From discrete to continuum: Lessons from the Gromov-Hausdorff space

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Based on an ongoing work with

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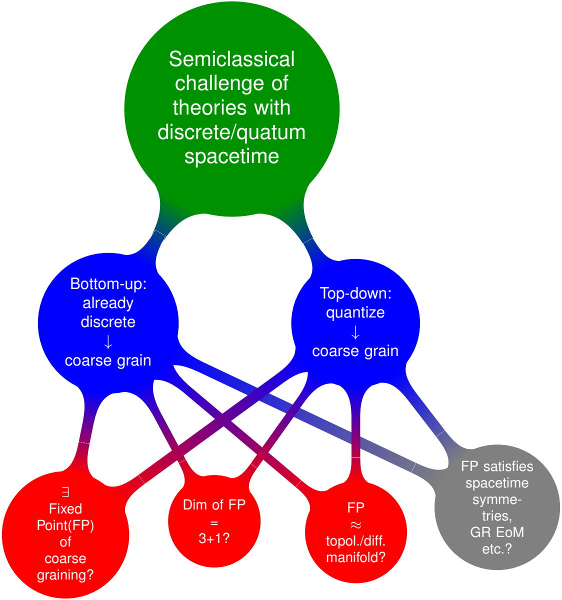
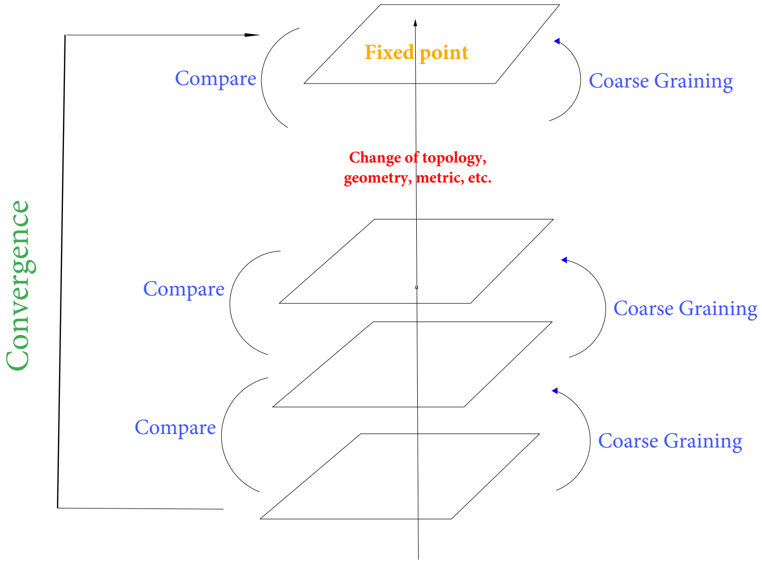
Introduction:

The Birds Eye View Of The Idea

Shared Semiclassical Limit Challenges



Bird's eye view



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2. $\dim(X) = 3(+1)$?
3. X a manifold (at least topological)?

Setting The Stage:
Some Graph Theoretic Definitions

Some basic graph concepts - 1

A **graph** $G = (V, E)$; **vertices** $v_i \in V$; **edges** $e_{ij} \in E \subset V \times V$; if $e_{ij} \neq e_{ji}$ **directed** graph.
Locally bounded valence: valence finite $\forall v_i \in G$
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Path γ: an edge sequence without repetition of vertices, except of initial/terminal vertex.
Connected graph: $\exists \gamma$ for each $v_i, v_j \in G$.
Length of path $l(\gamma)$: number of edges occurring in the path.
Geodesic path: path of minimal length between two vertices

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Ball of radius r centered at $x \in M$ in a metric space (M, d):

$$B(x, r) = \{y \in M | d(x, y) \leq r\}$$

Coarse Graining

- **Provide a scheme of coarse graining.**
- **How to ignore details, so to be able to check if two spaces are coarsely similar?**

A class of coarse grainings based on isometry

Isometric embedding: A distant preserving map between two metric spaces

$$f : (X, d_X) \to (Y, d_Y) \text{ such that } d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

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Quai-isometric embedding: A map f from a metric space,
$f : (M_1, d_1) \to (M_2, d_2)$, , i.e. if there exist constants, $\lambda \geq 1, \epsilon \geq 0$, such that

$$\forall x, y \in M_1 : \frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon,$$

i.e. $\forall x, y \in M_1$, the distance between their images is (up to the additive constant ϵ) within a factor of λ of their original distance.

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Quasi-isomery: a quasi-isometric embedding f in which

$$\forall z \in M_2 \ \wedge \ \exists x \in M_1 : d_2(z, f(x)) \leq C.$$

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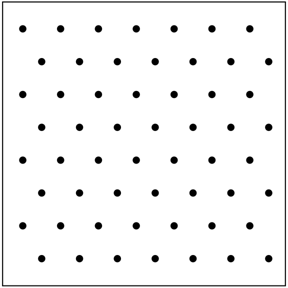
Quasi-isometry is an equivalence relation **on metric spaces** that **ignores their small-scale details in favor of their coarse structure**.

Quasi-isometry example:

Equilateral triangle lattice



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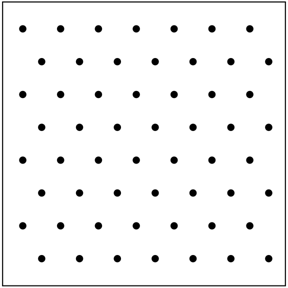

The integer lattice $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$:

$$\begin{aligned} f :& \mathbb{Z}^n \to \mathbb{R}^n \\ :& (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n), \qquad x_i \in \mathbb{Z} \end{aligned}$$

1. Distances are preserved.
2. n-tuples $\in \mathbb{R}^n$ are within $\sqrt{\frac{n}{4}}$ of n-tuples $\in \mathbb{Z}^n$.

Comparison And Fixed Point

- **Construct super metric space.**
- **Provide a measure of comparing different (coarse grained) spaces.**
- **A sequence of spaces are coarsely similar? Converge to a fixed point?**

d_H: Distance of metric subspaces

In (M, d), **Hausdorff Distance** $d_H(X, Y)$ of $X, Y \subset M \ \wedge\ X, Y \neq \emptyset$:

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ϵ-neighborhood of a subset

In (M, d), ϵ-neighborhood of $X \subset M$: union of all ϵ-balls around all $x \in X$

$$U_\epsilon(X) = \bigcup_{x \in X} \{z \in M | d(z, x) \leq \epsilon\}$$

All points within ϵ of the set X, or generalized ball of radius ϵ around X.

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- Make them subspace of another metric space.

d_{GH}: Distance of metric spaces, convergence

The **Gromov-Hausdorff Distance** d_{GH} of compact $(X, d_X), (Y, d_Y)$

$$d_{GH}(X, Y) = \inf d_H^Z(f(X), g(Y))$$

of all metric spaces Z and isometric embeddings $f : X \to Z$, $g : Y \to Z$

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How to deal with (i.e. define convergence for) non-compact spaces? Pointed convergence...

Pointed convergence, uniform compactness

Pointed convergence: Given a sequence $(X_i, x_i \in X_i)$ of (locally compact complete) metric spaces with distinguished points, it converges to (X, x) if for any $R > 0$ the sequence of closed R-balls, $B(x_i, R)$, converges to $B(x, R)$ in X in d_{GH} sense.

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Set (or sequence) of compact $\{(X_i, d_i)\}$ are uniformly compact if:

- Diameters D_i uniformly bounded: $\exists R \in \mathbb{R} | \, D_i \leq R, \, \forall X_i$.
- For each $\epsilon > 0$, X_i is coverable by $N_\epsilon < \infty$ balls of radius ϵ independent of the index i.

Growth

Growth function $\beta(G, v_i, r)$ in a graph G is the number of vertices in a ball of radius r:

$$\beta(G, v_i, r) := |B_G(v_i, r)|$$

G has **polynomial growth**: $\beta(G, v_i, r) \lesssim r^{\bar{D}} \approx Ar^{\bar{D}}$ for $\bar{D} \geq 0$.
G has **uniform polynomial growth** (uniform polynomial growth):

$$Ar^d \leq \beta(G, v_i, r) \leq Br^d$$

and $A, B, d > 0$. (for locally finite graph, is indep. of v_i)
Degree of polynomial growth:

$$\bar{D}(G) = \limsup_r \frac{\ln(\beta(G, r))}{\ln(r)}$$

may not be an integer.

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Fixed point: The sequence of $\{G_i\}$ with initial condition G_0 of uniform polynomial growth, and $G_i \stackrel{\text{quasi-isometry}}{\longrightarrow} G_{i+1}$ (has a subsequence that) converges in pointed d_{GH} sense.

Dimension

- **Introduce a candidate.**
- **Conditions for integer dimension?**
- **Behavior of dimension under coarse graining?**

A first look at dimension: a definition

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Dimension? Limiting behavior of growth function a good candidate: (upper and lower) **Internal scaling dimension**

$$\overline{D}_s(v_i) := \limsup_{r\to\infty} \frac{\ln\left(\beta\left(v_i, r\right)\right)}{\ln(r)}\,, \qquad \underline{D}_s(v_i) := \liminf_{r\to\infty} \frac{\ln\left(\beta\left(v_i, r\right)\right)}{\ln(r)}$$

in general different and non-integer.

Stability under coarse graining, integerness

Some properties of internal scaling dimension:

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- **Integer dimension**: Vertex transitive graphs with polynomial growth have integer dim

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Bottom line: starting from a vertex transitive graph G_0 with uniform polynomial growth, and coarse graining by $G_i \xrightarrow{\text{quasi-isometry}} G_{i+1}$, all G_i guaranteed to have integer dim.

Final word on fixed point and integ. dim

If

1. **Initial condition**: G_0 vertex transitive graph with uniform polynomial growth (or any graph with uniform polynomial growth or integer dim)

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A fixed point (Hausdorff limit) with an integer dim is guaranteed

Continuum Limit

- **An idea how to get a topological/PL manifold out of a graph.**

The main steps of the method

- **Main theorem**: A D-dimensional simplicial complex is a PL-manifold if the link of every vertex in the complex is topologically a $(D-1)$-sphere [Thurston]
  - Link: given a vertex v in a simplicial complex, consider the set of all simplices σ_i which have v on their boundary; then the link of v is the union of all other simplices on the boundary of those σ_i which do not contain v.

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Argument steps:

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   1. barycentric decomposition
3. Is link of every vertex in the complex is topologically a $(D-1)$-sphere?

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- Excitations 1D edges

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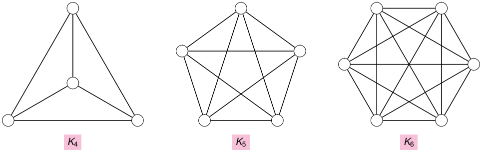

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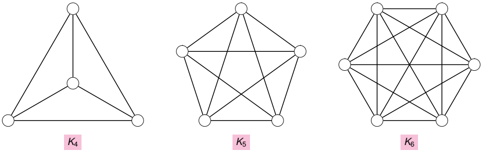

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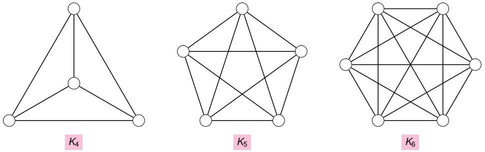

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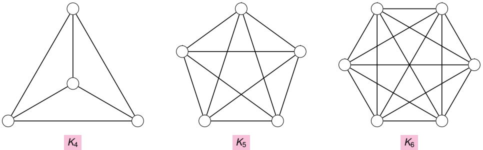

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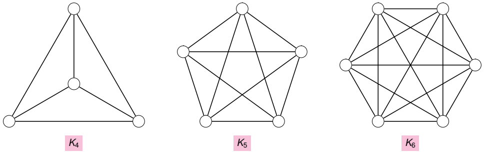

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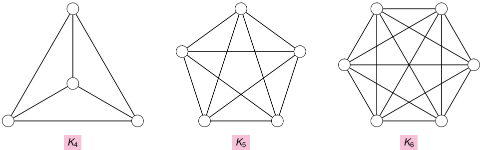

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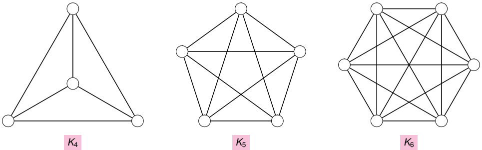

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- Points far w.r.t. smooth spacetime metric may internally be close
    - Entanglement and non-locality?
    - ER=EPR, BH info paradox?

Summary/Future directions

What is done up to now:

- In super space of all metric spaces, there are sequences of graphs that have a fixed point w.r.t. quasi-isometry and d_{GH}.
- Starting from a large class of graphs (vertex transitive & uniform polynomial growth), a fixed point (Hausdorff limit) with integer dim is guaranteed.
- Convergence and integer dim are compatible; seemingly go hand in hand
- Completely background independent, bottom-up and generic (no decomposition of a manifold)

Summary/Future directions

Future direction (a lot!):

- Explore smoothness, metric in fixed point resembles spacetime metric?
- Beyond polynomial growth? exponential growth etc.
- Apply to spin networks/foams. Edge color play a rule? Changes under coarse graining? LQG an effective theory?
- Explore relation to other coarse graining methods
- Expand to more methods of coarse graining? Also use more of Coarse Geometry methods
- Is metric space analysis enough? need to check more structure?
- Connections between graph of groups and Cayley graphs, and spin networks/foams?
- Random graphs
- Less restriction on dim (different fractal dim for each level)?