2+1 dimensional Loop Quantum Cosmology

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Feb 18, 2015

This talk is based on

X. Zhang, *Loop quantum cosmology in 2+1 dimension*, Phys. Rev. **D 90** 124018 (2014).

Outline

Motivations

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- Why 2+1 dimensions?
- From general relativity perspective: it is simple but not trivial, BTZ solution
- From loop quantum gravity side, 2+1 dimensional LQG is well established since 1989[Ashtekar et. al1989]

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- The difference is that the Lorentzian theory has a more difficult Hamiltonian constraint, while the Euclidean theory admits simpler ones.
- The Ashtekar formalism of 2+1 dimensional gravity constitutes a $SU(2)$ connections A_a^i and a densitized dyad $E_i^a = \epsilon^{ab} e_{bi}$ defined on an oriented two dimensional manifold S

Connection Dynamics

- The commutation relation for the canonical conjugate pairs satisfies

$$\{A_i^a(x), E_b^j(y)\} = \kappa\gamma\delta_i^j\delta_b^a\delta(x,y) \tag{1}$$

here $\kappa = 8\pi G$.

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- The 2+1 dimensional gravity also has three constraints similar with 3+1 dimensional general relativity

$$G_i = D_a E_i^a \tag{2}$$

$$V_a = \frac{1}{\kappa\gamma} F_{ab}^i E_i^b \tag{3}$$

$$H_{gr} = \frac{\epsilon^{ijk} E_i^a E_j^b}{2\kappa\sqrt{h}} F_{ab}^k - 2(\gamma^2+1)\frac{E_{[i}^a E_{j]}^b}{2\kappa\sqrt{h}} K_a^i K_b^j \tag{4}$$

Classical 2+1 dimensional Cosmology

- line element of 2+1 dimensional Friedman-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2\right)$$

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- As in 3+1 dimensional case, we introduce a massless scalar field ϕ as our matter field.

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- The reduced connection and densitized dyad as

$$A^i_a = cV_0^{-\frac{1}{2}} \; {}^o\omega^i_a, \qquad E^b_j = pV_0^{-\frac{1}{2}} \sqrt{\det({}^0 q)} \; {}^o e^b_j,$$

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- Poisson bracket

$$\{A_a^j(x), E_k^b(y)\} \quad = \quad \kappa\gamma\delta_a^b\delta_k^j\delta(x,y) \Rightarrow \{c,p\} = \frac{\kappa}{2}\gamma.$$

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- Note that the new variables are related to the old ones by $|p| = aV_0^{\frac{1}{2}}$ and $c = \gamma\dot{a}V_0^{\frac{1}{2}}$.

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- The evolution equation of the p,

$$\dot{p} = \{p, H_T\} = \frac{1}{\gamma}c \tag{9}$$

- By using the Hamiltonian constraint we can easily yield 2+1 dimensional Friedman equation

$$H^2 = \left(\frac{\dot{p}}{p}\right)^2 = \frac{1}{\gamma^2 p^2} c^2 = \kappa \frac{p_\phi^2}{2p^4} = \kappa\rho \tag{10}$$

where the $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the matter density in 2+1 dimension by definition is $\rho = \frac{p_\phi^2}{2p^4}$.

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- By using the 2+1 dimensional FRW line element, the three dimensional scalar curvature reads

$$R = 2\left(\frac{2\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{2\dot{N}\dot{a}}{N^3 a}\right) \tag{11}$$

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- Thus the action becomes

$$\begin{aligned} S &= \frac{V_0}{8\pi G}\int dt N a^2 \left(\frac{2\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{2\dot{N}\dot{a}}{N^3 a}\right) \\ &= -\frac{V_0}{8\pi G}\int dt \frac{\dot{a}^2}{N} \end{aligned} \tag{12}$$

Here we use the fact that $\sqrt{-g} = Na^2 r$ and the coordinate volume $V_0 = \int drd\theta r$. From now on, we will assume $V_0 = 1$ by a rescaling. In addition, we also fix the lapse function $N = 1$.

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- The Hamiltonian of the gravitational part is

$$H_{grav} = \dot{a}p_a - L_{grav} = -\frac{\kappa}{4}p_a^2 \tag{16}$$

- Introduce a new set of canonical variables $p = a$ and $c = \gamma \dot{a} = -\frac{\kappa\gamma}{2} p_a$.

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- The gravitational part of the Hamiltonian in terms of these new variables becomes

$$H_{grav} = -\frac{1}{\kappa\gamma^2}c^2 \tag{18}$$

Which is coincides with the Hamiltonian we obtained from symmetric reduction.

Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

- In order to implement the Hamiltonian constraint at quantum level, we need to rewrite the Hamiltonian constraint with a suitable manner. This is essentially because that the classical Hamiltonian constraint involves inverse of the determinate of 2-metric and thus can not be promoted as a well defined operator on the kinematic Hilbert space.

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- In 3+1 dimensional LQG this difficulty can be overcome through the well know classical identity $\frac{1}{2}\epsilon^{ijk}\frac{\epsilon_{abc}E^b_j E^c_k}{\sqrt{q}} = \frac{1}{\kappa\gamma}\{A^i_a, V\}$

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- In 2+1 dimension, A deformed formula will help us to resolve this problem. The key point is introducing a quantity called the degenerate vector[Thiemann,1998]

$$E^i = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}E^a_j E^b_k \tag{19}$$

which can be expressed via Thiemann trick as

$$E^i = \frac{1}{2(\kappa\gamma)^2}\epsilon_{ijk}\epsilon^{ab}\{A^j_a, V\}\{A^k_b, V\} \tag{20}$$

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- On the other hand the F^i_{ab} can be expressed by

$$F^i_{ab} \quad = \quad -2 \lim_{Ar_\square \to 0} \mathrm{Tr}\left(\frac{h^{\lambda}_{\square_{jk}} - 1}{\lambda^2} {}^0\omega^j_a {}^0\omega^k_b \tau^i \right) \tag{21}$$

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- With these two basic ingredients in hand, the original Hamiltonian constraint can be rewritten in a compact form

$$\begin{aligned} H_{\mathrm{gr}} &= -\frac{1}{\kappa\gamma^2}\frac{F_i E^i}{\sqrt{h}} = \frac{2}{\kappa^3\gamma^4\lambda^4}\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}\Big(h^\lambda_{\Box_{ij}} h_k\{h_k^{-1},\sqrt{V}\}h_l\{h_l^{-1},\sqrt{V}\}\Big) \\ &= \frac{2\sin^2(\lambda c)}{\kappa^3\gamma^4\lambda^4}\epsilon^{ij}\epsilon^{kl}\epsilon^m_{ij}\mathrm{Tr}\Big(\tau_m h_k\{h_k^{-1},\sqrt{V}\}h_l\{h_l^{-1},\sqrt{V}\}\Big) \end{aligned} \quad (22)$$

Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

- We adopt the polymer-like quantization for connection c. The kinematical Hilbert space for the geometry part can be defined as $\mathcal{H}_{\rm kin}^{\rm gr} := L^2(R_{Bohr}, d\mu_H)$, where R_{Bohr} and $d\mu_H$ are respectively the Bohr compactification of the real line and Haar measure on it.
- We choose Schrodinger representation for the scalar field. Thus the kinematical Hilbert space for the scalar field part is defined as in usual quantum mechanics, $\mathcal{H}_{\rm kin}^{\rm sc} := L^2(R, d\mu)$.
- The whole Hilbert space of 2+1 dimensional loop quantum cosmology is a direct product, $\mathcal{H}_{\rm kin}^{2+1} = \mathcal{H}_{\rm kin}^{\rm gr} \otimes \mathcal{H}_{\rm kin}^{\rm sc}$.
- Now let $|\mu\rangle$ be the eigenstates of $\hat{p}$ in the kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm gr}$ such that

$$\hat{p}|\mu\rangle = 2\pi G\gamma\hbar\mu|\mu\rangle = \frac{\hbar\kappa\gamma}{4}\mu|\mu\rangle.$$

Then these eigenstates satisfy orthonormal condition

$$\langle\mu_i|\mu_j\rangle = \delta_{\mu_i,\mu_j} \ , \tag{23}$$

Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

- In 2+1 dimensional quantum gravity, the spectrum of length operator is quantized[Livine,Freidel,Rovelli,2003]

$$Spectrum\{L\} = \kappa\hbar \sum_j \sqrt{j(j+1)} \tag{24}$$

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- We should shrink the loop $\Box_{ij}$ till the edge of the loop, which is measured by the physical metric q_{ab}, reaches the value of minimal length L. Since the physical length of the elementary cell is $|p|$ and each side of $\Box_{ij}$ is λ times the edge of the elementary cell, we use a specific function $\bar{\nu}(p)$ to denote λ, we have

$$\bar{\nu}(p)|p| = L \equiv 4\sqrt{3}\pi\ell_{\rm p} \tag{26}$$

Quantum Kinematic of 2+1 dimensional loop quantum Cosmology

- For the convenience of studying quantum dynamics, we define new variables

$$v := \frac{\sqrt{3}}{\gamma} sgn(p)\bar{\nu}^{-2}, \quad b := \bar{\nu}c,$$

where $\bar{\nu} = \frac{L}{|p|}$ with $L = 4\sqrt{3}\pi\ell_{\mathrm{p}}$ being a minimum nonzero eigenvalue of the length operator

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where $\bar{\nu} = \frac{L}{|p|}$ with $L = 4\sqrt{3}\pi\ell_{\mathrm{p}}$ being a minimum nonzero eigenvalue of the length operator

- Commutation relation between new variables reads

$$\{b, v\} = \frac{2}{\hbar} \ .$$

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- $|v\rangle$ constitutes an orthonormal basis in $\mathcal{H}^{\mathrm{gr}}_{\mathrm{kin}}$. Such that

$$\langle v_i | v_j \rangle = \delta_{v_i, v_j}$$

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- In (b, v) representation, the Hamiltonian constraint can be written as

$$\begin{aligned} H &= -\frac{1}{\kappa\gamma^2}c^2 + \frac{p_\phi^2}{2p^2} \\ &= -\frac{1}{\sqrt{3}\kappa\gamma}b^2|v| + \left(\frac{\sqrt{3}}{\gamma L^2}\right)\frac{p_\phi^2}{2|v|} \end{aligned} \tag{27}$$

Quantum Dynamics of 2+1 dimensional Loop Quantum Cosmology

- At the quantum level, we use the commutator to replace the Poisson bracket to get the exact expression of Hamiltonian constraint

$$\begin{aligned}\hat{H}_{\rm gr} &= \frac{4}{\kappa^3\gamma^4\bar{\nu}^4\hbar^2}\sin^2(\bar{\nu}c)\left(\sin(\frac{\bar{\nu}c}{2})\sqrt{V}\cos(\frac{\bar{\nu}c}{2})-\cos(\frac{\bar{\nu}c}{2})\sqrt{V}\sin(\frac{\bar{\nu}c}{2})\right)^2\\ &= \sin(\bar{\nu}c)\hat{F}\sin(\bar{\nu}c) \end{aligned} \quad (28)$$

- Hence its action on a quantum state $\Psi(\nu,\phi)\in\mathcal{H}^{2+1}_{\rm kin}$ is

$$\hat{H}_{\rm gr}\Psi(v) = f_+(v)\Psi(v+4)+f_0(v)\Psi(v)+f_-(v)\Psi(v-4) \quad (29)$$

where

$$\begin{aligned} f_+(v) &= -\frac{1}{4}F(v+2)=\frac{1}{8\sqrt{3}\kappa\gamma}(|v+2|)^2\left(\sqrt{|v+1|}-\sqrt{|v+3|}\right)^2\\ f_0(v) &= \frac{1}{4}F(v+2)+\frac{1}{4}F(v-2)\\ f_-(v) &= -\frac{1}{4}F(v-2) \end{aligned} \quad (30)$$

- Now we turn to the inverse volume operator. As such, we first have the following classical identity

$$|p|^{-1/2} = \mathrm{sgn}(p)\frac{8}{3\kappa\gamma\bar{\nu}}\mathrm{Tr}\left(\sum_j \tau^j h_j\{h_j^{-1}, V^{1/4}\}\right) \tag{31}$$

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- Note that $V^{-1} = |p|^{-2}$ and we takes the replacement $\{,\} \to \frac{1}{i\hbar}[,]$, to get the quantum version of the operator, then we can yield the action of inverse volume operator on a quantum state $\Psi(v)$ reads

$$\begin{aligned}\widehat{V^{-1}}\Psi(v) &= \left(\frac{4}{\kappa\gamma L\hbar}\right)^4\left(\frac{\gamma L^2}{\sqrt{3}}\right)^3 v^2\Big||v+1|^{\frac{1}{4}} - |v-1|^{\frac{1}{4}}\Big|^4\Psi(v) \\ &= \left(\frac{16\sqrt{3}}{\gamma L^2}\right) v^2\Big||v+1|^{\frac{1}{4}} - |v-1|^{\frac{1}{4}}\Big|^4\Psi(v) \\ &:= B(v)\Psi(v)\end{aligned} \tag{32}$$

- Thus, the Hamiltonian constraint has been successfully quantized in the cosmological model. The Hamiltonian constraint equation of 2+1 dimensional loop quantum cosmology reads

$$\hat{H}_{\mathrm{gr}}\Psi(\phi, v) = B(v)\frac{\hbar^2}{2}\frac{\partial^2\Psi(\phi, v)}{\partial\phi^2}. \tag{33}$$

- Based on this Klein-Gordon like equation, we can show that the matter density has a upper bound, namely $\rho \leq \frac{1}{\kappa\gamma^2 L^2} = \rho_c$ which implies a singularity resolution.

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- Based on this Klein-Gordon like equation, we can show that the matter density has a upper bound, namely $\rho \leq \frac{1}{\kappa\gamma^2 L^2} = \rho_c$ which implies a singularity resolution.
- Interestingly, the upper-bound of the matter density we obtained here coincides with critical matter density from the effective Friedmann equation.

Effective equations of 2+1 dimensional loop quantum cosmology

- By assuming $\mathcal{O}(\frac{1}{|v|}) \ll 1$, and using timeless path integral formalism, we can get the effective equation of 2+1 dimensional Loop quantum cosmology

$$H_F = -\frac{1}{\sqrt{3}\kappa\gamma}|v|\sin^2 b + \frac{\gamma L^2|v|}{\sqrt{3}}\rho,$$

where the matter density is defined by

$$\rho = \frac{p_\phi^2}{2|p|^4} = \frac{3p_\phi^2}{2v^2\gamma^2 L^4}. \tag{34}$$

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- Note that the above effective Hamiltonian can also be obtained form the classical Hamiltonian by the heuristic replacement $b \to \sin b$ or $c \to \frac{\sin(\bar{\nu}c)}{\bar{\nu}}$. Hence the classical Hamiltonian constraint can be recovered from the effective H_F in the large scale limit.

- Equations of motion for v and ϕ are given respectively as

$$\dot{v} = \{v, H_F\} = \frac{4}{\sqrt{3}\hbar\gamma\kappa}|v|\sin(b)\cos(b), \quad (35)$$

$$\dot{\phi} = \frac{\sqrt{3}p_\phi}{\gamma L^2|v|}. \quad (36)$$

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- The evolution of $\dot{p}_\phi = 0$. Thus ϕ can be viewed as an emergent time variable.

- Effective Friedman equation reads

$$H^2 = \kappa\rho\left(1 - \frac{\rho}{\rho_c}\right) \tag{37}$$

where the critical matter density is $\rho_c = \frac{1}{\kappa\gamma^2 L^2} = \frac{4}{3\kappa^3\gamma^2\hbar^2}$.

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- Moreover combining Eq. (37) with the continuity equation in 2+1 dimension, $\dot{\rho} + 2H(\rho + p) = 0$, we can obtain 2+1 dimensional Raychauduri equation with loop quantum correction

$$\frac{\ddot{a}}{a} = \frac{\ddot{v}}{2v} - H^2 = \kappa\rho\left(1 - \frac{\rho}{\rho_c}\right) - \kappa(\rho + p)\left(1 - \frac{2\rho}{\rho_c}\right) \quad (38)$$

- Comparison between 2+1 dimensional LQC and 3+1 dimensional LQC.

TABLE I. Some differences between (2 + 1)-dimensional LQC and (3 + 1)-dimensional LQC.

	2 + 1 dimension	3 + 1 dimension
Thiemann trick	$E^i = \frac{1}{2(\kappa\gamma)^2}\epsilon_{ijk}\epsilon^{ab}\{A_a^j, V\}\{A_b^k, V\}$	$\frac{1}{2}\epsilon^{ijk}\frac{\epsilon_{abc}E_j^b E_k^c}{\sqrt{q}} = \frac{1}{\kappa\gamma}\{A_a^i, V\}$
Geometric variable p	a	a^2
Representation	$\bar{\nu} = \frac{L}{p}$	$\bar{\mu} = \sqrt{\frac{\Delta}{p}}$
Poisson bracket between basic variables	$\{c, p\} = \frac{\kappa\gamma}{2}$	$\{c, p\} = \frac{\kappa\gamma}{3}$
Inverse volume V^{-1}	$(\frac{8}{3\kappa\gamma\bar{\nu}})^4 \mathrm{Tr}(\sum_j \tau^j h_j\{h_j^{-1}, V^{1/4}\})^4$	$(\frac{4}{\kappa\gamma\bar{\mu}})^3 \mathrm{Tr}(\sum_j \tau^j h_j\{h_j^{-1}, V^{1/3}\})^3$
Heuristic replacement	$c \to \frac{\sin(\bar{\nu}c)}{\bar{\nu}}$	$c \to \frac{\sin(\bar{\mu}c)}{\bar{\mu}}$

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- Extend LQC quantization scheme to 2+1 dimensioanal theory, effective equations which include a quantum bounce is obtained.
- Singularity resolution of 2+1 dimensional LQC indicates that the existence of quantum bounce might be a universal feature of loop quantum cosmological models in arbitrary spacetime dimensions.

Outlook

- Link LQC with LQG in 2+1 dimensions?

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- Many other issues........

Thank you very much!