Can chaos be observed in quantum gravity?

Philipp Höhn

Institute for Quantum Optics and Quantum Information, Vienna, and Vienna Center for Quantum Science and Technology

4th Tux Workshop on QG
18 Feb 2016

based on:
PH, M. Kubalova and A. Tsoobanjan, PRD 86 065014 (2012);
GR and ‘observables’

General Relativity is a gauge theory
⇒ physical observables should be diffeomorphism invariant
canonically:
  ■ observables should commute with constraints ⇒ Dirac observables as ‘constants of motion’
  ■ dynamics relationally ⇒ ‘evolving constants of motion’ [Wheeler 60’s; Rovelli 90’s; Dittrich ’06,’07…...]
  ■ important for quantum theory
⇒ notoriously difficult to construct

often overlooked: even absent in presence of chaos
  1 what then is observable?
  2 consequences for QT?
Illustration: closed FRW with (min. coupled) massive scalar

(a) typical solution, (b) close-up on (a), (c) defocussing of nearby trajectories in turning region

- Ham. constraint $C = p_{\phi}^2 - p_{\alpha}^2 - e^{4\alpha} + m^2 \phi^2 e^{6\alpha}$
- model chaotic and non-integrable [Page '84, Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]
- strong defocussing of classical solutions near $\alpha_{\text{max}}$
- has not been fully quantized in any canonical approach
Breakdown of relational dynamics and semiclassicality [PH, Kubalova, Tsobanjan, '12]

- devoid of good internal ‘clocks’
- treat model with effective method (‘effective WdW’)  
  ⇒ clock changes possible in QT  
  [method from Bojowald, PH, Tsobanjan '11a, '11b]
- in region of max. expansion hell breaks loose (chaotic scattering):  
  1  breakdown of semiclassicality  
    [also indep. observed in Kiefer '88]  
  2  relational observables only transient  
    ⇒ relational evolution breaks down
Chaos and constants of motion

integrable (unconstrained) systems:

- $N$ (smooth) constants of motion $F_1, \ldots, F_N$ for $2N$-phase space

- if $\{F_i, F_j\} = 0$, the $F_i$ form $N$-dim. surface

  \[ M_F \simeq T^k \times \mathbb{R}^{N-k} \]

non-integrable (unconstrained) systems:

- no global (smooth) constants of motion other than $H$ exist

  $\Rightarrow$ trajectories lie on $(2N - 1)$-dim. energy surface

- various characterizations:
  - ergodic
  - chaotic
  - ... $\Rightarrow$ distinction unimportant for us, important: non-integrability

- non-integrability generic, $\exists$ concrete theorems for absence of constants of motion [Arnold, Kozlov, Neishtadt book '07]
Non-integrability and constraints \cite{Dittrich, PH, Koslowski, Nelson '15; '16}

Consider system on $2N$-dim. phase space with $m_1$ 1st class constraints $C_i$.

weakly integrable if:

1. $\exists 2(N - m_1)$ Dirac observables $O_i$ indep. of $C_j$
2. $N - m_1$ of Dirac observables are weakly in involution $\{O_i, O_j\} \approx 0$

$\Rightarrow$ $\exists$ reduced phase space
$\Rightarrow$ trajectories on $N$-dim submanifolds of constraint surface

weakly non-integrable if:

$\not\exists$ different. Dirac observables indep. of $C_i$

$\Rightarrow$ $\not\exists$ reduced phase space
$\Rightarrow$ gauge invariant DoFs exist, but non-differentiable (or local)
$\Rightarrow$ trajectories not restricted to $N$-dim submanifolds

- generalize notion of ‘observable’: include non-differentiable ones
- how to represent in QT? $\Rightarrow$ no Poisson algebraic structure
GR a presumably weakly non-integrable

Plenty of evidence that GR weakly non-integrable:

- a generic dynamical system is chaotic

- Newtonian $N \geq 3$ body problem chaotic

- $k = 1$ FRW with min. coupled massive scalar chaotic
  
  [Page '84; Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]

- Mixmaster (Bianchi IX) universe chaotic
  
  [Misner '69; Cornish, Levin '97; Motter, Leterlier '01]

- BKL conjecture: generic cosmological solution features chaotic oscillations
  
  [Belinsky, Khalatnikov, Lifshitz '70]

- vacuum GR on closed spatial slices: no Dirac observables as spatial integrals of metric and its derivatives
  
  [Anderson, Torre '93; '96]

$\Rightarrow$ smooth Dirac observables and reduced phase space (probably) $\not\in$ in full GR

$\Rightarrow$ what are repercussions for QG?
Parametrized chaotic systems are weakly integrable

Let $H_{\text{chaos}}(q_i, p_i)$ be Hamiltonian of non-integrable unconstrained system.

- Parametrization yields constrained system

\[
C = p_t - H_{\text{chaos}}(q_i, p_i) \approx 0
\]

- BUT: weakly integrable because global gauge $t = \text{const}$ exists

difference:

unconstrained: do not need to solve dynamics

constrained: need to solve dynamics
to access physical DoFs
Toy model: free particles on a circle [Dittrich, PH, Koslowski, Nelson '15; '16]

Compactify free dynamics: \( x_i + 1 \sim x_i, \ i = 1, 2 \Rightarrow \text{conf. manf.} \ Q \simeq T^2 \)

\[
C = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - E \approx 0
\]

\( \exists \) solutions to EoMs

\[
\begin{align*}
    x_1(t) & = \frac{p_1}{m_1} t + x_{10} - n_1 \\
    x_2(t) & = \frac{p_2}{m_2} t + x_{20} - n_2
\end{align*}
\]

\( n_i := \lfloor \frac{p_i}{m_i} t + x_{i0} \rfloor \) winding number in \( x_i \)

if:

\[
\frac{m_2}{m_1} \frac{p_1}{p_2} \in \mathbb{Q}: \text{resonant torus, periodic orbits}
\]

\[
\frac{m_2}{m_1} \frac{p_1}{p_2} \notin \mathbb{Q}: \text{non-resonant torus, ergodic orbits}
\]
Absence of sufficiently many Dirac observables [Dittrich, PH, Koslowski, Nelson '15; '16]

- momenta $p_i$ are Dirac observables
- $\exists$ smooth Dirac observables $F(p_i; x_1, x_2)$ with $\partial_i F \neq 0$?

**NO:** $F$ constant on trajectories must be discontinuous in $x_i$

- trajectories on non-resonant torus fill it densely
  $\Rightarrow$ $F$ takes every value in every neighbourhood (of non-resonant torus)

- ergodicity destroys full integrability
  $\Rightarrow$ no reduced phase space, no (sufficient) algebra of observables
- even worse: space of solutions
  1. non-Hausdorff
  2. not a manifold

- failure of Marsden-Weinstein reduction
Generalization of Dirac observables

- can still have gauge invariant ‘observables’, however, either
  1. global and discontinuous, e.g.
     \[ M = (x_1 + n_1)p_2/m_2 - (x_2 + n_2)p_1/m_1 \]
  2. local [Bojowald, PH, Tsobanjan ‘11a; ’11b]
- also relational dynamics still meaningful, albeit implicitly
  ⇒ e.g.: choose \( x_1 \) as ‘clock’, obtain relational ‘observable’
  \[ x_2(\tau) = \frac{m_1}{m_2} \frac{p_2}{p_1} (\tau - x_1 + n_1(\tau, x_2(\tau), x_1, x_2)) + x_2 - n_2(\tau, x_2(\tau), x_1, x_2) \]

resonant torus: finitely many solutions
non-resonant torus: ‘densely many’ solutions

- but: locally, explicit solutions exist on each branch (for fixed \( n_1, n_2 \)
Quantization?

1. reduced quantization
2. ‘standard’ Dirac quantization
3. polymer quantization
Quantization?

1. reduced quantization $\times$
2. ‘standard’ Dirac quantization $\times$
3. polymer quantization $\checkmark$
Reduced quantization

outright impossible since no reduced phase space ×
Standard Dirac quantization

- $\mathcal{H}_{\text{kin}} = L^2(S^1 \times S^1)$
- $\hat{p}_i \psi = -i\hbar \partial_i \psi$
- basis:

$$\psi_{k_1,k_2}(x_1, x_2) = \exp(2\pi ik_1 x_1) \exp(2\pi ik_2 x_2), \quad (k_1, k_2) \in \mathbb{Z}^2$$

- constraint

$$\hat{C} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} - E$$

- solutions to constraint given by $k_1, k_2$ s.t.

$$k_1^2 + \frac{m_1}{m_2} k_2^2 = \frac{2m_1 E}{\hbar^2}$$

  difficult Diophantine problem

$\Rightarrow$ for $m_1/m_2 \notin \mathbb{Q}$

$$0 \leq \dim \mathcal{H}_{\text{phys}} \leq 4$$

- ‘few observables’ $\Rightarrow$ ‘few states’
Sick quantum theory: no semiclassics [Dittrich, PH, Koslowski, Nelson '15; '16]

\[ \dim \mathcal{H}_{\text{phys}} = 4: \]

NOT peaked on class. orbit for \( m_1/m_2 \notin \mathbb{Q} \) width/separation \( \approx 1 \) ⇒ similar for other cases

- physical transition amplitudes

\[
W(\vec{x}_1, \vec{p}_1; \vec{x}_2, \vec{p}_2) = \frac{\langle (\vec{x}_2, \vec{p}_2) | \hat{P} | (\vec{x}_1, \vec{p}_1) \rangle}{\sqrt{\langle (\vec{x}_1, \vec{p}_1) | \hat{P} | (\vec{x}_1, \vec{p}_1) \rangle \langle (\vec{x}_2, \vec{p}_2) | \hat{P} | (\vec{x}_2, \vec{p}_2) \rangle}}
\]

show no semiclassical behaviour either
An extreme example [Dittrich, PH, Koslowski, Nelson '15]

Consider on $S^1 \times S^1$

$$C = \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} \quad \text{and} \quad \sqrt{\frac{m_2}{m_1}} \notin \mathbb{Q}$$

$\Rightarrow$ all classical solutions ergodic

$\Rightarrow$ no configurational Dirac observable

$\Rightarrow$ solutions to quantum constraint equivalent to

$$k_2 = \pm \sqrt{\frac{m_2}{m_1}} k_1$$

$\Rightarrow$ no solutions for $\vec{k} \in \mathbb{Z}^2$

$\Rightarrow$ well-defined classical dynamics, but no ‘standard’ QT
Polymer type quantization: discrete topology [Dittrich, PH, Koslowski, Nelson ’15; ’16]

additional ‘observables’ discontinuous ⇒ try discrete topology on $T^2$

- $\mathcal{H}_{\text{kin}}$ given by (uncountable) basis

\[ \psi_{x_1',x_2'}(x_1,x_2) = \delta_{x_1',x_1} \delta_{x_2',x_2} \]

- no momenta, but translations

\[ (R_1^\mu \psi)(x_1,x_2) = \psi(x_1 + \mu, x_2), \quad (R_2^\mu \psi)(x_1,x_2) = \psi(x_1, x_2 + \mu) \]

⇒ $p_i^2/2$ replaced by

\[ S_i^\mu := -\frac{\hbar^2}{2\mu^2} (R_i^{\mu+} + R_i^{\mu-} - 2) \]

- constraint

\[ \hat{C}^\mu = S_1^\mu + S_2^\mu - E \]

Bohr compactification:

- eigenstates and eigenvalues $R_i^{\mu}$ for $\mu \notin \mathbb{Q}$ (continuous $\rho \in [0, 1]$):

\[ \phi_{x',\rho}(x) = \sum_{l \in \mathbb{Z}} e^{2\pi i l \rho} \delta_{x' + l \mu, x}, \quad \{ e^{2\pi i \rho} \in U(1) \} \]
Large physical Hilbert space and enough observables

- spectrum of constraint $\hat{C}^\mu$ for $\mu \notin \mathbb{Q}$:
  \[
  \left\{ \frac{\hbar^2}{\mu^2} \left( 2 - \cos(2\pi \rho_1) - \cos(2\pi \rho_2) \right) - E |\rho_1, \rho_2 \in [0, 1]\right\}
  \]

$\Rightarrow$ upon superselec. get $\infty$-dim. separable $\mathcal{H}_{\text{phys}}$ as $L^2$ over ‘momentum’ $\rho$

- on this $\mathcal{H}_{\text{phys}}$ have sufficiently many observables
  
  \[
  \hat{M} := \frac{i}{2\pi} \left( \sin(2\pi \rho_2) \frac{\partial}{\partial \rho_1} - \sin(2\pi \rho_1) \frac{\partial}{\partial \rho_2} \right)
  \text{ ('angular mom.')}
  \]

  \[
  [\hat{M}, e^{2\pi i \rho_1}] = -e^{2\pi i \rho_1}\sin(2\pi \rho_2)
  \]

- good semiclassical transition amplitudes

\[
W(\psi_1, \psi_2) = \frac{\langle \psi_1 | \hat{P} | \psi_2 \rangle}{\sqrt{\langle \psi_1 | \hat{P} | \psi_1 \rangle \langle \psi_2 | \hat{P} | \psi_2 \rangle}}
\]
Conclusions

- Chaos precludes smooth Dirac observables
  ⇒ probably no smooth Dirac observables and reduced phase space for full GR
- serious problem for ‘standard’ constraint quantization

what do we do?

- always ∃ generalized discontinuous ‘observables’
  ⇒ adapt method of quantization, refine topology until sufficiently many observables continuous
  ⇒ here: polymer quantization overcomes troubles of ‘standard’ quantization!

open questions

- path integral
- extend to field theory case