Quartics, sextics and beyond

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MD, Roger Penrose. Quadratic invariant of binary sextics (2015)
`arXiv:1504.07998` .

- Quadratic $\psi_0 x^2 + 2\psi_1 xy + \psi_2 y^2$
- Cubic, quartic, quintic, ...
- sextic

$$\psi_0 x^6 + 6\psi_1 x^5 y + 15\psi_2 x^4 y^2 + 20\psi_3 x^3 y^3 + 15\psi_4 x^2 y^4 + 6\psi_5 x y^5 + \psi_6 y^6$$

- Roots $\leftrightarrow$ Points on $\mathbb{CP}^1 = S^2$, where $[x, y] \in \mathbb{CP}^1$.

- Linear transformation

$$x = a\tilde{x} + b\tilde{y}, \qquad y = c\tilde{x} + d\tilde{y}, \qquad ad - bc \neq 0.$$

- $GL(2, \mathbb{C}) \subset GL(7, \mathbb{C})$

$$\begin{aligned} \widetilde{\psi}(\tilde{x}, \tilde{y}) &= \psi(x, y) = \psi_0 (a\tilde{x} + b\tilde{y})^6 + \dots \\ &= \tilde{\psi}_0 \tilde{x}^6 + \dots + \tilde{\psi}_6 \tilde{y}^6. \end{aligned}$$

- An invariant of a binary sextic is a polynomial $I = I(\psi_0, \dots, \psi_6)$ s. t.

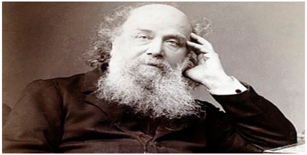
$$I(\tilde{\psi}_0, \dots, \tilde{\psi}_6) = (ad - bc)^w I(\psi_0, \dots, \psi_6).$$

- Quadratic: $\mathcal{I}_2 = {\psi_1}^2 - \psi_0 \psi_2$. $\mathcal{I}_2 = 0 \leftrightarrow$ Repeated roots.
- Sextic: $\mathcal{I}_2, \mathcal{I}_4, \mathcal{I}_6, \mathcal{I}_{10}$ (discriminant), $\mathcal{I}_{15}$

$$\mathcal{I}_2(\boldsymbol{\psi}) = 2\psi_0\psi_6 - 12\psi_1\psi_5 + 30\psi_2\psi_4 - 20\psi_3^2.$$

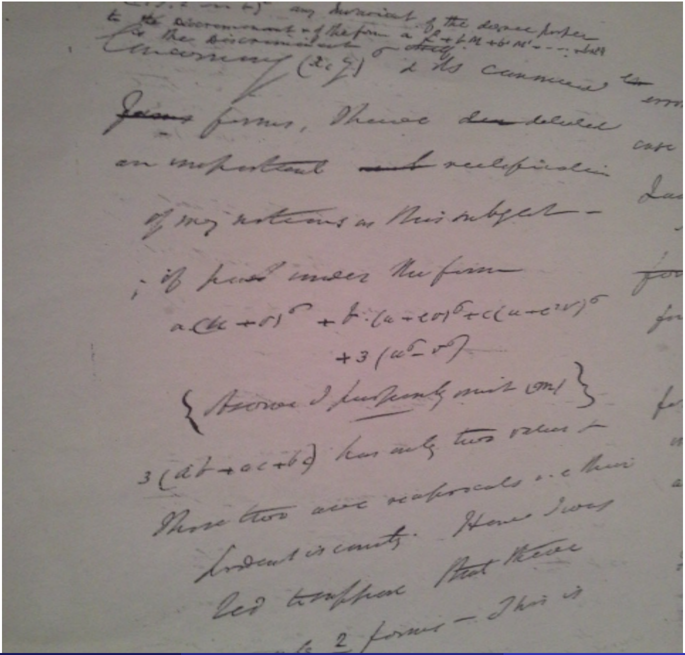
Question: What is special about sextics with $\mathcal{I}_2 = 0$?

Interlude: James Joseph Sylvester (1814-1897)



I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought - brain on fire, and feet feeling, or feelingless, as if plunged in an ice pail. That night we slept no more.

- Junior Wrangler (St John's College, Cambridge 1837).
- Actuary by day, mathematician by night.
- Appointed to the Savilian Chair of Geometry at Oxford in 1883.



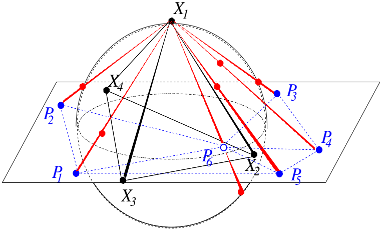
[illegible] any [illegible] of the degree [illegible]
to the discriminant of the form $a\xi + b\xi\eta + b'\eta^2 \ldots$
for the discriminant [illegible]
Concerning (x, y) & its connexion to
~~[illegible]~~ forms, I have deduced
an important ~~[illegible]~~ verification
of my notions on this subject —
; if put under the form

$$a(u+v)^6 + b(u+\varepsilon v)^6 + c(u+\varepsilon^2 v)^6 + 3(u^6 - v^6)$$

{ [illegible] }
$3(ab + ac + bc)$ has only two values &
these two are reciprocals i.e. their
product is unity. Hence I was
led to suppose that there
[illegible] 2 forms — This is

Assume that the sextic ψ is generic: it has six distinct roots.

Theorem (MD, Roger Penrose). Let X_1, X_2, X_3, X_4 be four points on a two–dimensional sphere such that the stereographic projection of one of the roots of the sextic from any of these four points lies in the centroid of the projections of the remaining five roots. Then $\mathcal{I}_2(\psi) = 0$ if and only if the points $X_1, \dots, X_4$ can be transformed into vertices of a regular tetrahedron by a Möbius transformation (if they are distinct), or if at least three of these points coincide.



- Let $P_1, \dots, P_k \in \mathbb{CP}^1$. Notation: $\psi = P_1 P_2 \dots P_k$ - quantic determined (up to a multiple) by a position of its roots.

$$P_1 \dots P_5 P_6 = (P_1 \dots P_5) P_6, \quad \psi = \kappa(\text{quintic}) \times \rho(\text{linear}).$$

- Transvectant:

$$\begin{aligned} \delta =< \kappa, \rho >: &= \frac{\partial \kappa}{\partial x}\frac{\partial \rho}{\partial y} - \frac{\partial \kappa}{\partial y}\frac{\partial \rho}{\partial x} \qquad \text{(quartic)} \\ &= \delta_0 x^4 + 4\delta_1 x^3 y + 6\delta_2 x^2 y^2 + 4\delta_3 x y^3 + \delta_4 y^4. \end{aligned}$$

- Calculate: $-\mathcal{I}_2(\psi) = \mathcal{I}_2(\delta) = 2\delta_0\delta_4 - 8\delta_1\delta_3 + 6(\delta_2)^2$.
- Quartics δ with $\mathcal{I}_2(\delta) = 0$ are projectively equivalent to a regular tetrahedron $(x-y)(x-\omega y)(x-\omega^2 y)y, \quad \omega^3 = 1$, or a pair of points $x^3 y$, or four points x^4.

Outline of the proof: Apolarity

- $V_m = \mathrm{Sym}^m(\mathbb{C}^2)$. Space of binary quantics of degree m.
- Transvectant. $< , >_k$: $V_m \times V_n \to V_{m+n-2k}$.

$$< \phi, \psi >_k := \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{\partial^k \phi}{\partial x^{k-j} \partial y^j} \frac{\partial^k \psi}{\partial x^j \partial y^{k-j}}.$$

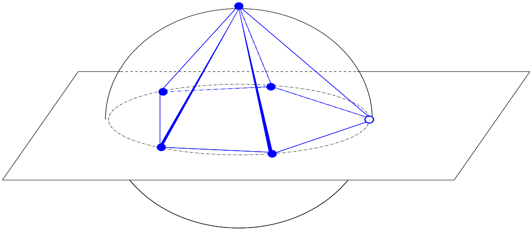
- Quantics $\phi \in V_n$ and $\psi \in V_m$ $(m \geq n)$ are apolar if $< \psi, \phi >_n = 0$.
- $\kappa = P_1 P_2 P_3 P_4 P_5$ and $\chi = P_6 X_i X_i X_i X_i$ are apolar for each i. Pick i, and set $X_i = [0, 1]$ (north pole), and $P_j = [1, p_j]$. Find

$$p_6 = \frac{1}{5} \sum_{k=1}^{5} p_k.$$

Example: Pentagonal Pyramid

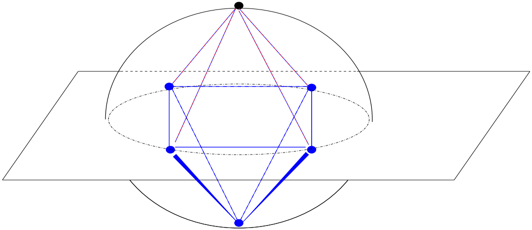
- The multiplicities of the elements of δ can depend on the choice of the point P_6.
- Pentagonal pyramid with one root at ∞ and $\mathcal{I}_2 = 0$.

$$\psi = (x-y)(x-\omega y)(x-\omega^2 y)(x-\omega^3 y)(x-\omega^4 y)y, \quad \text{where} \quad \omega^5 = 1$$



- $P_6 = \infty$ gives $\delta = x^4$ which has one quadrupole root.
- $P_6 = 1$ gives $\delta = x^4 + 6x^3 + 6x^2 + 6x + 6 = 0$ with $\mathcal{I}_2(\delta) = 0$.

- Given five distinct points $P_1, \ldots, P_5$ can one always find a distinct point P_6 such that $\mathcal{I}_2(\psi) = 0$, where $\psi = P_1 \ldots P_5 P_6$?
- Answer: No!
- **Proposition.** All quintics with five distinct roots which can not be extended to a sextic with $\mathcal{I}_2 = 0$ are projectively equivalent to a square based pyramid.



- $\kappa = y(x^4 - y^4)$. For any P_6 find $\mathcal{I}_2(\psi) = 1/3$.

$\mathcal{I}_2$ FOR GENERAL QUANTICS

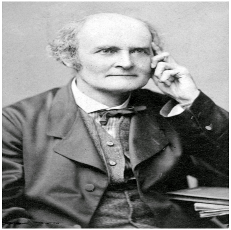
$$\psi(x,y) = \sum_{k=0}^{n} \binom{n}{k} \psi_k x^{n-k} y^k, \quad \mathcal{I}(\psi) = 2\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \psi_k \psi_{n-k}.$$

- $\mathcal{I}_2 = 0$ if n is odd.
- Given $\psi_{2n} = \{P_1, P_2, \ldots, P_{2n}\}$, let $\delta_{2n-2} = \{X_1, X_2, \ldots, X_{2n-2}\}$ be the points s. t. the stereographic projection of P_{2n} from any X_i is the centroid of the stereographic projections of $P_1, \ldots, P_{2n-1}$. Then

$$\mathcal{I}_2(\psi) = 0 \quad \text{iff} \quad \mathcal{I}_2(\delta) = 0.$$

- Algebraic characterisation: An even degree quantic $\psi \in V_{2n}$ with distinct roots is a sum of $(2n)^{th}$ powers of its factors iff $\mathcal{I}_2(\psi) = 0$.

Interlude: Arthur Cayley (1821-1895)



- Senior Wrangler (Trinity College, Cambridge 1842).
- Lawyer by day, mathematician by night.
- Elected to the Salderian Chair of Pure Mathematics at Cambridge in 1863.
- Tait on Cayley: *Is it not a shame that such an outstanding man puts his abilities to such entirely useless questions?*

Equity & Law Life Assurance Society,
26, Lincoln's Inn Fields,
London 11 April 1852

Dear Cayley.

I write to correct two mistakes one of my own, the other of yours.

For my own first.

The content of the □ of a curve having a double point or a surface

Thank You!