New action for simplicial gravity
Curvature and relation to Regge calculus

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Tux 3

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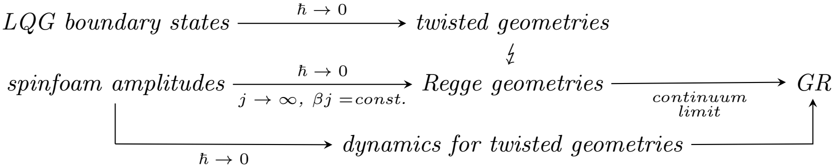
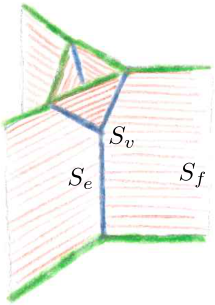


For large spins (large distances) and small Barbero–Immirzi parameter we seem to get the Regge action. What do we get for small spins—short distances—high energies?

The semi-classical limit of covariant LQG for arbitrary values of j and β should give a theory of discretized gravity in terms of Ashtekar–Barbero variables. Can we find such a classical theory without knowing the exact quantum theory behind?



- A spinfoam model assigns amplitudes A_e, A_f, A_v,... to the elementary building blocks of the simplicial complex.
- In the semi-classical limit these amplitudes turn into action functionals: $A_e \propto \mathrm{e}^{\mathrm{i}S_e}$, $A_f \propto \mathrm{e}^{\mathrm{i}S_f}$, $A_v \propto \mathrm{e}^{\mathrm{i}S_v}$,...
- The desired action will be of the form: $S_{\text{spinfoam}} = \sum_e S_e + \sum_f S_f + \sum_v S_v + \dots$

Outline of the talk

Main message: Covariant LQG suggests a new action for simplicial gravity with spinors as the fundamental configuration variables: The theory has a Hamiltonian and local gauge symmetries. Generic solutions represent twisted geometries, but the solution space contains also Regge configurations.

Table of contents

References:

*ww, New action for simplicial gravity in four dimensions, Class. Quant. Grav. 32 (2015), arXiv:1407.0025.
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*ww, Twistorial phase space for complex Ashtekar variables, Class. Quant. Grav. 29 (2012), arXiv:1107.5002.
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Worldline action for simplicial gravity

Plebański principle

The BF action is topological, and determines the symplectic structure of the theory:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_{\mathcal{M}} \underbrace{\frac{1}{2{\ell_{\mathrm{P}}}^2}\Big(* \Sigma_{\alpha\beta} - \frac{1}{\beta}\Sigma_{\alpha\beta}\Big)}_{\Pi_{\alpha\beta}} \wedge F^{\alpha\beta}[A]. \tag{1}$$

General relativity follows from the simplicity constraints added to the action:

$$\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}. \tag{2}$$

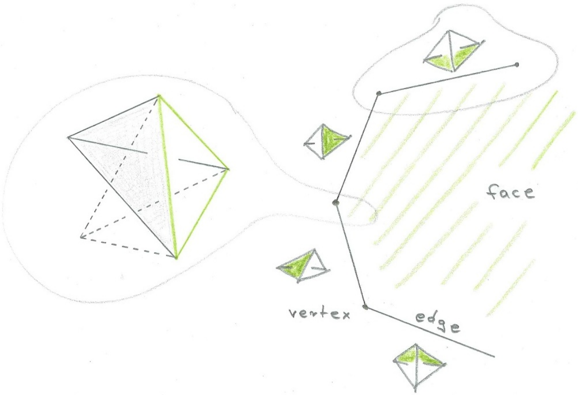
With the solutions:

$$\Sigma^{\alpha\beta} = \begin{cases} \pm e_\alpha \wedge e_\beta, \\ \pm * (e_\alpha \wedge e_\beta). \end{cases} \tag{3}$$

Notation:

- $\alpha, \beta, \gamma \dots$ are internal Lorentz indices.
- ${\Sigma^\alpha}_\beta$ is an $\mathfrak{so}(1,3)$-valued two-form.
- ${A^\alpha}_\beta$ is an $SO(1,3)$ connection, with ${F^\alpha}_\beta = \mathrm{d}{A^\alpha}_\beta + {A^\alpha}_\mu \wedge {A^\mu}_\beta$ denoting its curvature.
- e^α is the tetrad, diagonalizing the four-dimensional metric $g = \eta_{\alpha\beta} e^\alpha \otimes e^\beta$.
- ${\ell_{\mathrm{P}}}^2 = 8\pi/G$, and β is the Barbero–Immirzi parameter, $\hbar = 1 = c$.

Simplicial discretization



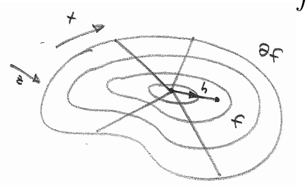
Faust: Das Pentagramma macht dir Pein? Ei sage mir, du Sohn der Hölle, Wenn das dich bannt, wie kamst du denn herein? Wie ward ein solcher Geist betrogen?	*Faust:* The pentagram prohibits thee? Why, tell me now, thou Son of Hades, If that prevents, how cam'st thou in to me? Could such a spirit be so cheated?
Mephistopheles: Beschaut es recht! es ist nicht gut gezogen: Der eine Winkel, der nach außen zu, Ist, wie du siehst, ein wenig offen.	*Mephistopheles:* Inspect the thing: the drawing's not completed. The outer angle, you may see, Is open left—the lines don't fit it.

Aus Goethes Faust.

Construction of the action, 1/2

- Step 1: Discretize the action:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_{\mathcal{M}} \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \approx \sum_{f:\mathrm{faces}} \int_{\tau_f} \Pi_{\alpha\beta} \int_f F^{\alpha\beta} \equiv \sum_{f:\mathrm{faces}} S_f .$$

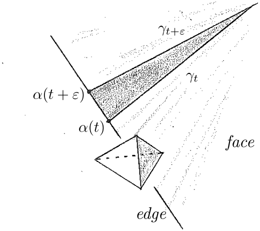


- Step 2: Employ the non-Abelian Stoke's theorem:

$$F_f = \int_f h^{-1} F h = \oint_{\partial f} h^{-1} D h.$$

- Step 3: Define the smeared flux:

$$\Pi_f = \int_{\tau_f} h^{-1} \Pi h.$$



- Step 4: Contract the two formulae in a common frame:

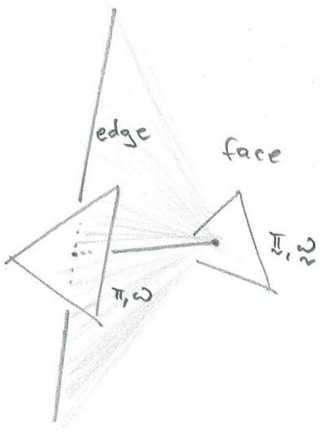
$$S_f = -\oint_{\partial f} \mathrm{d}t \left[h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)} \right]_{\alpha\beta} \Pi_f^{\alpha\beta}(t).$$

Construction of the action, 2/2

- Step 5: Introduce spinors to diagonalize both holonomies and fluxes:

$$\Pi_f^{\alpha\beta}(t) = \frac{1}{2}\bar{\epsilon}^{\bar{A}\bar{B}}\omega_f^{(A}(t)\pi_f^{B)}(t) + \text{cc.},$$

$$[h_{\gamma_t}]^A{}_B = \text{Pexp}\Big(-\int_{\gamma_t} A\Big)^A{}_B = \frac{\underset{\sim}{\omega}{}_f^A(t)\pi^f_B(t) - \underset{\sim}{\pi}{}_f^A(t)\omega^f_B(t)}{\sqrt{E_f(t)}\sqrt{\underset{\sim}{E}{}_f(t)}}.$$



We also need the area-matching constraint:

$$\varDelta_f := \underset{\sim}{\pi}{}^f_A\underset{\sim}{\omega}{}^A_f - \pi^f_A\omega^A_f \equiv \underset{\sim}{E}{}_f(t) - E_f(t).$$

Putting the pieces together yields the face action:

$$S_f[Z, \underset{\sim}{Z}, A, \zeta] = \\ = \oint_{\partial f} \mathrm{d}t\Big[\pi_A \frac{D}{\mathrm{d}t}\omega^A - \underset{\sim}{\pi}{}_A \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{\omega}{}^A - \zeta\varDelta\Big] + \text{cc.} \qquad (5)$$

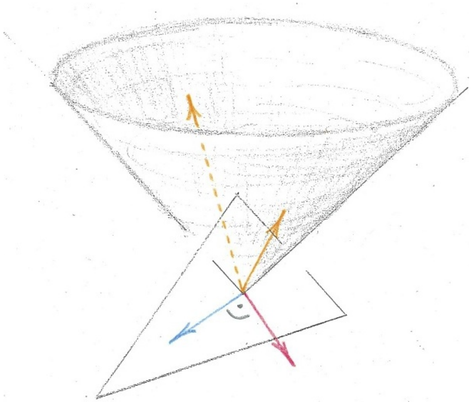
We define the complex null tetrad

$$\ell^{\alpha} = \mathrm{i}\pi^{A}\bar{\pi}^{\bar{A}}, \qquad k^{\alpha} = \mathrm{i}\omega^{A}\bar{\omega}^{\bar{A}},$$

$$\frac{1}{2}(x^{\alpha} + \mathrm{i}y^{\alpha}) = m^{\alpha} = \mathrm{i}\omega^{A}\bar{\pi}^{\bar{A}}, \qquad \bar{m}^{\alpha} = \mathrm{i}\pi^{A}\bar{\omega}^{\bar{A}}.$$

Area, and plane of the triangle:

$$(\beta + \mathrm{i})\,\mathrm{Ar} = \beta \ell_{\mathrm{P}}{}^{2}\pi_{A}\omega^{A}, \quad \Sigma_{\alpha\beta} \propto x_{[\alpha}y_{\beta]}. \tag{6}$$



Geometric interpretation of the spinors

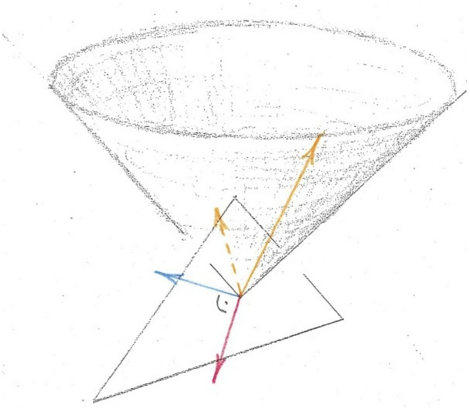
We define the complex null tetrad

$$\ell^\alpha = \mathrm{i}\pi^A\bar{\pi}^{\bar{A}}, \qquad k^\alpha = \mathrm{i}\omega^A\bar{\omega}^{\bar{A}},$$

$$\frac{1}{2}(x^\alpha + \mathrm{i}y^\alpha) = m^\alpha = \mathrm{i}\omega^A\bar{\pi}^{\bar{A}}, \qquad \bar{m}^\alpha = \mathrm{i}\pi^A\bar{\omega}^{\bar{A}}.$$

The area-matching constraint $\varDelta$ generates the transformations:

$$\pi^A \to \mathrm{e}^z\pi^A, \quad \omega^A \to \mathrm{e}^{-z}\omega^A, \quad z \in \mathbb{C}. \tag{7}$$



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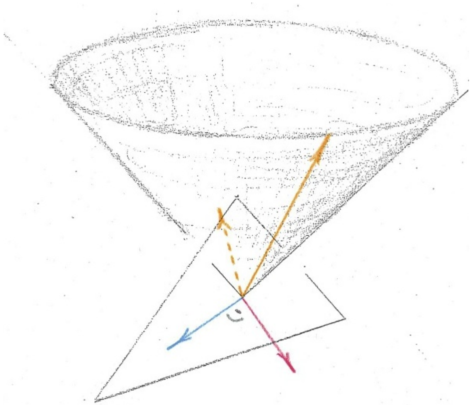
Rotations around the z-axis:

$$\pi^A \to \mathrm{e}^{-\frac{\mathrm{i}\varphi}{2}}\pi^A, \quad \omega^A \to \mathrm{e}^{\frac{\mathrm{i}\varphi}{2}}\omega^A, \quad \varphi \in [0, 4\pi). \tag{8}$$



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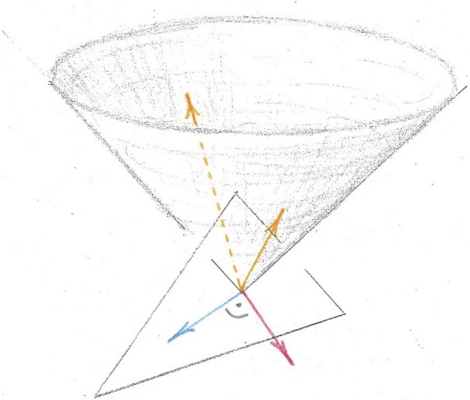
Boosts into the z-direction:

$$\pi^A \to \mathrm{e}^{-\frac{\xi}{2}}\pi^A, \quad \omega^A \to \mathrm{e}^{\frac{\xi}{2}}\omega^A, \quad \xi \in \mathbb{R}. \tag{9}$$



Instead of discretizing the quadratic simplicity constraints

$$\Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu}, \tag{10}$$

we will use the linear simplicity constraints:

For a tetrahedron T_e (dual to an edge e) there exist an internal future-oriented four-vector n_e^α such that the fluxes through the four bounding triangles τ_f (dual to a face f: $e \subset \partial f$) annihilate n_e^α:

$$\int_{\tau_f} \Sigma_{\alpha\beta} n_e^\beta = 0. \tag{11}$$

The spinorial parametrization turns the simplicity constraints into the following complex conditions:

$$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0, \tag{12a}$$

$$W_{ef} = n_e^{A\bar{A}} \pi^f_A \bar{\omega}^f_{\bar{A}} \stackrel{!}{=} 0. \tag{12b}$$

Adding the simplicity constraints

- The simplicity constraints reduce the $SO(1,3)$ spin connection $A^{\alpha}{}_{\beta}$ to the $SU(2)_n$ Asthekar–Barbero connection:

$$\mathcal{A}^{\alpha} = n^{\mu}\Big[\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\rho}A^{\nu}{}_{\rho} + \beta A^{\alpha}{}_{\mu}\Big]. \tag{13}$$

- We introduce Lagrange multipliers $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ and get the following constrained action for each face in the discretization:

$$S_{\text{face}}[Z,\underset{\sim}{Z}|\zeta,z,\lambda|\mathcal{A},n] = \oint_{\partial f}\Big(\pi_A \mathcal{D}\omega^A - \underset{\sim}{\pi}_A \mathrm{d}\underset{\sim}{\omega}^A - \zeta\big(\underset{\sim}{\pi}_A\underset{\sim}{\omega}^A - \pi_A\omega^A\big) + \\ - \frac{\lambda}{2}\Big(\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi_A\omega^A + \text{cc.}\Big) - z\, n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}}\Big) + \text{cc.}, \tag{14}$$

  where $\mathcal{D}\pi^A = \mathrm{d}\pi^A + \mathcal{A}^{\alpha}\tau^{A}{}_{B\alpha}\pi^B$ is the $SU(2)_n$ covariant differential.
- Problem: There is no term in the action that would determine the t-dependence of the normal n^{α}_e along the edges $e(t)$.
- We now have to make a proposal.

Four-dimensional closure constraint

Any proposal for the dynamics of the time normals must respect the closure constraint at the vertices (four-simplices):

We define the four-momenta:

$$p^e_\alpha = g\, n^e_\alpha \mathrm{Vol}(e). \tag{15}$$

At every four simplex we have the closure constraint:

$$\sum_{\substack{\text{outgoing edges } e \\ \text{at } v}} p^e_\alpha = \sum_{\substack{\text{incoming edges } e \\ \text{at } v}} p^e_\alpha. \tag{16}$$

The constant g is dimensionful:

$$[g] = \left[\frac{\Lambda}{8\pi G}\right] = \frac{\text{mass}}{\text{volume}} \tag{17}$$

Remarks:

- $\mathrm{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \text{cc.}$
- In a locally flat geometry, the closure constraint follows from the vanishing of torsion:

$$T^\alpha = De^\alpha = 0 \Rightarrow \frac{1}{3!}\epsilon_{\alpha\rho\mu\mu} D\big(e^\rho \wedge e^\mu e^\nu\big) = 0 \Rightarrow \sum_{\substack{\text{edges } e \\ \text{at } v}} \pm_e p^e_\alpha = 0.$$

Any proposal for the dynamics of the time-normals
- must respect the four-dimensional closure constraint, and
- be consistent with all symmetries of the action.

The following action fulfills these requirements:

$$S_{\text{edge}}[X, p|N, \text{Vol}(e)] = \int_e \Big(p_\alpha \mathrm{d}X^\alpha - \frac{N}{2}\big(g^{-1}\, p_\alpha p^\alpha + g\, \text{Vol}^2(e)\big)\Big). \tag{18}$$

We just need an additional boundary term at the vertices:

$$S_{\text{vertex}}[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}] = \sum_{e:e\ni v} \big(Y_v^\alpha - X_{ev}^\alpha\big) v_\alpha^{ev}. \tag{19}$$

Where N is a Lagrange multiplier imposing the mass-shell condition:

$$C := \frac{1}{2}\Big(g^{-1}\, p_\alpha p^\alpha + g\, \text{Vol}^2(e)\Big) \stackrel{!}{=} 0. \tag{20}$$

Putting the pieces together – defining the action

Adding the face, edge and vertex contributions gives us a proposal for an action for discretized gravity in first-order variables:

$$\begin{aligned} S_{\text{spinfoam}} = & \sum_{f:\text{faces}} S_{\text{face}}\big[Z_f, \underset{\sim}{Z}_f \big| \zeta_f, z_f, \lambda_f \big| \mathcal{A}_{\partial f}, n_{\partial f}\big] + \\ & + \sum_{e:\text{edges}} S_{\text{edge}}\big[X_e, p_e \big| N_e, \text{Vol}(e)\big] + \\ & + \sum_{v:\text{vertices}} S_{\text{vertex}}\big[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}\big]. \end{aligned} \tag{21}$$

Notation:

- Z_f and $\underset{\sim}{Z}_f$ are the twistors $Z_f : \partial f \rightarrow \mathbb{T} \simeq \mathbb{C}^4$ parametrizing the $SL(2,\mathbb{C})$ holonomy-flux variables.
- ζ_f, λ_f and z_f are Lagrange multipliers imposing the area-matching constraint and simplicity constraints respectively.
- $\mathcal{A}$ is the $SU(2)_n$ Ashtekar–Barbero connection along the edges of the discretization.
- n denotes the time normal of the elementary tetrahedra.
- p_e is the volume-weighted time-normal, of the tetrahedron dual to the edge e.
- $\text{Vol}(e)$ denotes the corresponding three-volume.
- N is a Lagrange multiplier imposing the mass-shell condition $C = 0$.

- A dictionary:

spinfoam formalism	–	auxiliary particles
tetrahedra	–	particles
four-simplices	–	interaction vertices
three-volume	–	mass
tetrahedral shapes	–	internal $SU(2)$ DOF
torsion=0	–	conservation of four-momentum

Immediate question

Is this a reasonable model for discretized gravity?

1. The Equations of motion generate twisted geometries: Every triangle has a unique area, but the shape of a triangle depends on whether we compute it from the metric in one adjacent four-simplex or the other.
2. Relation to Regge calculus: We can restrict ourselves to Regge-like solutions.
3. The model has curvature: There is a deficit angle once we go around a triangle.

Regge solutions

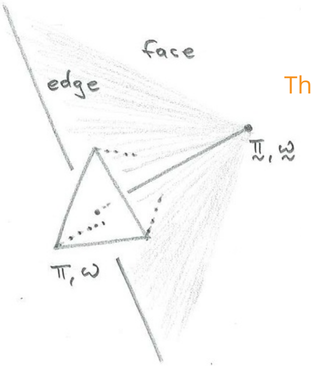
Hamiltonian formulation

The Hamiltonian:

$$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + z^f W_{ef} + \bar{z}^f \bar{W}_{ef} + \lambda^f V_f \right) + NC, \quad (22)$$

generates the t-evolution along the edges of the discretization:

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_f^A = \{H, \omega_f^A\}. \quad (23)$$



The fundamental Poisson brackets are:

$$\{p_\alpha, X^\beta\} = \delta_\alpha^\beta,$$
$$\{\pi_A^f, \omega_{f'}^B\} = +\delta_{ff'}\delta_A^B,$$
$$\{\underset{\sim}{\pi}_A^f, \underset{\sim}{\omega}_{f'}^B\} = -\delta_{ff'}\delta_A^B.$$

Dirac anaysis and reduced Hamiltonian

$$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + z^f W_{ef} + \bar{z}^f \bar{W}_{ef} + \lambda^f V_f \right) + NC. \quad (24)$$

- The Hamiltonian preserves all constraints for $z_f = 0$.
- The W_{ef} simplicity constraint is second class, all other constraints first class.
- There are no secondary constraints.
- $\zeta_f = 0$ wlog.

$$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \lambda^f V_f + NC. \quad (25)$$

Relevant constraints of the system:

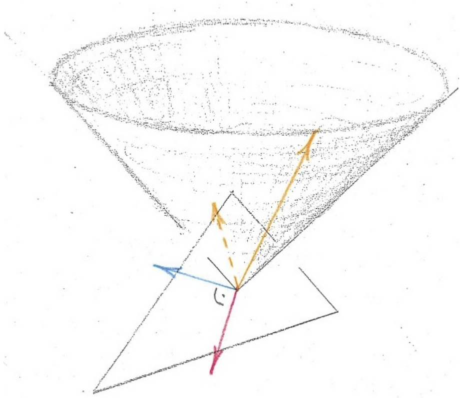
- mass-shell: $C = \frac{1}{2}\left(g^{-1} p_\alpha p^\alpha + g\,\mathrm{Vol}^2\right)$
- first-class simplicity: $V = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi_A \omega^A + \mathrm{cc}.$
- Gauß constraint: $G_\alpha = -\sum_{f:\partial f \subset e} L^f_\alpha$ generates $SU(2)_n$ transformations, with $L^f_\alpha = -\tau^{AB}{}_\alpha \pi^f_A \omega^f_B + \mathrm{cc}.$, and $\mathfrak{su}(2)_n$ generators $\tau_\alpha : [\tau_\alpha, \tau_\beta] = \epsilon_{\alpha\beta}{}^\mu \tau_\nu$.

Action of the first-class simplicity constraints

The first-class simplicity constraints $V = 0$ generates a four-screw:

$$\{V, \omega^A\} = \frac{\mathrm{i}}{\beta + \mathrm{i}}\omega^A, \quad \{V, \pi^A\} = -\frac{\mathrm{i}}{\beta + \mathrm{i}}\pi^A.$$

A combination of a rotation and a boost preserving the triangle's plane:



*F Hellmann and W Kamiński, Holonomy spin foam models: Asymptotic geometry of the partition function , JHEP 1310 (2013), arXiv:1307.1679.
*V Bonzom, Spin foam models for quantum gravity from lattice path integrals, Phys. Rev D. 80 (2009), arXiv:0905.1501.

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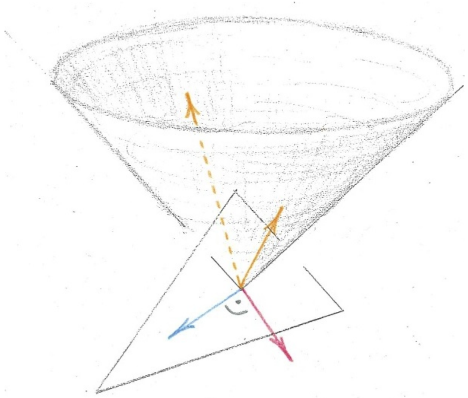

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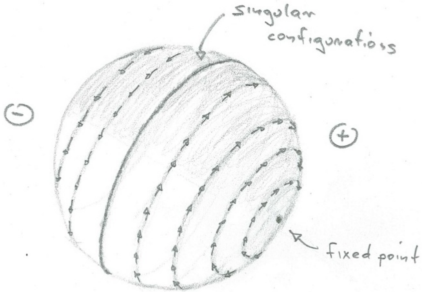
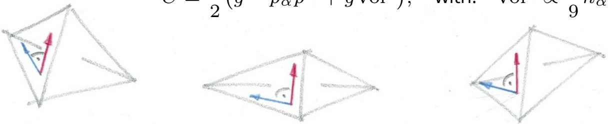
Action of the mass-shell condition

Volume flow in the shape space of all tetrahedra with fixed areas:



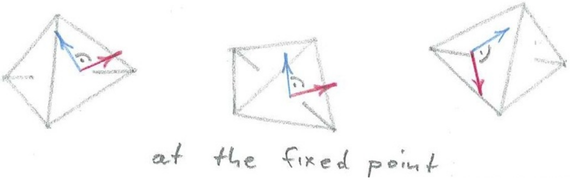
- The simplicity constraints impose that the $\Sigma_{\alpha\beta}$ fluxes define planes in internal Minkowski space.
- The Gauß constraint G_α tells us that these planes close to form a tetrahedron.
- The mass-shell condition generates a globale rotation plus a shear plus a $U(1)$ rotation in the plane of every triangle.

$$C = \frac{1}{2}\big(g^{-1}p_\alpha p^\alpha + g\mathrm{Vol}^2\big), \quad \text{with:} \quad \mathrm{Vol}^2 \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu .$$



*E Bianchi and HM Haggard, Bohr-Sommerfeld Quantization of Space, Phys.Rev. D 86 (2012), `arXiv:1208.2228`.

- We restrict ourselves to tetrahedra that are stable under the volume flow:

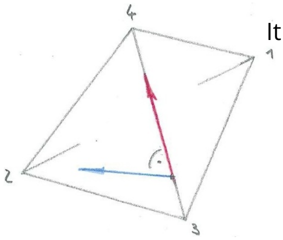


- The mass shell condition generates residual $U(1)$ transformation of the spinors in the triangles:

$$\pi^A \to e^{-\frac{i\varphi}{2}}\pi^A, \quad \omega^A \to e^{+\frac{i\varphi}{2}}\omega^A. \tag{26}$$

- We can tune these angles in such a way that they cancel the unwanted rotation from the simplicity constraints.
- We are then left with a pure boost.

Gauge fixing



It works as follows (for e.g. the spinors in the 1-plane):

- We align the dyade in the 1-plane with the (34)-edge,
- and require that the Hamiltonian preserves this additional condition.

$$H = \mathcal{A}^{\alpha} G_{\alpha} + \sum_{f=1}^{4} \lambda^{f} V_{f} + NC. \tag{27}$$

- This fixes the Lagrange multiplier λ^f in terms of N:

$$\lambda^{1}(N) = \frac{g{\ell_{\mathrm{P}}}^{2}}{4}(1+\beta^{2})\frac{\mathrm{Vol}^{2}}{\sin^{2}\vartheta_{12}}\frac{\mathrm{Ar}_{2}+\mathrm{Ar}_{1}\cos\vartheta_{12}}{\mathrm{Ar}_{1}\,\mathrm{Ar}_{2}}N \tag{28}$$

- There are three possible choices to align the spinors to an edge. The result (28) is independent of this ambiguity.

Remark:
ϑ_{IJ} is the dihedral angle, Vol denotes the volume and Ar_I is the area of the I-th triangle.

Effective equations of motion

Our alignment brings the evolution equations into a simple form:

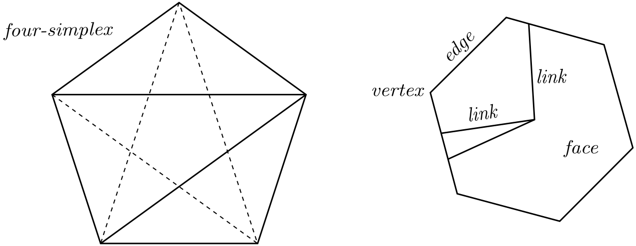
$$\frac{\nabla}{\mathrm{d}t}\omega_f^A = \frac{\mathrm{d}}{\mathrm{d}t}\omega_f^A + \Gamma^A{}_B\omega_f^B = +\frac{\xi_f}{2}\omega_f^A, \quad (29a)$$

$$\frac{\nabla}{\mathrm{d}t}\pi_f^A = \frac{\mathrm{d}}{\mathrm{d}t}\pi_f^A + \Gamma^A{}_B\pi_f^B = -\frac{\xi_f}{2}\pi_f^A, \quad (29b)$$

Geometric interpretation:

- $\Gamma \in \mathfrak{su}(2)_n$ is the Levi-Civita connection along the edges.
- $\xi_f \in \mathbb{R}$ measures the extrinsic curvature in the face f.
- The parallel transport around the face is a pure boost:

$$[h_f]^A{}_B = \left[\mathrm{Pexp}\left(-\oint_{\partial f}\mathrm{d}t\,\Gamma\right)\right]^A{}_B = \\ = \frac{1}{\pi_C\omega^C}\left[\mathrm{e}^{-\frac{1}{2}\oint_{\partial f}\mathrm{d}t\xi}\omega^A\pi_B + \mathrm{e}^{+\frac{1}{2}\oint_{\partial f}\mathrm{d}t\xi}\pi^A\omega_B\right] \quad (30)$$



Inter-tetrahedral angles:

$$\cosh \Xi_{vf} = -\eta^{\mu\nu} n^{e}_{\mu} n^{e'}_{\nu}, \quad \text{with: } e \cap e' = v, \text{ and: } e,\, e' \subset \partial f. \tag{31}$$

Deficit angle around a triangle:

$$\Xi_f := \sum_{v\text{: vertices in } f} \Xi_{vf} = \oint_{\partial f} \mathrm{d}t \xi. \tag{32}$$

The on-shell action is the Regge action

We evaluate the on-shell action for volume-stable geometries.

- The action consists of a symplectic potential plus constraints.
- The constraints vanish on-shell.
- The symplectic potential $p_\alpha \mathrm{d}X^\alpha$ does not contribute either, because of:

$$\int_e p_\alpha \mathrm{d}X^\alpha \overset{\text{EOM}}{=} p_\alpha \left[X^\alpha\big|_{e(1)} - X^\alpha\big|_{e(0)} \right], \quad \text{and} \quad \sum_{\substack{\text{edges } e \\ \text{at } v}} \pm_e p^e_\alpha = 0 \tag{33}$$

- All contributions come from the symplectic potential for the spinors.

Regge action at the fixed point of the volume flow

$$S_{\text{spinfoam}}[\underline{\pi}, \underline{\omega}, \underline{p}, \underline{X}] \overset{\text{EOM}}{=} \sum_{f:\text{faces}} \oint_{\partial f} \pi^f_A \mathrm{d}\omega^A_f + \text{cc.} \overset{\text{EOM}}{=} \frac{1}{2} \sum_{f:\text{faces}} \pi^f_A \omega^A_f \Xi_f + \text{cc.} =$$

$$\overset{\text{EOM}}{=} \frac{1}{2} \sum_{f:\text{faces}} \frac{(\beta + \mathrm{i})}{\beta {\ell_{\mathrm{P}}}^2} \mathrm{Ar}_f \, \Xi_f + \text{cc.} = \frac{1}{8\pi G} \sum_{f:\text{faces}} \mathrm{Ar}_f \, \Xi_f = S_{\text{Regge}}\big[\{\ell_b\}_{b:\text{bones}}\big].$$

Solutions of the equations of motion extremize the spinfoam action, hence they also bring the Regge action to an extremum.

Conclusion

Summary

General picture: The simplicial edges turn into the worldlines of a system of auxiliary particles scattering in a flat auxiliary manifold. Every tetrahedron carries a conserved four-momentum. Its norm is not mass but volume.

Results:

- Generic solutions represent twisted geometries that are the boundary data of loop quantum gravity.
- Regge configurations appear at the fixed point of the volume flow, where the on-shell action S_{spinfoam} turns into the Regge action S_{Regge}.

Open issues of the model

- Role of local Lorentz invariance: The addition of the $p_\alpha \mathrm{d}X^\alpha$ term breaks local $SL(2,\mathbb{C})$ invariance down to the little group $SU(2)_n$.
- What is the physical role of the X^α-background geometry with flat Minkowski metric $\eta_{\alpha\beta}$? Is there a relation to teleparallelism?
- We have only shown local existence of Regge-like solutions. Open problem: Find explicit solutions that triangulate physical (Ricci flat) spacetime geometries.

Relevance for loop quantum gravity

- Quantum kinematics (simple problem): The instantaneous Hilbert space is the Hilbert space of projected spin network functions.

  Area $\leftrightarrow$ norm of the Pauli–Lubanski vector. Quantization of area $\leftrightarrow$ quantization of spin in the auxiliary particle model.

- Quantum dynamics (hard problem): Take the spinfoam action and reformulate loop quantum gravity as a one-dimensional QFT over the edges of the discretization. The spinfoam amplitudes turn into the S-matrix amplitudes of an auxiliary worldline model.

*S Alexandrov and ER Livine , SU(2) loop quantum gravity seen from covariant theory , Phys. Rev. D 67 (2003), arXiv:gr-qc/0209105.

Thank you for your attention