Einstein-Weyl Spaces and Near Horizon Geometry

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Black hole topology

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- Not true in $D + 1$ dimensions if $D > 3$. **Emparan–Reall 2001**: AF Black Ring in 4 + 1 dimensions. Horizon topology $S^2 \times S^1$. 

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Galloway–Schoen 2006: Horizon cross–section admits a metric of positive scalar curvature. $D = 4$ : $S^3$ (or quotient), $S^2 \times S^1$, connected sums.
$D + 1$–dimensional manifold $M$, Lorentzian metric $g$, Maxwell potential $A$, stationary Killing vector $U$. 

Moncrief–Isenberg 1983: In the neighbourhood of the Killing horizon $g(U,U) = 0 \exists$ Gaussian coordinates $(u,r,y^i)$ s. t. $U = \partial/\partial u$ and

The horizon is a surface $r = 0$. $y^i, i = 1, \ldots, D - 1$ are coordinates on a Riemannian cross–section $\Sigma$.

$g = 2 du (dr + rh - \frac{1}{2}r^2 \Delta du) + \gamma$, $A = r \Phi du + B$, where $\gamma = \gamma_{ij}(r,y^i) dy^i dy^j, h = h^i(r,y^i) dy^i, B = B^i(r,y^i) dy^i, \Delta = \Delta(r,y^i), \Phi = \Phi(r,y^i)$ are all real–analytic in $r$.

Near–horizon limit (NHL) Reall 2003, Lewandowski–Pawlowski 2003 $u \to u/\epsilon, r \to r\epsilon$, limit $\epsilon \to 0$ ($\gamma, h, B, \Delta, \Phi$). Riemannian metric, one–forms, functions on $\Sigma$. 

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- $(\gamma, h, B, \Delta, \Phi)$. Riemannian metric, one–forms, functions on $\Sigma$. 

Results

- In NHL, the field equations on \((M, g, A)\) reduce to elliptic equations on \(\Sigma\). This talk:
  
  - **Theorem 1**: Einstein–Maxwell–Chern–Simons equations (minimal supergravity in five dimensions) on \(M \rightarrow\) Einstein–Weyl equations on \(\Sigma\).
  
  - **Theorem 2**: Compact \((\Sigma, [\gamma])\): squashed \(S^3\), product metric on \(S^2 \times S^1\), or flat torus. Reconstruct \((M, g, A)\) from its NHL? Too though. But
  
  - **Theorem 3**: The moduli space of transverse infinitesimal deformations of a compact near–horizon geometry is finite–dimensional.

All theorems assume supersymmetry.

Unexpected spin-off: conformal invariance and integrability (via twistor transform) on \(\Sigma\).
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Minimal supergravity in 5 dimensions

- $\mathcal{R}$ Ricci scalar of $g$. Maxwell field $H = dA$

$$S = \int_M \mathcal{R} \text{vol}_M - \frac{3}{2} H \wedge \star_5 H - H \wedge H \wedge A.$$
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$$dH = 0, \quad d\ast_5 H + H \wedge H = 0,$$

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$$*3(d\Phi + h\Phi) = dh, \quad (\text{Maxwell})$$

$$d *3h = 0, \quad (\text{Einstein ur})$$

$$R_{ij} + \nabla_{(i}h_{j)} + h_i h_j = \left( \frac{1}{2} \Phi^2 + h^k h_k \right) \gamma_{ij} \quad (\text{Einstein ij})$$
A Weyl structure \((\Sigma, [\gamma], D)\)
- Riemannian conformal structure \([\gamma] = \{e^{2\Omega}\gamma, \Omega: \Sigma \to \mathbb{R}\}\).
- Torision–free connection \(D\) on \(T\Sigma\).
- Compatibility \(D_i\gamma_{jk} = 2h_i\gamma_{jk}\) for some \(h \in \Lambda^1(\Sigma)\).
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3D Einstein–Weyl geometry

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  - In general (real analytic): 4 arbitrary functions of 2 variables.
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- Example (Berger sphere)

\[
\gamma = (\sigma_1)^2 + (\sigma_2)^2 + a^2(\sigma_3)^2, \quad h = a\sqrt{1 - a^2}\sigma_3
\]

where \(d\sigma_1 + \sigma_2 \wedge \sigma_3 = 0\), etc.
An Einstein–Weyl space is Hyper–CR iff there exists $\Phi : \Sigma \to \mathbb{R}$ s. t.

$$\star_3 (d\Phi + h\Phi) = dh, \quad W = \frac{3}{2}\Phi^2,$$

where $W$ is the Ricci scalar of $D$. 

**Conformal weights:**

$W \in \Gamma(\mathcal{E}(-2))$, $\Phi \in \Gamma(\mathcal{E}(-1))$. 

Gauduchon–Tod 1999: Berger sphere, product metric on $S^2 \times S^1$ and flat torus are all compact examples. 

Integrable system MD-Tod 2001: There exist local coordinates $z : \Sigma \to \mathbb{C}$, $v : \Sigma \to \mathbb{R}$ and a function $F : \Sigma \to \mathbb{R}$ such that

$$\gamma = dzd\bar{z} + \frac{1}{16}((Fdv - i(Fzdz - F\bar{z}d\bar{z})) + dF)^2,$$ 

$h = \ldots$, 

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where $F = F(z, \bar{z}, v)$ satisfies

$$F_{\bar{z}z}(F + F_{vv}) - (F_z + iF_{vz})(F_{\bar{z}} - iF_{v\bar{z}}) = 4.$$
Let \((\gamma, h)\) be a hyper–CR Einstein–Weyl structure on \(\Sigma\) and let \(\Omega : \Sigma \to \mathbb{R}^+\) satisfy \(d \ast_3 (de^\Omega) + d \ast_3 (e^\Omega h) = 0\). Then

\[
    g = e^{2\Omega}(2du(dr + rh - \frac{1}{3}r^2Wdu) + \gamma + 6rdud\Omega)
\]

\[
    A = \sqrt{\frac{2}{3}}e^\Omega r\sqrt{W}du + \alpha \quad (\ast)
\]

is a solution to the 5D Einstein–Maxwell–Chern–Simons supergravity. Here \(\alpha \in \Lambda^1(\Sigma)\) is such that \(d\alpha = -e^\Omega \ast_3 (h + d\Omega)\).

- All near–horizon geometries for 5D SUSY back holes/rings/strings are locally of the form \((\ast)\).
- If \(\Sigma\) is compact then \(\gamma\) is a metric on the Berger sphere, a product metric on \(S^1 \times S^2\) or a flat metric on \(T^3\).
Taylor expand the near-horizon data in $r$

\[
\Delta = \Delta_0(y) + r\delta\Delta(y) + O(r^2), \\
h = h_0(y) + r\delta h(y) + O(r^2), \\
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\]
**Extension to the bulk**

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- Gauge freedom $\delta \gamma_{ij} \rightarrow \delta \gamma_{ij} + \nabla_i \nabla_j f - h_{(i} \nabla_{j)} f$, etc.
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- **Theorem**: The moduli space of supersymmetric transverse deformations of supersymmetric near horizon solutions with compact spatial sections, corresponding to the moduli $(\delta\Delta, \delta h, \delta\gamma)$, modulo the gauge transformations is finite dimensional.
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- **Idea of proof:** Find a global gauge s.t. $\nabla^2 \delta \gamma_{ij} = C_{ij}$ (RHS linear in $\delta \gamma, \nabla \delta \gamma$). Use ellipticity.
Summary and Outlook

- Supersymmetric near horizon geometry of \((4 + 1)\)-dimensional minimal supergravity.
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  - Integrable Einstein–Weyl geometry on spatial cross–sections.

One–form \(h \in \Lambda^1(\Sigma)\) satisfies
\[
\nabla_i (h_j + h_i h_j) = -\frac{1}{2} R_{ij}^{\gamma}.
\]
This can be solved explicitly if \(\gamma\) is axi–symmetric (Kerr).

Conjecture. Let \(\gamma\) be a Riemannian metric on \(\Sigma = S^2\), such that \((\ast)\) holds. Then there exists a Killing vector on \((\Sigma, \gamma)\).
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- Near horizon geometry of $(3 + 1)$ vacuum Einstein equations.
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  - Spatial cross section \((\Sigma, \gamma)\) is Einstein: \(R_{ij} = R\gamma_{ij}/2\)
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  - One–form $h \in \Lambda^1(\Sigma)$ satisfies

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\nabla(i h_j) + h_i h_j = \frac{1}{2} R\gamma_{ij}. \quad (*)
\]

This can be solved explicitly if $\gamma$ is axi–symmetric (Kerr).

Conjecture. Let $\gamma$ be a Riemannian metric on $\Sigma = S^2$, such that $(*)$ holds. Then there exists a Killing vector on $(\Sigma, \gamma)$.

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