Emergent continuous spacetime via a geometric renormalization method

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In collaboration with
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Essense of Renormalization - 1

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  - F at equilibrium: the statistical weight of $\{s_i\}$
  - F out of equilibrium: evolution of configuration $\{s_i\}$ (e.g. Hamiltonian)

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Different schemes of renormalization, e.g.

- Kadanoff: renormalisation within a given ensemble of parameters K,

$$F_k \to F_{k'} \text{ with renormalized parameters } K' = R_b(K)$$

- RG: functional ensemble $\mathcal{F} = \{F\}$, where $R_b : \mathcal{F} \to \mathcal{F}$ changes form of F

$$F \to F' = R_b(\phi)$$

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- This succession of models: trajectories in SPM, renormalization flow

Renormalization as symmetry: Covariance and Invariance

Covariance under renormalization:

- Physical reality not altered if change manner of observing it
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  - RG: symmetry group leaves the system under study invariant

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- Invariance by renormalization:
  - Property A remains invariant: $A = Ren_b(A)$.
    - Selfsimilarity of A.
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- Part of $C(M^*)$ converging (flowing) to M^*: basin of attraction of M^*.

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Graph distance

$d(v_i, v_j)$= Min # of edges connecting v_i, v_j.

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Why a graph?

G_i: network of elementary entities (vertices) interacting (edges)

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    1. Coarse graining $\mathcal{K} : (G_i, d_i) \to (G_{i+1}, d_{i+1})$ where $\mathcal{K}^j(G_0) = G_j$

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5. Upper chain: continuum limit chain

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Two flows

Two parts of the model:

1. Horizontal discrete SPM
2. Vertical rescaling flow

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- End in a fixed point, or a set of accumulation points (non-generic scenario)

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$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon, \qquad \forall x_1, x_2 \in X$$

and

$$\forall y \in Y : \ \exists x \in X : \ d_Y(y, f(x)) \leq C.$$

is a quasi-isometry, i.e.

- distance of the images under f, within a factor λ, and up to a constant, of their original distances, and
- every point $y \in Y$ lies within a constant distance $C \geq 0$ of an image point.

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Coarse graining

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1. Quai-isometry, or
2. Rough isometry

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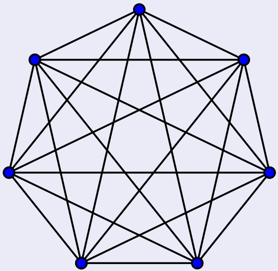
Clique

Clique: a complete subgraph (maximally connected). Here $K7$:



Examples of $\mathcal{K}$

Pure quasi-isometry: k-local insertion/deletion of edges – preserves k-neighborhoods of vertices.

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- Alternative: edge if $v_i^{\mathcal{C}}, v_j^{\mathcal{C}} \in V(\mathcal{C}(G))$ have minimal internal # of edge between them in G.

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- In a metric space (M, d), Hausdorff distance $d_H(X, Y)$ of $X, Y \subset M \ \wedge X, Y \neq \emptyset$:

$$d_H(X, Y) = \inf \{\epsilon \geq 0 | X \subseteq U_\epsilon(Y), Y \subseteq U_\epsilon(X)\}$$

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- The Gromov-Hausdorff distance d_{GH} of two compact metric spaces $(X, d_X), (Y, d_Y)$

$$d_{GH}(X, Y) = \inf \, d_H^Z(f(X), g(Y))$$

  of all metric spaces Z and all isometric embeddings $f : X \to Z$, $g : Y \to Z$.

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Convergence criteria can used horizontally (for coarse graining sequence), or vertically (e.g. existence of a continuum limit)

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Convergence and uniform compactness

Theorem: If for all r and $\epsilon > 0$ the balls $B(x_i, r)$ of a given sequence of proper metric spaces $\{(X_i, x_i \in X_i)\}$ are uniformly compact, then a subsequence of spaces converges in the pointed GH sense.

Set (or sequence) of compact $\{(X_i, d_i)\}$ are uniformly compact if:

- Diameters $\mathrm{diam}(X_i) = \sup\{d_i(x, y) : x_i, y_i \in X_i\}$ are uniformly bounded: $\exists R \in \mathbb{R} | \mathrm{diam}(X_i) \leq R, \forall X_i$.
- For each $\epsilon > 0$, X_i is coverable by $N_\epsilon < \infty$ balls of radius ϵ independent of the index i.

Rescaling ϕ_λ and Continuum Limit Properties

Rescaling map:

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- $\lim_{\lambda \to 0} \phi_\lambda$ corresponds to the large scale structure of X:

$$\lim_{\lambda \to 0} \phi_\lambda \left((X, d_X) \right) = \lim_{\lambda \to 0} (X, \lambda d_X) = \left(X_\infty, d_{X,\infty} \right),$$

important for us!

Some Properties of the Continuum Limit

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$$Ar^d \leq \beta(G, v_i, r) \leq Br^d$$

and $A, B, d > 0$. (for locally finite graph, is independent of v_i)

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Full Picture and Summary

Combination of two operations

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\begin{array}{ccccccccc}
[G_0]_\infty & & [\mathcal{K}(G_0)]_\infty & & [\mathcal{K}^2(G_0)]_\infty & & [\ldots]_\infty & & [\mathcal{K}^n(G_0)]_\infty \\
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- Implication: different levels of spacetime have different distance functions, even if they are the same set (entanglement explanation? ER=EPR?)

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2. Coarse graining chain reaches a stable fixed point/set of accumulation points
   1. If the spaces are uniformly compact, (Gromov's compactness theorem). Not generic

Brief Comments on Dimension

For graphs with polynomial growth, a dimension is

$$D(G) = \lim_{r \to \infty} \frac{\log \beta(G, v_i, r)}{\log r}$$

coincides with

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- D stable under quasi-isometry $\mathcal{K}$
- Change of D under renormalization? use not quasi-isometric $\mathcal{K}$, but translocal, i.e. change k-neighborhoods

Future Directions

Future: color, dynamics, connecting to other approaches, ...

- Add color (internal DoF of vertices and edges)
- Color makes it possible to introduce dynamics
- Makes it possible to connect with LQG, etc.
- Makes it possible to connect with coarse graining methods of using projective Hilbert space
- Emergent color? Emergent symmetries?
- How it affects distance?

More details: "*Emergent Space-Time via a Geometric Renormalization Method*", SR, M. Requardt, Phys. Rev. D 94, 124019 (2016).

Rescaling ϕ_λ and continuum limit properties

Take lattice $\mathbb{Z}^n$ embedded in $\mathbb{R}^n$, take the scaling limit

$$\phi_l : (\mathbb{Z}^n, d_{\mathbb{Z}^n}) \longmapsto (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}) \,, \qquad \lambda = 2^{-l}$$

with $d_{\mathbb{Z}^n}$ a suitable metric on $\mathbb{Z}^n$. Then

$$\lim_{\lambda \to 0} (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}) = \mathbb{R}^n,$$

in pointed GH-sense. For a fixed ball around $x = 0$, and for $l \to \infty$ the ball is more and more filled with points stemming from lattices having edge length 2^{-l}.