## Hamiltonian spinfoam gravity EFI winter conference on canonical and covariant LQG Tux, Austria

#### Wolfgang Wieland Centre de Physique Theorique de Luminy, Marseille

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## Motivation

#### Conceptual unity of loop quantum gravity?

- Canonical loop quantum gravity is an inductive (bottom-up) approach. It starts from classical GR, follows the Dirac program in the "right" variables and defines the solution space of the constraints.
- Spinfoam gravity is a deductive (top-down) approach. It postulates a gravitational path-integral, tries to prove finiteness of the theory and studies the semi-classical limit to recover general relativity.
- Both approaches share their kinematical structure, the Hilbertspace with operators representing area, angles and parallel transport.
- Does this relation extend beyond kinematics? Are these two approaches just different views of the same underlying quantum theory?

Within the reduced setting of a fixed discretisation of space-time, this talk will answer this question in the affirmative. Canonical quantisation techniques can be used to recover the spinfoam transition amplitudes.

#### Three points:

- Spinors as covariant variables for loop quantum gravity
- 2 Partial continuum limit and Hamiltonian formulation
- Quantisation

We will use the spinorial framework of the theory. This is useful for us because it embeds the non-linear loop gravity phase space into a large phase space with canonical Darboux coordinates.

# 1. Spinors as covariant variables for loop quantum gravity

## BF-theory—the symplectic potential of GR

We start from the following topological action

$$S[\Sigma, A] = \frac{i\hbar}{\ell_{\rm P}^2} \frac{\beta + i}{\beta} \int_M \Sigma_{AB} \wedge F^{AB}[A] + \text{cc.}$$
(1)

Peforming a 3+1 split we obtain the symplectic structure

$$\{\Pi_{i}{}^{a}(p), A^{j}{}_{b}(q)\} = \delta^{j}_{i}\delta^{a}_{b}\delta(p,q) = \{\bar{\Pi}_{i}{}^{a}(p), \bar{A}^{j}{}_{b}(q)\}.$$
(2)

With

$$\Pi_{AB} = -\frac{\hbar}{\ell_{\rm P}^2} \frac{\beta + i}{2i\beta} \Sigma_{AB}.$$
(3)

This theory is totally trivial, but it is important for us since it shares the symplectic structure of general relativity. Notation:

- $\Pi^{A}{}_{B}{}^{a} = \Pi_{i}{}^{a}\tau^{A}{}_{B}{}^{i}$  is an  $\mathfrak{sl}(2,\mathbb{C})$ -valued vector density.
- $A^{A}{}_{Ba}$  is the selfdual (Ashtekar)  $SL(2, \mathbb{C})$  connection.
- $A, B, C, \dots = 0, 1$  are spinor indices, the complex conjugate representation carries an overbar  $\overline{A}, \overline{B}, \overline{C}, \dots = \overline{0}, \overline{1}$ , and all indices are moved by  $\epsilon_{AB}, \overline{\epsilon}^{\overline{AB}}, \dots$

## Holonomies and fluxes



- Take a simplicial decomposition of space-time.
- Consider the triangulation induced on the spatial slices.
- To each triangle  $\tau, \tau', \ldots$  we assign the dual link  $\gamma, \gamma', \ldots$

Parallel transport: 
$$h^{A}{}_{B}[\tau] = \operatorname{Pexp}\left(-\int_{\gamma}A\right)^{A}{}_{B} \in SL(2,\mathbb{C}).$$
 (4a)

Flux: 
$$\Pi^{A}{}_{B}[\tau] = \int_{p\in\tau} \left(h_{p}^{-1}\Pi_{p}h_{p}\right)^{A}{}_{B} \in \mathfrak{sl}(2,\mathbb{C}).$$
 (4b)

The continuos Poisson brackets induce the commutation relations of  $(T^*SL(2,\mathbb{C}))^L$ , i.e.:

$$\left\{\Pi_{i},\Pi_{j}\right\} = -\epsilon_{ik}{}^{k}\Pi_{k}, \quad \left\{\Pi_{i},h\right\} = -h\tau_{i}.$$
(5)

There is also the anti-selfdual sector, which is the complex conjugate of the former. Both sectors mutually commute, just as in the continuum.

The elementary building block is the phase space of  $T^*SL(2,\mathbb{C}) \ni (\Pi,h)$  on a link.

On the initial point we can always find a pair  $(\pi^A, \omega^A) \in \mathbb{C}^2 \times \mathbb{C}^2$  of diagonalising spinors:

$$\Pi_{AB} = -\frac{1}{2}\omega_{(A}\pi_{B)}.$$
(6)

The holonomy (parallel transport) maps these spinors into the frame of the final point:

$$\underline{\pi}^{A} = h^{A}{}_{B}\pi^{B}, \quad \underline{\omega}^{A} = h^{A}{}_{B}\omega^{B}.$$
(7)

For  $\pi_A \omega^A \neq 0$  the pair  $(\pi, \omega)$  forms a basis and we can reverse the logic. Start with a quadruple of spinors and parametrize both flux and holonomy:

$$h^{A}{}_{B} = \frac{\omega^{A} \pi_{B} - \pi^{A} \omega_{B}}{\sqrt{\pi \omega} \sqrt{\pi \omega}}.$$
(8)

With  $\pi\omega = \pi_A \omega^A$ .

The parametrisation is not unique. There are the symmetries

$$Z_2:(\underline{\pi},\underline{\omega},\pi,\omega)\mapsto(\underline{\omega},\underline{\pi},\omega,\pi),\tag{9a}$$

$$\mathbb{C} - \{0\} : (\underline{\pi}, \underline{\omega}, \pi, \omega) \mapsto (z\underline{\omega}, z\underline{\pi}, z^{-1}\omega, z^{-1}\pi).$$
(9b)

To recover flux and holonomy we also need constraints:

non-degeneracy : 
$$\pi_A \omega^A \neq 0$$
, (10a)

area matching constraint : 
$$C = \pi_A \omega^A - \pi_A \omega^A = 0.$$
 (10b)

- Non-degeneracy is automatically fulfilled once we assume all triangles are spacelike (or timelike).
- The matching constraint guarantees the area of the triangle is the same as seen from the two sides.

We introduce Poisson brackets:

$$\{\pi_A, \omega^B\} = \delta^B_A = -\{\underline{\pi}_A, \underline{\omega}^B\},\tag{11a}$$

$$\{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\} = \delta_{\bar{A}}^{\bar{B}} = -\{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\}.$$
 (11b)

On the constarint hypersurface C = 0 we recover the commutation relations of  $T^*SL(2, \mathbb{C})$ .



- The area-matching constraint generates a flow  $X_C$  on the hypersuface C = 0.
- This flow realises the scaling symmetry introduced on the last slice.
- Performing a symplectic quotient we end up with the original phase space  $T^*SL(2, \mathbb{C})$  removed by its null configurations  $\Pi^{AB}\Pi_{AB} = 0$ .

- To recover general relativity we need additional constraints that guarantee the metric be real.
- Some of those are the linear simplicity constraints.
- These constraints demand the bivectors  $\Sigma_{\alpha\beta}[\tau]$  to define planes in local Minkowski space orthogonal to a time-like normal attached to each tetrahedron, i.e.  $\Sigma_{\alpha\beta}n^{\beta} = 0$ .

Using spinors they turn into the following three independent constraints:

$$D = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi_A \omega^A + \mathrm{cc.} = 0, \tag{12a}$$

$$F_n = n^{A\bar{A}} \pi_A \bar{\omega}_{\bar{A}} = n^{A\bar{A}} m_{A\bar{A}} = 0.$$
 (12b)

- D = 0 is first-class.
- $\blacksquare$  D = 0 guarantees the area of the triangle is real.
- $F_n = 0$  is second class, and gives an additional  $\mathfrak{su}(2)$  structure.
- The triangle is spanned by the vectors  $m_{\alpha}$ ,  $\bar{m}_{\alpha}$  in complexified Minkowski space, orthogonal to  $n^{\alpha}$

On the solution space of the simplicity constraints we can introduce SU(2) spinors:

$$z^{A} = \sqrt{2J} \frac{\omega^{A}}{\|\omega\|^{i\beta+1}}, \quad \underline{z}^{A} = \sqrt{2J} \frac{\underline{\omega}^{A}}{\|\underline{\omega}\|^{i\beta+1}}.$$
 (13)

That obey commutation relations of the Harmonic oscillator:

$$\{\bar{z}^{\bar{A}}, z^A\}^* = -i\delta^{A\bar{A}} = -i\{\bar{z}^{\bar{A}}, \bar{z}^A\}^*.$$
 (14)

- The SU(2) spinors transform linearly only under rotations but not under boosts.
- The Lorentz conncetion maps the  $\omega$ ,  $\tilde{\omega}$ -spinors into one-another. For the SU(2) spinors, on the other hand, there is a lattice version of the Ashtekar–Barbero connection that relates z with z.
- This reduction leads us back to the original SU(2) spinorial loop gravity framework developed by Dupuis, Freidel, Speziale, Livine.

see also: L. Freidel and S. Speziale; From twistors to twisted geometries; Phys. Rev. D 82:084041, (2010); arXiv:1001.2748.

- The gravitational phase space on a triangulation of a spatial slice is  $T^*SL(2,\mathbb{C})$ .
- 2 We can go to a spinor representation where this phase space is parametrised by four spinors per link.
- The symplectic structure simplifies, the spinors are canonical Darboux coordiantes on phase space. This is useful to solve the simplicity constraints but also important for quantum theory.

# 2. Continuum action on a fixed triangulation and Hamiltonian formulation

## The topological action in terms of spinors

The last section was about kinematics. This section is about the dynamics on a simplicial manifold.



We write the discretised action as a sum over wedges:

$$S[\Sigma, A] = \frac{i\hbar}{\ell_P{}^2} \frac{\beta + i}{\beta} \int_M \Sigma_{AB} \wedge F^{AB} + cc. = \sum_{w:wedges} S_w.$$
(15)

## Wedge action

We use:

$$h^{AB}[\partial w] \approx -\epsilon^{AB} + \int_{w} F^{AB}, \quad (16a)$$

$$\Sigma_{AB}[\tau_w] = \frac{\ell_{\rm P}^2}{\hbar} \frac{\mathrm{i}\beta}{\beta + \mathrm{i}} \omega_{(A} \pi_{B)}. \quad (16b)$$

$$\text{Vertex}$$

And find the following approximation for the topological action on a wedge:

$$S_{w} = -\frac{1}{2} M_{w} \left( g \underline{g}^{-1} \right)^{AB} \left( \omega_{A} \underline{\pi}_{B} + \pi_{A} \underline{\omega}_{B} \right) + \text{cc.},$$
  
with:  $M_{w} = \frac{1}{2} \left( \frac{\sqrt{\pi \omega}}{\sqrt{\pi \underline{\omega}}} + \frac{\sqrt{\pi \underline{\omega}}}{\sqrt{\pi \omega}} \right).$  (17)

## Continuum limit on a spinfoam face

We now split every wedge into N auxiliary wedges and take the continuum limit  $N \to \infty$ . We set  $\varepsilon = N^{-1}$  and expand all variables in  $\varepsilon$ . With  $\underline{\pi}(t) = \pi(t + \varepsilon)$ , and  $E = \pi_A \omega^A$  we find:

$$M_{w} = \frac{1}{2} \left( \frac{\sqrt{E(t)}}{\sqrt{E(t+\varepsilon)}} + \frac{\sqrt{E(t+\varepsilon)}}{\sqrt{E(t)}} \right) =$$

$$= 1 + \mathcal{O}(\varepsilon).$$

$$gg^{-1} = \operatorname{Pexp} \left( -\int_{t}^{t+\varepsilon} A_{e(s)}(\dot{e}) \right) =$$

$$= 1 - \varepsilon A_{e(t)}(\dot{e}) + \mathcal{O}(\varepsilon^{2}).$$

$$\pi_{B} = \pi_{B} + \varepsilon \dot{\pi}_{B} + \mathcal{O}(\varepsilon^{2}).$$

Putting everything together we end up with the following action on a wedge:

$$S_w = \frac{1}{2} \int_e \mathrm{d}t \left( \omega_A D_{\partial_t} \pi^A + \pi_A D_{\partial_t} \omega^A \right) + \mathrm{cc.}$$
 (19)

$$\square D_{\partial_t} \pi^A = \dot{\pi}^A + A^A{}_B(\dot{e})\pi^B.$$

This action is just a covariant symplcetic potential for the spinors.

The action of the topological theory is stationary if the spinors are covaraintly constant along the edge:

$$D_{\partial_t} \pi^A = 0 = D_{\partial_t} \omega^A \tag{20}$$

#### Comments:

- We will later see this constraint reflects the vanishing of curvature in *BF*-theory.
- We have seen spinors correspond to holonomy-flux variables only if the area matching constraint is staisfied.
- Where does this constraint show up here?

The  $\varepsilon \to 0$  limit of the area matching constraint reveals the conservation law:

$$C = \pi_A \omega^A - \pi_A \omega^A = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} (\pi_A \omega^A)$$
(21)

But the equations of motion automatically preserve this constraint:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pi_A\omega^A) = D_{\partial_t}(\pi_A\omega^A) = 0 \tag{22}$$

## Where is the Gauß law in this picture hiding?



**Edge-action** 

- The Gauß law belongs to tetrahedra.
- It guarantees local Lorentz invariance.
- In the dual picture tetrahedra are edges bounding the spinfoam face.

$$S_e = \frac{1}{2} = \sum_{I=1}^{4} \int_0^1 \mathrm{d}t \left( \omega_A^{(I)} D_{\partial_t} \pi_{(I)}^A + \pi_A^{(I)} D_{\partial_t} \omega_{(I)}^A \right) + \mathrm{cc.}$$
(23)

Variation of this action with respect to the Lagrange multiplier  $\Phi^{A}{}_{B}(t) = A^{A}{}_{Be(t)}(\dot{e})$  reveals Gauß's law for each edge (tetrahedron):

$$G_{AB} = -\frac{1}{2} \sum_{I=1}^{4} \omega_{(A}^{(I)} \pi_{B)}^{(I)} = 0.$$
 (24)

#### So far we have just dealt with the topological theory

We now introduce Lagrange multipliers  $\lambda \in \mathbb{R}$ , and  $z \in \mathbb{C}$  and add the constraints to the edge action:

$$S_{e} = \frac{1}{2} \sum_{I=1}^{4} \int_{0}^{1} dt \Big[ \omega_{A}^{(I)} D_{\partial_{t}} \pi_{(I)}^{A} + \pi_{A}^{(I)} D_{\partial_{t}} \omega_{(I)}^{A} + -2z_{(I)} F_{n(t)}[\pi_{(I)}, \omega_{(I)}] - \lambda_{(I)} D[\pi_{(I)}, \omega_{(I)}] \Big] + cc.$$
(25)

In this action there appears the normal of the tetrahedron n(t) as a function of t. In spinfoam gravity geometry is assumed to be locally flat. So it makes sense to put:

$$D_{\partial_t} n^{\alpha}(t) = 0. \tag{26}$$

## Dirac analysis of the constraints

For each pair of spinors the equation of motion assume the Hamiltonian form:

$$D_{\partial_t}\omega^A = \{H', \omega^A\}, \quad D_{\partial_t}\pi^A = \{H', \pi^A\}.$$
 (27a)

With the primary Hamiltonian

$$H' = z(t)F_{n(t)}[\pi,\omega] + \frac{\lambda(t)}{2}D[\pi,\omega] + cc.$$
 (28)

We now have to check whether time evolution preserves the constarints of the theory. We get:

- For the second-class constraints  $F_n = 0$  to hold for all times the multiplier z must vanish.
- Time evolution preserves the first class constraint D = 0 and the Gauß law  $G_{AB} = 0$ .
- The area matching constraint, i.e. the conservation law  $\frac{d}{dt}(\pi_A \omega^A) = 0$  is also always satisfied.

Time evolution along the edge is governed by the secondary Hamiltonian:

$$H'' = \lambda(t)D[\pi, \omega].$$
<sup>(29)</sup>

## Solving the EOMs for the spinors

The equations of motion for the spinors are:

$$D_{\partial_t}\omega^A = \frac{\mathrm{i}}{\beta + \mathrm{i}}\lambda\omega^A, \quad D_{\partial_t}\pi^A = -\frac{\mathrm{i}}{\beta + \mathrm{i}}\lambda\pi^A.$$
 (30)

The solution of the equations of motion is for e.g.  $\omega$ :

$$\omega(t) = e^{\frac{i}{\beta+i} \int_0^t ds \lambda(s)} \operatorname{Texp}\left(-\int_0^t ds A_{e(s)}(\dot{e})\right) \omega(0).$$
(31)

The boundary conditions constrain the holonomy around the face:

$$U^{A}{}_{B}(0,N) = (\pi\omega)^{-1} \left[ e^{-\frac{i}{\beta+i}\Lambda} \omega^{A}(0)\pi_{B}(0) - e^{\frac{i}{\beta+i}\Lambda}\pi^{A}(0)\omega_{B}(0) \right].$$
(32)  
With  $\Lambda = \int_{0}^{N} dt\lambda$ .





The extrinsic curvature along the link is measured as the bitetrahedral angle

$$\begin{split} & \ch{\Xi}(t,t') = -n_{A\bar{A}}(t') \\ & \cdot h^{A}{}_{B}(t,t') \bar{h}^{\bar{A}}{}_{\bar{B}}(t,t') n^{B\bar{B}}(t). \end{split}$$
 (33)

There are two cases to distinguish, (i) t and t' belong to the same edge, (ii) they don't.

This gives the physical intepretation of the Lagrange multiplier  $\lambda$  enforcing D = 0.

- For t, t' on the same edge using  $D_{\partial_t} n^{\alpha} = 0$  we get  $\Xi(t, t') = -\frac{2}{\beta^2 + 1} \int_t^{t'} ds \lambda(s).$
- At a vertex  $v_i$  there is a residual angle  $\Xi_i$ .
- The gauge trafos generated by *D* can shift  $\lambda \mapsto \lambda + \dot{\varepsilon}$ .
- For generic *t*-values  $\Xi(t, t')$  is not an observable.
- The overall angle  $\Lambda = \int_0^N dt \lambda(t)$  is an observable due to the periodic boundary conditions.

## Intrinsic curvature on a spinfoam face, 1/2

Let us now further uncover the geometric meaning of the spinors, consider the variation of the holonomy:

$$\delta h_{\gamma} = -A_{\gamma(1)}(\delta\gamma)h_{\gamma} + h_{\gamma}A_{\gamma(0)}(\delta\gamma) + \int_0^1 \mathrm{d}s \, h_{\gamma(1)}h_{\gamma(s)}^{-1}F_{\gamma(s)}(\dot{\gamma},\delta\gamma)h_{\gamma(s)}.$$
 (34)

The spinor  $\pi(t+\varepsilon)$  is the parallel transport of  $\pi(t)$  along the spike  $\gamma_{t+\varepsilon}^{-1} \circ \gamma_t$ .



$$D_{\partial_t}\pi^A = F^A{}_B(t)\pi^B, \qquad (35a)$$

$$D_{\partial_t}\omega^A = F^A{}_B(t)\omega^B.$$
 (35b)

With:

$$F^{A}{}_{B}(t) = \int_{0}^{1} \mathrm{d}s \left[ h_{\gamma(s)}^{-1} F_{\gamma_{t}(s)} \left( \frac{\mathrm{d}}{\mathrm{d}s} \gamma_{t}(s), \frac{\mathrm{d}}{\mathrm{d}t} \gamma_{t}(s) \right) h_{\gamma_{t}(s)} \right]^{A}{}_{B} \in \mathfrak{sl}(2, \mathbb{C}).$$
(36)

The equations of motion for the spinors constitute such a variation, and we can read off the curvature form the equations of motion:

$$F^{AB}(t) = \frac{2\hbar}{\beta\ell_{\rm P}^2} \frac{1}{J} \frac{\lambda}{\beta+{\rm i}} \Sigma^{AB}.$$
(37)

With  $\lambda$  being gauge dependend, we have to integrate over the whole face to end up with something observable:



## Why are there no secondary constraints, do we miss something?

Where torsion can hide in a discretised theory of gravity:

1. On triangles, this one we do not have:

$$De^{\alpha} = 0 \Rightarrow \sum_{b \in \partial \tau} e^{\alpha}[b] = 0.$$
 (40)

2. At tetrahedra, this one we have in our theory:

$$D(e^{\alpha} \wedge e^{\beta}) = 0 \Rightarrow \sum_{\tau \in \partial \mathcal{T}} \Sigma^{\alpha \beta}[\tau] = 0.$$
(41)

Half of it is first class, the rest is second class.

**3**. At 4-simplices, this one we also have, but it holds in the weakest possible way. Only if the equations of motion (constraints+evolution equations) are satisfied we find the right hand side to vanish:

$$D(e^{\alpha} \wedge e^{\beta} \wedge e^{\nu}) = 0 \Rightarrow \sum_{\mathcal{T} \in \partial v} n_{\alpha}[\mathcal{T}]^{3} \operatorname{vol}[\mathcal{T}] = 0.$$
(42)

In quantum theory, at the saddle point, this equation is satisfied, but this may be too weak. When going beyond the saddle point approximation we may need to impose this constraint more strongly.

- We wrote the gravitational action as a one-dimensional line integral along a spinfoam edge.
- The equations of motion admit a Hamiltonian formulation. The Hamiltonian flow preserves all constraints of the theory without any secondary constraints needed.
- The evolution equations have a geometrical intepretation and show the model carries curvature.
- We discussed the role of torsion in the discrete geometry. The torsional constraints are imposed in different strenght. The four dimensional closure constraint may hold too weakly.

## 3. Quantum theory

- We now want take our spinorial description as the starting point for the quantisation problem.
- With a linear phase-space, canonical coordinates, and a Hamiltonian generating the "time"-evolution, quantisation on a fixed triangulation becomes straight forward.
- This will lead us to a new representation of loop quantum gravity in between the loop- and the Baratin–Oriti flux-representation.
- The quantum states are functions on  $\mathbb{C}^2$ .

The "primary" phase space on a half link is  $\mathbb{C}^2 \oplus \mathbb{C}^2 \ni (\pi^A, \omega^A)$ , we take a position representation and define the auxiliary Hilbert-space

$$\mathcal{H}_{\mathrm{aux}} := L^2(\mathbb{C}^2, d^4\omega) = \int_{\mathbb{R}}^{\oplus} \mathrm{d}\rho \sum_{k \in \mathbb{Z}} \mathcal{H}_{\rho,k}.$$
(43)

Where  $\mathcal{H}_{\rho,k}$  denotes an irreducible subspace under the unitary  $SL(2,\mathbb{C})$ action  $(\mathcal{D}(g)f)(\omega) = f(g^{-1}\omega)$  on  $L^2(\mathbb{C}^2, d^4\omega)$ . The "position" operator acts by multiplication, the momentum becomes a derivative:

$$\pi_A \to \frac{\hbar}{i} \frac{\partial}{\partial \omega^A}, \quad \bar{\pi}_{\bar{A}} \to \frac{\hbar}{i} \frac{\partial}{\partial \bar{\omega}^{\bar{A}}}$$
 (44)

We introduce the "canonical" basis  $\{f_{jm}^{(\rho,k)}\}$  simultaneously diagonalising the Casimirs  $\vec{L}\vec{K}$ ,  $\vec{L}^2 - \vec{K}^2$  of  $SL(2, \mathbb{C})$  together with  $\vec{L}^2$  and  $L_3$ . The first-class constraint D = 0 becomes an operator diagonal in the canonical basis:

$$\hat{D}f_{jm}^{(\rho,k)} = \frac{2\hbar}{\beta^2 + 1} \left(\rho - \beta(k+1)\right) f_{jm}^{(\rho,k)}$$
(45)

The second class constraints  $F_n = 0$  act like the step operators for  $\mathfrak{su}(2)$ :

$$\hat{F}_{n_o} f_{jm}^{(\rho,k)} = -\frac{\hbar}{\sqrt{2}} \sqrt{(j-k)(j+k+1)} f_{jm}^{(\rho,(k+1))}$$
(46a)

$$\hat{F}_{n_o}^{\dagger} f_{jm}^{(\rho,k)} = -\frac{\hbar}{\sqrt{2}} \sqrt{(j+k)(j-k+1)} f_{jm}^{(\rho,(k+1))}$$
(46b)

## Solution-space and finite inner product

The *D*-constraint is first-class, we can impose it strongly, with the solution space spanned by functions

$$\mathcal{H}_D = \text{span} \{ f_{jm}^{(\beta(k+1),k)} : k, j, m \}.$$
(47)

 $f_{jm}^{(\beta(k+1),k)}$  are distributions in  $\mathbb{C}^2$ , but they are orthogonal and properly normalised with respect to the inner product on the orbits:

$$\langle f, f' \rangle_{\mathbb{C}^2/C} = \int_{\mathbb{C}^2/C} X_C \lrcorner d^4 \omega \, \bar{f} f' \tag{48}$$

The *F*-constraint is second class, with Gupta–Bleuler we only impose  $\hat{F}$  strongly while  $\hat{F}^{\dagger}$  maps the solution space into its orthogonal complement. (The spurious part of  $\mathcal{H}_{auc}$ ). The resulting Hilbert-space is spanned by

$$\mathcal{H}_{\text{simpl}} = \text{span} \left\{ f_{jm}^{(\beta(j+1),j)} : j,m \right\}$$
(49)

The Gauß-constraint is again first class and we can impose it strongly revealing the physical Hilbertspace (of a quantised tetrahedron):

$$\Psi(\omega_{(1)},\ldots,\omega_{(4)}) \in \mathcal{H}_{phys} = \left(\bigotimes^{4} \mathcal{H}_{simpl}\right) / SU(2)$$
(50)

## Transition amplitudes on a spinfoam face

- The area matching constraint glues these Hilbertspaces to form the space of spinnetwork states on multiples of C<sup>2</sup>.
- $\blacksquare$  The resulting Hilbertspace matches the space of Penrose's SU(2) spinnetwork functions.
- What about the dynamics?

Time evolution along an edge is governed by the Hamilton equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}O_t = \left\{ \left( \Phi^{AB}(t)\pi_A \omega_B + \mathrm{cc.} \right) + \lambda(t)D, O_t \right\}$$
(51)

In quantum theory this becomes the Schrödinger equation on an edge:

$$i\hbar \frac{d}{dt}\psi_t = \left(\Phi^{AB}(t)\hat{\pi}_A\hat{\omega}_B + h.c.\right)\psi_t + \lambda(t)\hat{D}\psi_t$$
(52)

- The *D*-constraint annihilates  $\mathcal{H}_{simpl}$ , and only the first term survives.
- The first term acts as an infinitesimal Lorentz generator and matches Bianchi's boundary Hamiltonian.
- $\exp \Phi$  is nothing but the parallel transport along the edge.

see also: E. Bianchi; Entropy of Non-Extremal Black Holes from Loop Gravity, (2012); arXiv:1204.5122.

Consider now a scattering process at a vertex.



- The initial state be  $\Psi_{t_0} = f_{jm}^{(\beta(j+1),j)}$ , the final state  $\Psi_{t_1} = f_{jm'}^{(\beta(j+1),j)}$ .
- The transition amplitude for this process is the inner product:

$$A(\Psi_{t_o} \to \Psi_{t_1}) = \langle f_{j_{m'}}^{(\beta(j+1),j)}, \mathcal{D}(\underline{g}g^{-1}) f_{j_m}^{(\beta(j+1),j)} \rangle_{\mathbb{C}^2/D}$$
(53)

Glueing the amplitudes for the individual processes together we find the EPRL-amplitude on a spinfoam face *f*:

$$A_f(g_{e_o,e_1},g_{e_1,e_2},\dots) = \sum_{j=0}^{\infty} \sum_{m_1=-j}^{j} \cdots \sum_{m_N=-j}^{j} \prod_{i=1}^{N} \langle jm_{i+1} | g_{e_i,e_{i+1}} | jm_i \rangle$$
(54)

How can we glue these amplitudes together to find the full spinfoam amplitude?

- By just integrating the product of all face amplitudes against the Haar measure *dg* we would get the usual EPRL amplitude.
- Adding the torsional constraint proposed, we would have to insert an additional delta function  $\delta_{\mathbb{R}^4}(\sum_{\mathcal{T}} n^{\alpha}[\mathcal{T}]^3 \operatorname{vol}[\mathcal{T}])$ .
- This would remove four integrals per vertex.

The key question is now rather clear and well-posed:

Do we need the additional torsional constraints to make the model less divergent?

see also: A. Riello; Self-Energy in the Lorentzian EPRL-FK Spin Foam Model of Quantum Gravity, (2013); arXiv: 1302.1781.

- We used a spinor representation, and recovered the loop gravity Hilbert-space of spinnetwork functions after imposing the simplicity constraints.
- The dynamics on a spinfoam face matches both the EPRL model and Bianchi's boundary Hamiltonian.
- It is not so clear to me of how to glue the face-amplitudes to form a spinfoam. I mentioned two proposals, the choice of EPRL and another "torsional" glueing that may be less divergent.

## Conclusion

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#### definite answers:

- The EPRL proposal for the loop gravity transition amplitudes results from the canonical quantisation of a classical theory with a finite number of degrees of freedom.
- The underlying classical theory is a truncation of general relativity to a fixed triangulation parametrised by a field of spinors living on the edges of the triangulation.
- The spinorial framework is powerfull enough to complete the canonical analysis. All constraints are preserved in the "time" variable around a spinfoam face.
- But still, we may miss additional torsional constraints that the model imposes yet too weakly.

Spinors are useful for three reason: (i) They are canonical Darboux coordinates taking care of the non-linearities of the loop gravity phase space. (ii) They transform covariantly under the local symmetry group of general relativity. (iii) Dynamics on a fixed discretisation of space-time simplifies.

## Thanks for the invitation!

This talk is based on the papers:

- W. Wieland; Twistorial phase space for complex Ashtekar variables; Class. Quantum Grav. 29, (2012); arXiv:1104.3683.
- S. Speziale and W. Wieland; The twistorial structure of loop-gravity transition amplitudes; Phys. Rev. D 86, (2012); arXiv:1207.6348.
- W. Wieland; Hamiltonian spinfoam gravity; (2013); arXiv:1301.5859.

See also:

- L. Freidel and S. Speziale; From twistors to twisted geometries; Phys. Rev. D 82:084041, (2010); arXiv:1001.2748.
- M. Dupuis and E. Livine; Holomorphic Simplicity Constraints for 4d Spinfoam Models; Class. Quantum Grav. 28, (2011); arXiv:1001.2748.
- E. Bianchi; Entropy of Non-Extremal Black Holes from Loop Gravity, (2012); arXiv:1204.5122.