

# 2+1 dimensional Loop Quantum Cosmology

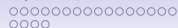
Xiangdong Zhang

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Feb 18, 2015

This talk is based on

X. Zhang, *Loop quantum cosmology in 2+1 dimension*, Phys. Rev. **D 90** 124018 (2014).



# Outline

Motivations

Connection dynamics and symmetric reduction

Loop Quantum Cosmology in 2+1 dimensions  
Cosmological difference-differential equation  
Effective equations

Conclusions

Outlook

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- Nowadays all the researches of loop quantum cosmology are focus on 3+1 dimensions, is it possible to generalize it to the spacetime dimensions other than four?

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- Why 2+1 dimensions?
- From general relativity perspective: it is simple but not trivial, BTZ solution
- From loop quantum gravity side, 2+1 dimensional LQG is well established since 1989[Ashtekar et. al1989 ]

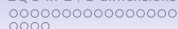
# Connection Dynamics

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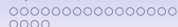
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- The difference is that the Lorentzian theory has a more difficult Hamiltonian constraint, while the Euclidean theory admits simpler ones.
- The Ashtekar formalism of 2+1 dimensional gravity constitutes a  $SU(2)$  connections  $A_a^i$  and a densitized dyad  $E_i^a = \epsilon^{ab} e_{bi}$  defined on an oriented two dimensional manifold  $S$

# Connection Dynamics

- The commutation relation for the canonical conjugate pairs satisfies

$$\{A_i^a(x), E_b^j(y)\} = \kappa \gamma \delta_i^j \delta_b^a \delta(x, y) \quad (1)$$

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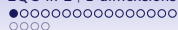
here  $\kappa = 8\pi G$ .

- The 2+1 dimensional gravity also has three constraints similar with 3+1 dimensional general relativity

$$G_i = D_a E_i^a \quad (2)$$

$$V_a = \frac{1}{\kappa\gamma} F_{ab}^i E_i^b \quad (3)$$

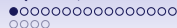
$$H_{gr} = \frac{\epsilon^{ijk} E_i^a E_j^b}{2\kappa\sqrt{h}} F_{ab}^k - 2(\gamma^2 + 1) \frac{E_{[i}^a E_{j]}^b}{2\kappa\sqrt{h}} K_a^j K_b^i \quad (4)$$



# Classical 2+1 dimensional Cosmology

- line element of 2+1 dimensional Friedman-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\theta^2 \right)$$



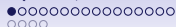
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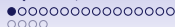
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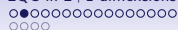
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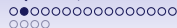
- As in 3+1 dimensional case, we introduce a massless scalar field  $\phi$  as our matter field.





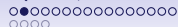
## Symmetric Reduction

- We first introduce an “elemental cell”  $\mathcal{V}$  on the homogeneous spatial manifold  $\mathbb{R}^2$  and restrict all integrals to this elemental cell.



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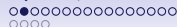
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- Chose fiducial Euclidean metric  ${}^o q_{ab}$  on  $\mathbb{R}^2$  as well as the orthonormal dyad and co-dyad  $({}^o e_i^a; {}^o \omega_a^i)$ , such that  ${}^o q_{ab} = {}^o \omega_a^i {}^o \omega_b^i$ .



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- The reduced connection and densitized dyad as

$$A_a^i = cV_0^{-\frac{1}{2}} {}^o \omega_a^i, \quad E_j^b = pV_0^{-\frac{1}{2}} \sqrt{\det({}^o q)} {}^o e_j^b,$$



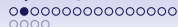
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- Poisson bracket

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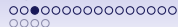
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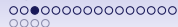
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- Note that the new variables are related to the old ones by  $|p| = aV_0^{\frac{1}{2}}$  and  $c = \gamma\dot{a}V_0^{\frac{1}{2}}$ .

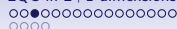


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- The evolution equation of the  $p$ ,

$$\dot{p} = \{p, H_T\} = \frac{1}{\gamma} c \quad (9)$$







- By using the 2+1 dimensional FRW line element, the three dimensional scalar curvature reads

$$R = 2 \left( \frac{2\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{2\dot{N}\dot{a}}{N^3 a} \right) \quad (11)$$

- Thus the action becomes

$$\begin{aligned} S &= \frac{V_0}{8\pi G} \int dt N a^2 \left( \frac{2\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{2\dot{N}\dot{a}}{N^3 a} \right) \\ &= -\frac{V_0}{8\pi G} \int dt \frac{\dot{a}^2}{N} \end{aligned} \quad (12)$$

Here we use the fact that  $\sqrt{-g} = Na^2 r$  and the coordinate volume  $V_0 = \int dr d\theta r$ . From now on, we will assume  $V_0 = 1$  by a rescaling. In addition, we also fix the lapse function  $N = 1$ .

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- The Hamiltonian of the gravitational part is

$$H_{grav} = \dot{a}p_a - L_{grav} = -\frac{\kappa}{4}p_a^2 \quad (16)$$



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- The gravitational part of the Hamiltonian in terms of these new variables becomes

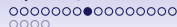
$$H_{grav} = -\frac{1}{\kappa\gamma^2} c^2 \quad (18)$$

Which coincides with the Hamiltonian we obtained from symmetric reduction.



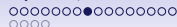
# Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

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- In 2+1 dimension, A deformed formula will help us to resolve this problem. The key point is introducing a quantity called the degenerate vector [Thiemann, 1998]

$$E^i = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}E_j^aE_k^b \quad (19)$$

which can be expressed via Thiemann trick as

$$E^i = \frac{1}{2(\kappa\gamma)^2}\epsilon_{ijk}\epsilon^{ab}\{A_a^j, V\}\{A_b^k, V\} \quad (20)$$



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- On the other hand the  $F_{ab}^i$  can be expressed by

$$F_{ab}^i = -2 \lim_{Ar_{\square} \rightarrow 0} \text{Tr} \left( \frac{h_{\square}^{\lambda} - 1}{\lambda^2} {}^0\omega_a^{j0} \omega_b^k \tau^i \right) \quad (21)$$

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- With these two basic ingredients in hand, the original Hamiltonian constraint can be rewritten in a compact form

$$\begin{aligned} H_{\text{gr}} &= -\frac{1}{\kappa \gamma^2} \frac{F_i E^i}{\sqrt{h}} = \frac{2}{\kappa^3 \gamma^4 \lambda^4} \epsilon^{ij} \epsilon^{kl} \text{Tr} \left( h_{\square}^{\lambda}{}_{ij} h_k \{h_k^{-1}, \sqrt{V}\} h_l \{h_l^{-1}, \sqrt{V}\} \right) \\ &= \frac{2 \sin^2(\lambda c)}{\kappa^3 \gamma^4 \lambda^4} \epsilon^{ij} \epsilon^{kl} \epsilon_{ij}^m \text{Tr} \left( \tau_m h_k \{h_k^{-1}, \sqrt{V}\} h_l \{h_l^{-1}, \sqrt{V}\} \right) \end{aligned} \quad (22)$$



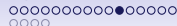
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- We adopt the polymer-like quantization for connection  $c$ . The kinematical Hilbert space for the geometry part can be defined as  $\mathcal{H}_{\text{kin}}^{\text{gr}} := L^2(R_{\text{Bohr}}, d\mu_H)$ , where  $R_{\text{Bohr}}$  and  $d\mu_H$  are respectively the Bohr compactification of the real line and Haar measure on it.
- We choose Schrodinger representation for the scalar field. Thus the kinematical Hilbert space for the scalar field part is defined as in usual quantum mechanics,  $\mathcal{H}_{\text{kin}}^{\text{sc}} := L^2(R, d\mu)$ .
- The whole Hilbert space of 2+1 dimensional loop quantum cosmology is a direct product,  $\mathcal{H}_{\text{kin}}^{2+1} = \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\text{sc}}$ .
- Now let  $|\mu\rangle$  be the eigenstates of  $\hat{p}$  in the kinematical Hilbert space  $\mathcal{H}_{\text{kin}}^{\text{gr}}$  such that

$$\hat{p}|\mu\rangle = 2\pi G\gamma\hbar\mu|\mu\rangle = \frac{\hbar\kappa\gamma}{4}\mu|\mu\rangle.$$

Then these eigenstates satisfy orthonormal condition

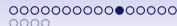
$$\langle\mu_i|\mu_j\rangle = \delta_{\mu_i,\mu_j}, \quad (23)$$



# Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

- In 2+1 dimensional quantum gravity, the spectrum of length operator is quantized[Livine,Freidel,Rovelli,2003]

$$\text{Spectrum}\{L\} = \kappa\hbar \sum_j \sqrt{j(j+1)} \quad (24)$$



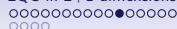
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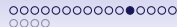
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- We should shrink the loop  $\square_{ij}$  till the edge of the loop, which is measured by the physical metric  $q_{ab}$ , reaches the value of minimal length  $L$ . Since the physical length of the elementary cell is  $|p|$  and each side of  $\square_{ij}$  is  $\lambda$  times the edge of the elementary cell, we use a specific function  $\bar{\nu}(p)$  to denote  $\lambda$ , we have

$$\bar{\nu}(p)|p| = L \equiv 4\sqrt{3}\pi \ell_P \quad (26)$$

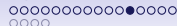


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- For the convenience of studying quantum dynamics, we define new variables

$$v := \frac{\sqrt{3}}{\gamma} \text{sgn}(p) \bar{\nu}^{-2}, \quad b := \bar{\nu} c,$$

where  $\bar{\nu} = \frac{L}{|p|}$  with  $L = 4\sqrt{3}\pi\ell_p$  being a minimum nonzero eigenvalue of the length operator



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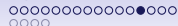
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where  $\bar{\nu} = \frac{L}{|p|}$  with  $L = 4\sqrt{3}\pi\ell_p$  being a minimum nonzero eigenvalue of the length operator

- Commutation relation between new variables reads

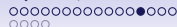
$$\{b, v\} = \frac{2}{\hbar}.$$



# Quantum Kinematic of 2+1 dimensional Loop Quantum Cosmology

- $|v\rangle$  constitutes an orthonormal basis in  $\mathcal{H}_{\text{kin}}^{\text{gr}}$ . Such that

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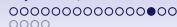
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- In  $(b, v)$  representation, the Hamiltonian constraint can be written as

$$\begin{aligned} H &= -\frac{1}{\kappa\gamma^2}c^2 + \frac{p_\phi^2}{2p^2} \\ &= -\frac{1}{\sqrt{3}\kappa\gamma}b^2|v| + \left(\frac{\sqrt{3}}{\gamma L^2}\right)\frac{p_\phi^2}{2|v|} \end{aligned} \quad (27)$$



# Quantum Dynamics of 2+1 dimensional Loop Quantum Cosmology

- At the quantum level, we use the commutator to replace the Poisson bracket to get the exact expression of Hamiltonian constraint

$$\begin{aligned}\hat{H}_{\text{gr}} &= \frac{4}{\kappa^3 \gamma^4 \bar{V}^4 \hbar^2} \sin^2(\bar{\nu}c) \left( \sin\left(\frac{\bar{\nu}c}{2}\right) \sqrt{V} \cos\left(\frac{\bar{\nu}c}{2}\right) - \cos\left(\frac{\bar{\nu}c}{2}\right) \sqrt{V} \sin\left(\frac{\bar{\nu}c}{2}\right) \right)^2 \\ &= \sin(\bar{\nu}c) \hat{F} \sin(\bar{\nu}c)\end{aligned}\quad (28)$$

- Hence its action on a quantum state  $\Psi(\nu, \phi) \in \mathcal{H}_{\text{kin}}^{2+1}$  is

$$\hat{H}_{\text{gr}} \Psi(\nu) = f_+(\nu) \Psi(\nu + 4) + f_0(\nu) \Psi(\nu) + f_-(\nu) \Psi(\nu - 4) \quad (29)$$

where

$$\begin{aligned}f_+(\nu) &= -\frac{1}{4} F(\nu + 2) = \frac{1}{8\sqrt{3}\kappa\gamma} (|\nu + 2|)^2 \left( \sqrt{|\nu + 1|} - \sqrt{|\nu + 3|} \right)^2 \\ f_0(\nu) &= \frac{1}{4} F(\nu + 2) + \frac{1}{4} F(\nu - 2) \\ f_-(\nu) &= -\frac{1}{4} F(\nu - 2)\end{aligned}\quad (30)$$

- Now we turn to the inverse volume operator. As such, we first have the following classical identity

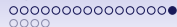
$$|\rho|^{-1/2} = \text{sgn}(\rho) \frac{8}{3\kappa\gamma\bar{\nu}} \text{Tr} \left( \sum_j \tau^j h_j \{h_j^{-1}, V^{1/4}\} \right) \quad (31)$$

- Now we turn to the inverse volume operator. As such, we first have the following classical identity

$$|p|^{-1/2} = \text{sgn}(p) \frac{8}{3\kappa\gamma\bar{v}} \text{Tr} \left( \sum_j \tau^j h_j \{h_j^{-1}, V^{1/4}\} \right) \quad (31)$$

- Note that  $V^{-1} = |p|^{-2}$  and we takes the replacement  $\{, \} \rightarrow \frac{1}{i\hbar} [, ]$ , to get the quantum version of the operator, then we can yield the action of inverse volume operator on a quantum state  $\Psi(v)$  reads

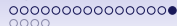
$$\begin{aligned} \widehat{V}^{-1}\Psi(v) &= \left( \frac{4}{\kappa\gamma L\hbar} \right)^4 \left( \frac{\gamma L^2}{\sqrt{3}} \right)^3 v^2 \left| |v+1|^{\frac{1}{4}} - |v-1|^{\frac{1}{4}} \right|^4 \Psi(v) \\ &= \left( \frac{16\sqrt{3}}{\gamma L^2} \right) v^2 \left| |v+1|^{\frac{1}{4}} - |v-1|^{\frac{1}{4}} \right|^4 \Psi(v) \\ &:= B(v)\Psi(v) \end{aligned} \quad (32)$$



- Thus, the Hamiltonian constraint has been successfully quantized in the cosmological model. The Hamiltonian constraint equation of 2+1 dimensional loop quantum cosmology reads

$$\hat{H}_{\text{gr}}\Psi(\phi, \nu) = B(\nu)\frac{\hbar^2}{2}\frac{\partial^2\Psi(\phi, \nu)}{\partial\phi^2}. \quad (33)$$

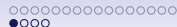
- Based on this Klein-Gordon like equation, we can show that the matter density has an upper bound, namely  $\rho \leq \frac{1}{\kappa\gamma^2 L^2} = \rho_c$  which implies a singularity resolution.



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- Interestingly, the upper-bound of the matter density we obtained here coincides with critical matter density from the effective Friedmann equation.



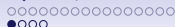
# Effective equations of 2+1 dimensional loop quantum cosmology

- By assuming  $\mathcal{O}(\frac{1}{|v|}) \ll 1$ , and using timeless path integral formalism, we can get the effective equation of 2+1 dimensional Loop quantum cosmology

$$H_F = -\frac{1}{\sqrt{3}\kappa\gamma} |v| \sin^2 b + \frac{\gamma L^2 |v|}{\sqrt{3}} \rho,$$

where the matter density is defined by

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- Note that the above effective Hamiltonian can also be obtained from the classical Hamiltonian by the **heuristic replacement**  $b \rightarrow \sin b$  or  $c \rightarrow \frac{\sin(\bar{v}c)}{\bar{v}}$ . Hence the classical Hamiltonian constraint can be recovered from the effective  $H_F$  in the large scale limit.





- Equations of motion for  $v$  and  $\phi$  are given respectively as

$$\dot{v} = \{v, H_F\} = \frac{4}{\sqrt{3}\hbar\gamma\kappa} |v| \sin(b) \cos(b), \quad (35)$$

$$\dot{\phi} = \frac{\sqrt{3}p_\phi}{\gamma L^2 |v|}. \quad (36)$$

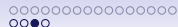


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- The evolution of  $\dot{p}_\phi = 0$ . Thus  $\phi$  can be viewed as an emergent time variable.



- Effective Friedman equation reads

$$H^2 = \kappa\rho \left(1 - \frac{\rho}{\rho_c}\right) \quad (37)$$

where the critical matter density is  $\rho_c = \frac{1}{\kappa\gamma^2 L^2} = \frac{4}{3\kappa^3\gamma^2\hbar^2}$ .



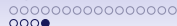
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- Moreover combining Eq. (37) with the continuity equation in 2+1 dimension,  $\dot{\rho} + 2H(\rho + p) = 0$ , we can obtain 2+1 dimensional Raychaudhuri equation with loop quantum correction

$$\frac{\ddot{a}}{a} = \frac{\ddot{v}}{2v} - H^2 = \kappa\rho \left(1 - \frac{\rho}{\rho_c}\right) - \kappa(\rho + p) \left(1 - \frac{2\rho}{\rho_c}\right) \quad (38)$$



- Comparison between 2+1 dimensional LQC and 3+1 dimensional LQC.

TABLE I. Some differences between (2 + 1)-dimensional LQC and (3 + 1)-dimensional LQC.

	2 + 1 dimension	3 + 1 dimension
Thiemann trick	$E^i = \frac{1}{2(\kappa\gamma)^2} \epsilon_{ijk} \epsilon^{ab} \{A_a^j, V\} \{A_b^k, V\}$	$\frac{1}{2} \epsilon^{ijk} \frac{\epsilon_{abc} E_a^b E_c^k}{\sqrt{q}} = \frac{1}{\kappa\gamma} \{A_a^i, V\}$
Geometric variable $p$	$a$	$a^2$
Representation	$\tilde{v} = \frac{L}{p}$	$\tilde{\mu} = \sqrt{\frac{\Delta}{p}}$
Poisson bracket between basic variables	$\{c, p\} = \frac{\kappa\gamma}{2}$	$\{c, p\} = \frac{\kappa\gamma}{3}$
Inverse volume $V^{-1}$	$(\frac{8}{3\kappa\gamma\tilde{v}})^4 \text{Tr}(\sum_j \tau^j h_j \{h_j^{-1}, V^{1/4}\})^4$	$(\frac{4}{\kappa\gamma\tilde{\mu}})^3 \text{Tr}(\sum_j \tau^j h_j \{h_j^{-1}, V^{1/3}\})^3$
Heuristic replacement	$c \rightarrow \frac{\sin(\tilde{\nu}c)}{\tilde{v}}$	$c \rightarrow \frac{\sin(\tilde{\mu}c)}{\tilde{\mu}}$

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- Extend LQC quantization scheme to 2+1 dimensional theory, effective equations which include a quantum bounce is obtained.
- **Singularity resolution** of 2+1 dimensional LQC indicates that the existence of quantum bounce might be a **universal** feature of loop quantum cosmological models in arbitrary spacetime dimensions.

# Outlook

- Link LQC with LQG in 2+1 dimensions?





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- LQC with arbitrary spacetime dimensions. (work in progress)



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- Link LQC with LQG in 2+1 dimensions?
- LQC with arbitrary spacetime dimensions. (work in progress)
- Many other issues.....

Thank you very much!