Can chaos be observed in quantum gravity?

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based on:
PH, M. Kubalova and A. Tsobanjan, PRD **86** 065014 (2012);
B. Dittrich, PH, T. Koslowski and M. Nelson arXiv:1508.01947, and
arXiv:1602.03237

GR and 'observables'

General Relativity is a gauge theory

$\Rightarrow$ physical observables should be diffeomorphism invariant

canonically:

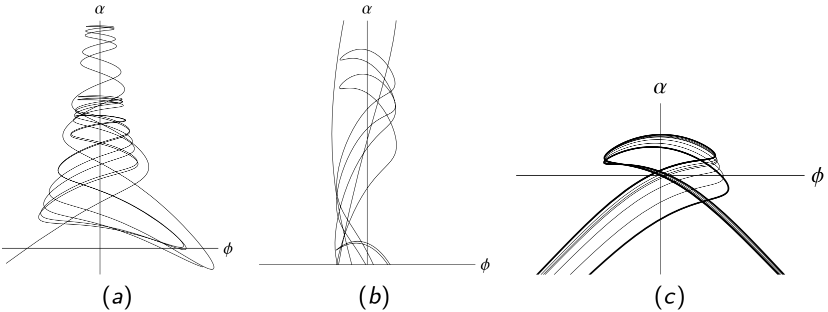
- observables should commute with constraints $\Rightarrow$ Dirac observables as 'constants of motion'
- dynamics relationally $\Rightarrow$ 'evolving constants of motion' **[Wheeler 60's; Rovelli 90's; Dittrich '06,'07......]**
- important for quantum theory

$\Rightarrow$ notoriously difficult to construct

often overlooked: even absent in presence of chaos

1. what then is observable?
2. consequences for QT?

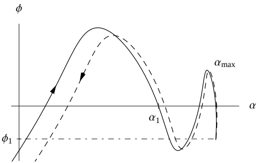
Illustration: closed FRW with (min. coupled) massive scalar



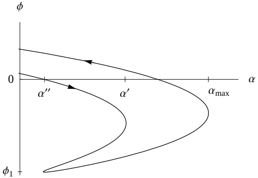
(a) typical solution, (b) close-up on (a), (c) defocussing of nearby trajectories in turning region

- Ham. constraint $C = p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\,\phi^2\,e^{6\alpha}$
- model chaotic and non-integrable **[Page '84, Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]**
- strong defocussing of classical solutions near α_{max}
- has not been fully quantized in any canonical approach

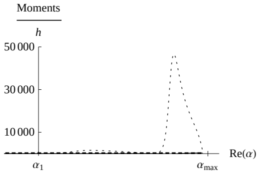
Breakdown of relational dynamics and semiclassicality [PH, Kubalova, Tsobanjan, '12]



classical solution



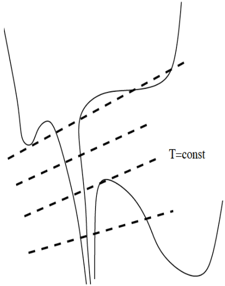
close-up on α_{max}



moments in initial α–time

- devoid of good internal 'clocks'
- treat model with effective method ('effective WdW')
  $\Rightarrow$ clock changes possible in QT
  [method from Bojowald, PH, Tsobanjan '11a, '11b]
- in region of max. expansion hell breaks loose (chaotic scattering):
  1. breakdown of semiclassicality
     [also indep. observed in Kiefer '88]
  2. relational observables only transient

     $\Rightarrow$ relational evolution breaks down



Chaos and constants of motion

integrable (unconstrained) systems:

- N (smooth) constants of motion $F_1, \dots, F_N$ for $2N$-phase space
- if $\{F_i, F_j\} = 0$, the F_i form N-dim. surface

$$M_F \simeq T^k \times \mathbb{R}^{N-k}$$

non-integrable (unconstrained) systems:

- no global (smooth) constants of motion other than H exist

$\Rightarrow$ trajectories lie on $(2N-1)$-dim. energy surface

- various characterizations:
  - ergodic
  - chaotic
  - ... $\Rightarrow$ distinction unimportant for us, important: non-integrabiliity
- non-integrability generic, $\exists$ concrete theorems for absence of constants of motion [Arnold, Kozlov, Neishtadt book '07]

Non-integrability and constraints [Dittrich, PH, Koslowski, Nelson '15; '16]

Consider system on $2N$-dim. phase space with m_1 1st class constraints C_i.

weakly integrable if:

1. $\exists\ 2(N - m_1)$ Dirac observables O_i indep. of C_j
2. $N - m_1$ of Dirac observables are weakly in involution $\{O_i, O_j\} \approx 0$

$\Rightarrow$ $\exists$ reduced phase space

$\Rightarrow$ trajectories on N-dim submanifolds of constraint surface

weakly non-integrable if:

$\nexists$ different. Dirac observables indep. of C_i

$\Rightarrow$ $\nexists$ reduced phase space

$\Rightarrow$ gauge invariant DoFs exist, but non-differentiable (or local)

$\Rightarrow$ trajectories not restricted to N-dim submanifolds

- generalize notion of 'observable': include non-differentiable ones
- how to represent in QT? $\Rightarrow$ no Poisson algebraic structure

GR a presumably weakly non-integrable

Plenty of evidence that GR weakly non-integrable:

- a generic dynamical system is chaotic
- Newtonian $N \geq 3$ body problem chaotic
- $k = 1$ FRW with min. coupled massive scalar chaotic
  [Page '84; Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]
- Mixmaster (Bianchi IX) universe chaotic
  [Misner '69; Cornish, Levin '97; Motter, Leterlier '01]
- BKL conjecture: generic cosmological solution features chaotic oscillations
  [Belinsky, Khalatnikov, Lifshitz '70]
- vacuum GR on closed spatial slices: no Dirac observables as spatial integrals of metric and its derivatives
  [Anderson, Torre '93; '96]

$\Rightarrow$ smooth Dirac observables and reduced phase space (probably) $\nexists$ in full GR
$\Rightarrow$ what are repercussions for QG?

Parametrized chaotic systems are weakly integrable

Let $H_{chaos}(q_i, p_i)$ be Hamiltonian of non-integrable unconstrained system.

- Parametrization yields constrained system

$$C = p_t - H_{chaos}(q_i, p_i) \approx 0$$

- BUT: weakly integrable because global gauge $t = const$ exists

difference:

unconstrained: do not need to solve dynamics

constrained: need to solve dynamics

to access physical DoFs

Toy model: free particles on a circle [Dittrich, PH, Koslowski, Nelson '15; '16]

Compactify free dynamics: $x_i + 1 \sim x_i$, $i = 1, 2 \Rightarrow$ conf. manf. $\mathcal{Q} \simeq T^2$

$$C = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - E \approx 0$$

- solutions to EoMs

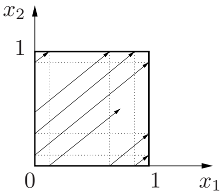
$$\begin{aligned} x_1(t) &= \frac{p_1}{m_1} t + x_{10} - n_1 \\ x_2(t) &= \frac{p_2}{m_2} t + x_{20} - n_2 \end{aligned}$$

$n_i := \lfloor \frac{p_i}{m_i} t + x_{i0} \rfloor$ winding number in x_i

if:

$\frac{m_2}{m_1} \frac{p_1}{p_2} \in \mathbb{Q}$: resonant torus, periodic orbits

$\frac{m_2}{m_1} \frac{p_1}{p_2} \notin \mathbb{Q}$: non-resonant torus, ergodic orbits



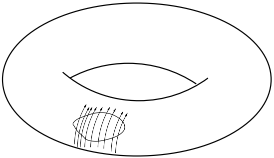
Absence of sufficiently many Dirac observables [Dittrich, PH, Koslowski, Nelson '15; '16]

- momenta p_i are Dirac observables
- $\exists$ smooth Dirac observables $F(p_i; x_1, x_2)$ with $\partial_i F \neq 0$?

NO: F constant on trajectories must be discontinuous in x_i

- trajectories on non-resonant torus fill it densely

$\Rightarrow$ F takes every value in every neighbourhood (of non-resonant torus)



- ergodicity destroys full integrability

$\Rightarrow$ no reduced phase space, no (sufficient) algebra of observables

- even worse: space of solutions
  1. non-Hausdorff
  2. not a manifold

- failure of Marsden-Weinstein reduction

Generalization of Dirac observables

- can still have gauge invariant 'observables', however, either
  1. global and discontinuous, e.g.

$$M = (x_1 + n_1)p_2/m_2 - (x_2 + n_2)p_1/m_1$$

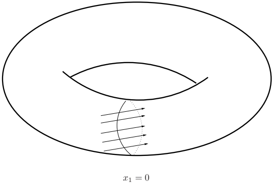
  2. local [Bojowald, PH, Tsobanjan '11a; '11b]
- also relational dynamics still meaningful, albeit implicitly

⇒ e.g.: choose x_1 as 'clock', obtain relational 'observable'

$$x_2(\tau) = \frac{m_1}{m_2}\frac{p_2}{p_1}(\tau - x_1 + n_1(\tau, x_2(\tau), x_1, x_2)) + x_2 - n_2(\tau, x_2(\tau), x_1, x_2)$$

resonant torus: finitely many solutions

non-resonant torus: 'densely many' solutions



- but: locally, explicit solutions exist on each branch (for fixed n_1, n_2)

Quantization?

1. reduced quantization
2. ‘standard’ Dirac quantization
3. polymer quantization

Quantization?

1. reduced quantization ×
2. ‘standard’ Dirac quantization ×
3. polymer quantization ✓

Reduced quantization

outright impossible since no reduced phase space ×

Standard Dirac quantization

- $\mathcal{H}_{\rm kin} = L^2(S^1 \times S^1)$
- $\hat{p}_i\psi = -i\hbar\partial_i\psi$
- basis:

$$\psi_{k_1,k_2}(x_1,x_2) = \exp(2\pi i k_1 x_1)\exp(2\pi i k_2 x_2), \qquad (k_1,k_2) \in \mathbb{Z}^2$$

- constraint

$$\hat{C} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} - E$$

- solutions to constraint given by k_1, k_2 s.t.

$$k_1^2 + \frac{m_1}{m_2}k_2^2 = \frac{2m_1E}{\hbar^2}$$

difficult Diophantine problem

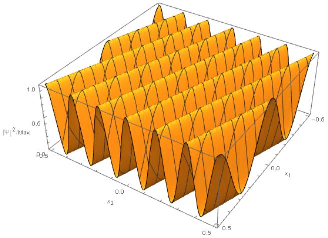
$\Rightarrow$ for $m_1/m_2 \notin \mathbb{Q}$

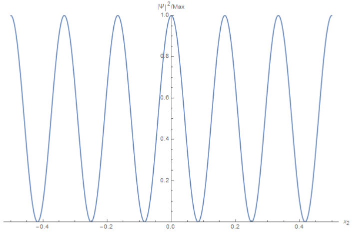
$$0 \leq \dim \mathcal{H}_{\rm phys} \leq 4$$

- 'few observables' $\Rightarrow$ 'few states'

Sick quantum theory: no semiclassics [Dittrich, PH, Koslowski, Nelson '15; '16]

$\dim \mathcal{H}_{\text{phys}} = 4$:





NOT peaked on class. orbit for $m_1/m_2 \notin \mathbb{Q}$ width/separation ≈ 1

$\Rightarrow$ similar for other cases

- physical transition amplitudes

$$W(\vec{x}_1, \vec{p}_1; \vec{x}_2, \vec{p}_2) = \frac{\langle(\vec{x}_2, \vec{p}_2)| \hat{P} |(\vec{x}_1, \vec{p}_1)\rangle}{\sqrt{\langle(\vec{x}_1, \vec{p}_1)| \hat{P} |(\vec{x}_1, \vec{p}_1)\rangle \langle(\vec{x}_2, \vec{p}_2)| \hat{P} |(\vec{x}_2, \vec{p}_2)\rangle}}$$

show no semiclassical behaviour either

An extreme example [Dittrich, PH, Koslowski, Nelson '15]

- Consider on $S^1 \times S^1$

$$C = \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} \qquad \text{and} \qquad \sqrt{m_2/m_1} \notin \mathbb{Q}$$

$\Rightarrow$ all classical solutions ergodic

$\Rightarrow$ no configurational Dirac observable

- solutions to quantum constraint equivalent to

$$k_2 = \pm\sqrt{m_2/m_1}\, k_1$$

$\Rightarrow$ no solutions for $\vec{k} \in \mathbb{Z}^2$

$\Rightarrow$ well-defined classical dynamics, but no 'standard' QT

Polymer type quantization: discrete topology [Dittrich, PH, Koslowski, Nelson '15; '16]

additional 'observables' discontinuous $\Rightarrow$ try discrete topology on T^2

- $\mathcal{H}_{\rm kin}$ given by (uncountable) basis

$$\psi_{x'_1,x'_2}(x_1,x_2)=\delta_{x'_1,x_1}\delta_{x'_2,x_2}$$

- no momenta, but translations

$$(R_1^\mu\,\psi)(x_1,x_2)=\psi(x_1+\mu,x_2),\qquad (R_2^\mu\,\psi)(x_1,x_2)=\psi(x_1,x_2+\mu)$$

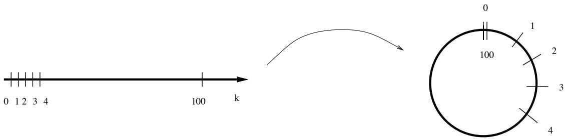
$\Rightarrow p_i^2/2$ replaced by

$$S_i^\mu:=-\frac{\hbar^2}{2\mu^2}(R_i^{+\mu}+R_i^{-\mu}-2)$$

- constraint

$$\hat{C}^\mu=S_1^\mu+S_2^\mu-E$$

Bohr compactification:



- eigenstates and eigenvalues R_i^μ for $\mu\notin\mathbb{Q}$ (continuous $\rho\in[0,1)$):

$$\phi_{x',\rho}(x)=\sum_{l\in\mathbb{Z}}e^{2\pi il\rho}\,\delta_{x'+l\mu,x},\qquad \{e^{2\pi i\rho}\in\mathrm{U}(1)\}$$

Large physical Hilbert space and enough observables

- spectrum of constraint $\hat{C}^\mu$ for$\mu \notin \mathbb{Q}$:

$$\{\frac{\hbar^2}{\mu^2}(2 - \cos(2\pi\rho_1) - \cos(2\pi\rho_2)) - E|\rho_1, \rho_2 \in [0,1)\}$$

$\Rightarrow$ upon superselec. get ∞-dim. separable $\mathcal{H}_{\rm phys}$ as L^2 over 'momentum' ρ

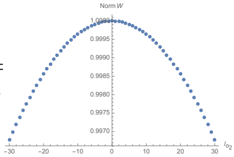
- on this $\mathcal{H}_{\rm phys}$ have sufficiently many observables

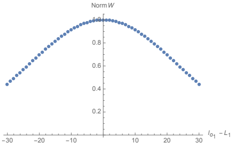
$$\hat{M} := \frac{i}{2\pi}\left(\sin(2\pi\rho_2)\frac{\partial}{\partial\rho_1} - \sin(2\pi\rho_1)\frac{\partial}{\partial\rho_2}\right) \qquad \text{('angular mom.')}$$

$$[\hat{M}, e^{2\pi i \rho_1}] = -e^{2\pi i \rho_1}\sin(2\pi\rho_2)$$

- good semiclassical transition amplitudes

$$W(\psi_1,\psi_2) = \frac{\langle\psi_1|\,\hat{P}\,|\psi_2\rangle}{\sqrt{\langle\psi_1|\,\hat{P}\,|\psi_1\rangle\,\langle\psi_2|\,\hat{P}\,|\psi_2\rangle}}$$





Conclusions

- Chaos precludes smooth Dirac observables

$\Rightarrow$ probably no smooth Dirac observables and red. phase space for full GR

- serious problem for ‘standard’ constraint quantization

what do we do?

- always $\exists$ generalized discontinuous ‘observables’

$\Rightarrow$ adapt method of quantization, refine topology until sufficiently many observables continuous

$\Rightarrow$ here: polymer quantization overcomes troubles of ‘standard’ quantization!

open questions

- path integral
- extend to field theory case