1}(t) \frac{d}{dt} g(t) + g^{-1}(t) A(t) g(t), \\
\tilde{z}(t) &= g^{-1}(t) z(t), \\
\tilde{w}(t) &= w(t).
\end{align*}
\]

Nota bene: There is no obvious representation of the shift symmetry.
Path integral quantisation
Bargmann quantisation of the harmonic oscillator

Representation of the canonical commutation relations \( \{ z_A^\dagger, z_B \} = \frac{i}{\hbar} \delta_B^A \) in the space of \( \mathbb{C}^2 \)-analytic functions \( f \in \mathcal{H} \):

\[
\frac{\partial}{\partial \bar{z}^A} f(z) = 0, \quad (\hat{z}^A f)(z) = z^A f(z), \quad (\hat{z}^A f)(z) = \frac{\partial}{\partial z^A} f(z). \quad (23)
\]

The inner product is:

\[
\langle f, f' \rangle = \frac{1}{\pi^2} \int_{\mathbb{C}} d^4 z e^{-\delta_{A\bar{A}} z^A \bar{z}^A} f(z) f'(z). \quad (24)
\]

There is an orthonormal basis:

\[
\langle z | j, m \rangle = \frac{1}{\sqrt{(j - m)!(j + m)!}} (z^0)^{j-m} (z^1)^{j+m}. \quad (25)
\]

With \( j = 0, \frac{1}{2}, 1, \ldots, m = -j, \ldots, j \).

- This gives the Hilbert space for a single bone.
- The constraints glue these individual Hilbert spaces together.
- The result agrees with the LQG Hilbert space on a fixed graph.

Definition of the path integral

We define the vacuum-to-vacuum amplitude $\langle \Omega | \Omega \rangle = Z_M$ for the discretised (closed) manifold $M$ as the path integral over the continuous spinor action:

$$Z_M = \int_{\text{all spinors be periodic in } \partial f} \prod_{f: \text{faces}} \mathcal{D}[f z] \mathcal{D}[f w] \mathcal{D}[\varphi_f] \Delta^\psi_{\text{FP}}[\varphi_f] \delta(\psi[\varphi_f])$$

$$\times \prod_{e: \text{edges}} \mathcal{D}[A_e] \Delta^\Psi_{\text{FP}}[A] \delta(\Psi[A]) e^{\frac{i}{\hbar} S_M[z,w,\varphi,A]}. \quad (26)$$

Notation:

- Each face contributes to $S_M = \sum_{f: \text{faces}} S_f$ through:
  $$S_f = -i\hbar \int_{\partial f} (z_A^\dagger D_z^A - w_A^\dagger d w^A - i \varphi \ dt (||z||^2 - ||w||^2)).$$

- $\mathcal{D}$ denotes formal Lebesgue measures, e.g. $\mathcal{D}[z] = \prod_t \frac{d^4 z(t)}{\pi^2}$.

- There is a gauge fixing for the gauge potentials $\varphi$ and $A$:
  - $\psi[\varphi]$ and $\Psi[A]$ are the gauge fixing conditions.
  - $\Delta^\psi_{\text{FP}}[\varphi]$ and $\Delta^\Psi_{\text{FP}}[A]$ are the corresponding Faddeev–Popov determinants.
Integral over the spinors

The integral over the spinors factorises into products over faces. This defines the face amplitude:

\[
Z_f [A, \varphi] := \int \mathcal{D}[z] \mathcal{D}[w] e^{\int_{\partial f} dt \left( z_A^\dagger \frac{D}{dt} z^A - w_A^\dagger \frac{d}{dt} w^A - i \varphi(\|z\|^2 - \|w\|^2) \right)}.
\]  

(27)

\[z^A(0) = z^A(1), \quad w^A(0) = w^A(1)\]

The Gaußian integral turns into the trace over Hilbert space:

\[
Z_f [A, \varphi] = \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \left[ \text{Pexp} \left( - \int_{\partial f} dt \left( A^i(t) \tau^{AB} i \hat{z}_A \hat{z}_B^\dagger + i \varphi(t) (\|\hat{z}\|^2 : - : \|\hat{w}\|^2 : ) \right) \right) \right].
\]  

(28)

(29)

In terms of the canonical basis:

\[
Z_f [A, \varphi] = \sum_{2 j = 0}^{\infty} \sum_j \sum_{2 l = 0}^{\infty} (2l + 1) \langle j, m | \text{Pexp} \left( i \int_{\partial f} dt A^i(t) \hat{L}_i \right) | j, m \rangle \\
\times e^{-i \int_{\partial f} dt \varphi(t)(2j - 2l)}.
\]  

(30)

Notation:

- Generators of angular momentum: \( \hat{L}_i = i \tau^{AB} A^i \hat{z}_A \hat{z}_B^\dagger \)
- The spinor’s norm: \( :\|\hat{z}\|^2 : = \frac{1}{2} (\hat{z}_A^\dagger \hat{z}_A + \hat{z}_A \hat{z}_A^\dagger) = z_A^\dagger \frac{\partial}{\partial z_A} + 1 \),
Integral over the $U(1)$ gauge potential

Also the $\varphi$-integration factorises into a product over spinfoam faces. We choose the gauge fixing condition:

$$
\psi[\varphi](t) = \frac{d}{dt}\varphi(t) = 0.
$$

(31)

Infinitesimal gauge transformations are $\varphi^\lambda(t) = \varphi(t) + \dot{\lambda}(t)$, and $\Delta^\psi_{FP}[\varphi]$ is the determinant of the operator:

$$
\hat{m}[\lambda] := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \psi[\varphi^{\varepsilon\lambda}](t) = \frac{d^2}{dt^2}\lambda(t).
$$

(32)

All eigenvalues are $\varphi$-independent, and $\Delta^\psi_{FP}[\varphi]$ just gives an irrelevant overall constant.

$$
Z_f[A] = \int \mathcal{D}[\varphi] \Delta^\psi_{FP}[\varphi] \delta(\delta\psi[\varphi]) Z_f[\varphi, A] =
$$

$$
= \sum_{2j=0}^{\infty} \sum_{m=-j}^{j} (2j + 1) \langle j,m | \text{Pexp} \left( i \int_{\partial f} dt A^i(t) L_i \right) | j,m \rangle =
$$

$$
= \delta_{SU(2)} \left( \text{Pexp} \left( - \int_{\partial f} dt A^i(t) \tau_i \right) \right).
$$

(33)
Integral over the $SU(2)$ gauge potential

The most difficult step is the integral over the gauge potential $A$. We choose the following gauge fixing on the edges:

$$\forall e(t) : \Psi[A](t) = \frac{d}{dt} A_e(t) = 0.$$  \hspace{1cm} (34)

- We can achieve this gauge globally, one every edge of the discretisation.
- It is only a partial gauge fixing: For a single edge we can map $A_e$ to any other constant $\tilde{A}_e \in su(2)$.

The infinitesimal gauge transformations are $\delta_\Lambda A = \frac{D}{dt} \Lambda = \dot{\Lambda} + [A, \Lambda]$. This yields the Faddeev–Popov operator:

$$\hat{M}^{ij} \Lambda^j(t) = \delta_\Lambda \Psi[A](t) = \frac{d^2}{dt^2} \Lambda^i(t) + \epsilon_{lm} A^l \frac{d}{dt} \Lambda^m(t).$$  \hspace{1cm} (35)

We diagonalise this operator, under the condition that the gauge elements $\Lambda(t)$ are everywhere continuous. This yields the Faddeev–Popov determinant:

$$\Delta_{\text{FP}}[A] \propto \det \hat{M} \propto \prod_{e: \text{edges}} \prod_{n \in \mathbb{Z} - \{0\}} \left(1 - \frac{|A_e|}{2\pi n}\right)^2 = \prod_{e: \text{edges}} \frac{4 \sin^2 \frac{|A_e|}{2}}{|A_e|^2}. \hspace{1cm} (36)$$

The path integral over the connection turns into an ordinary integral:

\[ \int \prod_{e: \text{edges}} D[A_e] \Delta_{FP}^\Psi [A] \delta(\Psi[A]) \cdots \propto \prod_{e: \text{edges}} \int_{\mathbb{R}^3} d^3 A_e \frac{4 \sin^2 \frac{|A_e|}{2}}{|A_e|^2} \cdots \] \hspace{1cm} (37)

\[ \propto \prod_{e: \text{edges}} \int_{SU(2)} d\mu_{\text{Haar}}(U_e) \cdots \] \hspace{1cm} (38)

The Ponzano–Regge as a path integral in spinor space

\[ Z_M = \int \prod_{f: \text{faces}} D[f_z] D[f_w] D[\varphi_f] \Delta_{FP}^\psi [\varphi_f] \delta(\psi[\varphi_f]) \]

\[ \times \prod_{e: \text{edges}} D[A_e] \Delta_{FP}^\Psi [A] \delta(\Psi[A]) e^{i \hbar S_M[z,w,\varphi,A]} = \]

\[ = \prod_{e: \text{edges}} \int_{SU(2)} d\mu_{\text{Haar}}(U_e) \prod_{f: \text{faces}} \delta_{SU(2)} \left( \prod_{e \in \partial f} U_e \right). \] \hspace{1cm} (39)

where \( \prod \prod \) denotes the path ordered product.
Conclusion
Summary

- There is a one-dimensional action for simplicial gravity in three dimensions.
- All fields are continuous, but have support only on the one-dimensional edges of the discretisation.
- Once we have an action, we can define a canonical path integral.
- The resulting amplitudes reproduce the Ponzano–Regge model.

Spinors are useful for the following reasons: (i) They are canonical Darboux coordinates for loop gravity. (ii) Dynamics on a fixed discretisation of spacetime simplifies. (iii) The action turns into a bilinear in the spinors, this greatly simplifies the evaluation of the path integral.
Thanks for the attention!

In progress:


See also: