From discrete to continuum: Lessons from the Gromov-Hausdorff space

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Based on an ongoing work with Manfred Requardt (University of Göttingen)

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Introduction:
The Birds Eye View Of The Idea
Shared Semiclassical Limit Challenges

Semiclassical challenge of theories with discrete/quantum spacetime

Bottom-up: already discrete  \(\downarrow\) coarse grain

Top-down: quantize  \(\downarrow\) coarse grain

\[\exists \text{ Fixed Point (FP) of coarse graining?}\]

\[\text{Dim of FP = 3+1?}\]

\[\text{FP \approx topol./diff. manifold?}\]

FP satisfies spacetime symmetries, GR EoM etc.?
Bird’s eye view

Convergence

Compare

Fixed point

Coarse Graining

Change of topology, geometry, metric, etc.

Compare

Convergence

Coarse Graining

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Bird’s eye view

Take the superspace of all metric spaces \( S = \{ s_i \mid (s_i, d_i) \text{ metric space} \} \).

- Define relevant dimension for relevant \( s_i \in S \).
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- Propose a coarse graining scheme in $(S, d_S)$:
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1. Existence of a fixed point: \( \{(G_i, d_{G_i})\} \) converges to \( (X, d_X) \) w.r.t. \( d_S \)?
2. \( \dim (X) = 3(+1) \)?
3. \( X \) a manifold (at least topological)?
Setting The Stage:
Some Graph Theoretic Definitions
Some basic graph concepts - 1

A graph $G = (V, E)$; vertices $v_i \in V$; edges $e_{ij} \in E \subset V \times V$; if $e_{ij} \neq e_{ji}$ directed graph.

Locally bounded valence: valence finite $\forall v_i \in G$

Globally bounded valence: valence $< \infty$ on $G$
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Path $\gamma$: an edge sequence without repetition of vertices, except of initial/terminal vertex.

Connected graph: $\exists \gamma$ for each $v_i, v_j \in G$.

Length of path $l(\gamma)$: number of edges occurring in the path.

Geodesic path: path of minimal length between two vertices

$$d(v_i, v_j) := \min_{\gamma} \{l(\gamma), \gamma \text{ connects } v_i \text{ with } v_j\}$$
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Ball of radius $r$ centered at $x \in M$ in a metric space $(M, d)$:

$$B(x, r) = \{ y \in M | d(x, y) \leq r \}$$
Coarse Graining

- Provide a scheme of coarse graining.
- How to ignore details, so to be able to check if two spaces are coarsely similar?
A class of coarse grainings based on isometry

**Isometric embedding**: A distant preserving map between two metric spaces

\[ f : (X, d_X) \to (Y, d_Y) \text{ such that } d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) \]
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**Quasi-isometric embedding:** A map \( f \) from a metric space,

\[ f : (M_1, d_1) \to (M_2, d_2), \] i.e. if there exist constants, \( \lambda \geq 1, \epsilon \geq 0 \), such that

\[ \forall x, y \in M_1 : \frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon, \]

i.e. \( \forall x, y \in M_1 \), the distance between their images is (up to the additive constant \( \epsilon \)) within a factor of \( \lambda \) of their original distance.
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**Quasi-isomery:** a quasi-isometric embedding \( f \) in which

\[ \forall z \in M_2 \land \exists x \in M_1 : d_2(z, f(x)) \leq C. \]

i.e. every \( z \in M_2 \) is within the constant distance \( C \) of an image point.
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Quasi-isometry: a quasi-isometric embedding \( f \) in which

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**Quasi-isometry** is an equivalence relation on metric spaces that ignores their small-scale details in favor of their coarse structure.
Quasi-isometry example:

Equilateral triangle lattice
**Quasi-isometry example:**

**Equilateral triangle lattice**

The integer lattice \( \mathbb{Z}^n \) is quasi-isometric to \( \mathbb{R}^n \):

\[
f : \mathbb{Z}^n \to \mathbb{R}^n
\]

\[
: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n), \quad x_i \in \mathbb{Z}
\]

1. Distances are preserved.
2. \( n \)-tuples \( \in \mathbb{R}^n \) are within \( \sqrt{\frac{n}{4}} \) of \( n \)-tuples \( \in \mathbb{Z}^n \).
Comparison And Fixed Point

- Construct super metric space.
- Provide a measure of comparing different (coarse grained) spaces.
- A sequence of spaces are coarsely similar? Converge to a fixed point?
$d_H$: Distance of metric subspaces

In $(M, d)$, **Hausdorff Distance** $d_H(X, Y)$ of $X, Y \subset M \land X, Y \neq \emptyset$:

$$d_H(X, Y) = \inf \{\epsilon \geq 0 | X \subseteq U_\epsilon(Y), Y \subseteq U_\epsilon(X)\}$$
$d_H$: Distance of metric subspaces

In $(M, d)$, **Hausdorff Distance** $d_H(X, Y)$ of $X, Y \subset M$ and $X, Y \neq \emptyset$:

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**$\epsilon$-neighborhood of a subset**

In $(M, d)$, $\epsilon$-neighborhood of $X \subset M$: union of all $\epsilon$-balls around all $x \in X$

$$U_\epsilon(X) = \bigcup_{x \in X} \{ z \in M \mid d(z, x) \leq \epsilon \}$$

All points within $\epsilon$ of the set $X$, or generalized ball of radius $\epsilon$ around $X$. 
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- Set of all non-empty compact (non-compact) subsets of a metric space + $d_H \Rightarrow$ metric (pseudometric) space
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How to compare two metric spaces? Gromov-Hausdorff distance

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- Make them subspace of another metric space.
The **Gromov-Hausdorff Distance** $d_{GH}$ of compact $(X, d_X), (Y, d_Y)$

$$d_{GH} (X, Y) = \inf d_Z^H (f(X), g(Y))$$

of all metric spaces $Z$ and isometric embeddings $f : X \to Z$, $g : Y \to Z$
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- Measures how far two compact metric spaces are from being isometric:
  Two compact spaces, $X, Y$, are isometric iff $d_{GH}(X, Y) = 0$. 

$d_{GH}$: Distance of metric spaces, convergence
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How to deal with (i.e. define convergence for) non-compact spaces?
Pointed convergence...
Pointed convergence, uniform compactness

Pointed convergence: Given a sequence \((X_i, x_i \in X_i)\) of (locally compact complete) metric spaces with distinguished points, it converges to \((X, x)\) if for any \(R > 0\) the sequence of closed \(R\)-balls, \(B(x_i, R)\), converges to \(B(x, R)\) in \(X\) in \(d_{GH}\) sense.
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Especially useful in uniformly compact spaces.

Set (or sequence) of compact \(\{(X_i, d_i)\}\) are uniformly compact if:

- Diameters \(D_i\) uniformly bounded: \(\exists R \in \mathbb{R} \mid D_i \leq R, \forall X_i\).
- For each \(\varepsilon > 0\), \(X_i\) is coverable by \(N_{\varepsilon} < \infty\) balls of radius \(\varepsilon\) independent of the index \(i\).
Growth

**Growth function** \( \beta(G, v_i, r) \) in a graph \( G \) is the number of vertices in a ball of radius \( r \):

\[
\beta(G, v_i, r) := |B_G(v_i, r)|
\]

\( G \) has **polynomial growth**: \( \beta(G, v_i, r) \lesssim r^\bar{D} \approx Ar^\bar{D} \) for \( \bar{D} \geq 0 \).

\( G \) has **uniform polynomial growth** (uniform polynomial growth):

\[
Ar^d \leq \beta(G, v_i, r) \leq Br^d
\]

and \( A, B, d > 0 \). (for locally finite graph, is indep. of \( v_i \))

**Degree of polynomial growth**:

\[
\bar{D}(G) = \limsup_r \frac{\ln(\beta(G, r))}{\ln(r)}
\]

may not be an integer.
Existence of fixed point: chain of argument

1 Start from a graph $G_0$ with unif. polyn. growth (e.g. Cayley; local. fin. vert. transit.)
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   - 1. Quasi-isometry preserves growth deg.
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3. If $G_0$ has unif. polyn. grow., and $G_i \xrightarrow{\text{quasi-isometry}} G_{i+1}$: The balls $B(x_i, R)$ of a sequence $\{G_i\}$ are uniformly compact.
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4. If $\forall R$ and $\epsilon > 0$, balls $B(x_i, R)$ of a given sequence $\{(X_i, x_i)\}$ are uniformly compact, then (a subsequence of spaces of) $\{(X_i, x_i)\}$ converges in pointed $d_{GH}$ sense.
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4. If $\forall R$ and $\epsilon > 0$, balls $B(x_i, R)$ of a given sequence $\{(X_i, x_i)\}$ are uniformly compact, then (a subsequence of spaces of) $\{(X_i, x_i)\}$ converges in pointed $d_{GH}$ sense.

**Fixed point:** The sequence of $\{G_i\}$ with initial condition $G_0$ of uniform polynomial growth, and $G_i \xrightarrow{\text{quasi-isometry}} G_{i+1}$ (has a subsequence that) converges in pointed $d_{GH}$ sense.
Dimension

- Introduce a candidate.
- Conditions for integer dimension?
- Behavior of dimension under coarse graining?
A first look at dimension: a definition

A few simple observations about dimension:
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Dimension? Limiting behavior of growth function a good candidate: (upper and lower) **Internal scaling dimension**

\[
\overline{D}_s(v_i) := \limsup_{r \to \infty} \frac{\ln(\beta(v_i, r))}{\ln(r)}, \quad \underline{D}_s(v_i) := \liminf_{r \to \infty} \frac{\ln(\beta(v_i, r))}{\ln(r)}
\]

in general different and non-integer.
Stability under coarse graining, integerness

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**Stability of dimension under coarse graining:** Degree of polynomial growth and thus internal scaling dimension, preserved under quasi-isometry: \( G_i \xrightarrow{\text{quasi-isometry}} G_{i+1} \)
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Some properties of internal scaling dimension:

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**Integer dimension**: Vertex transitive graphs with polynomial growth have integer dim
Dimension round up

Under this coarse graining scheme:

- Internal scaling dimension is stable under quasi-isometry (coarse graining)
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Bottom line: starting from a vertex transitive graph $G_0$ with uniform polynomial growth, and coarse graining by $G_i \xrightarrow{\text{quasi-isometry}} G_{i+1}$, all $G_i$ guaranteed to have integer dim.
If

**Initial condition**: $G_0$ vertex transitive graph with uniform polynomial growth (or any graph with uniform polynomial growth or integer dim)
Final word on fixed point and integ. dim

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A fixed point (Hausdorff limit) with an integer dim is guaranteed
Continuum Limit

- An idea how to get a topological/PL manifold out of a graph.
The main steps of the method

**Main theorem:** A $D$-dimensional simplicial complex is a PL-manifold if the link of every vertex in the complex is topologically a $(D - 1)$-sphere [Thurston]

- **Link:** given a vertex $v$ in a simplicial complex, consider the set of all simplices $\sigma_i$ which have $v$ on their boundary; then the link of $v$ is the union of all other simplices on the boundary of those $\sigma_i$ which do not contain $v$. 

Essentially, prove that each point has a neighborhood in the complex that is homeomorphic to a ball.
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   - barycentric decomposition
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**Argument steps:**

1. Given $G$, find the corresponding (Voronoi) cell complex.
2. Produce a simplicial complex that is PL equivalent to this cell complex
   - barycentric decomposition
3. Is link of every vertex in the complex is topologically a $(D - 1)$-sphere?
Brief review of SDCN

- Start from a graph with simple dynamics (more flexible version of cellular automata)
- Excitations 1D edges
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Corase graining (Ising-like): specific type of quasi-isometry, $k$-local edge deletion:

1. In $G_i$, find complete graphs $K_n^{(i)}$. 

![Diagram showing $K_3$, $K_5$, and $K_6$ graphs]
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- Points far w.r.t. smooth spacetime metric may internally be close
  - Entanglement and non-locality?
  - ER=EPR, BH info paradox?
Summary/Future directions

What is done up to now:

- In super space of all metric spaces, there are sequences of graphs that have a fixed point w.r.t. quasi-isometry and $d_{GH}$.
- Starting from a large class of graphs (vertex transitive & uniform polynomial growth), a fixed point (Hausdorff limit) with integer dim is guaranteed.
- Convergence and integer dim are compatible; seemingly go hand in hand.
- Completely background independent, bottom-up and generic (no decomposition of a manifold)
Summary/Future directions

Future direction (a lot!):

- Explore smoothness, metric in fixed point resembles spacetime metric?
- Beyond polynomial growth? exponential growth etc.
- Apply to spin networks/foams. Edge color play a rule? Changes under coarse graining? LQG an effective theory?
- Explore relation to other coarse graining methods
- Expand to more methods of coarse graining? Also use more of Coarse Geometry methods
- Is metric space analysis enough? need to check more structure?
- Connections between graph of groups and Cayley graphs, and spin networks/foams?
- Random graphs
- Less restriction on dim (different fractal dim for each level)?