Linking loop quantum gravity quantization ambiguities with phenomenology

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Introduction

We derive modified dispersion relations (MDR) from effective loop quantum gravity:

- Loop Quantum Gravity (LQG) [1,2] non-perturbative (background independent) quantization of gravity, almost complete formalism, in-principle valid for all regimes, BUT very difficult to extract testable predictions!
- Deformed Poincaré symmetries / Spacetime Noncommutativity [3,4] way to characterize a non-classical spacetime, maybe a first step toward QG, confined to the Minkowski (flat) limit, BUT there is a phenomenology[5]!

Still no direct link between them!

[1] C. Rovelli, Living Rev. Rel. 1, (1998) 1.

[2] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53-R152 (2004).

[3] H. S. Snyder, Phys. Rev. 71, 38 (1947)

[4] S. Doplicher, K. Fredenhagen, J. E. Roberts, Phys. Lett. B331, 39 (1994)

[5] G. Amelino-Camelia, Living Rev. Rel. 16, (2013) 5.

Motivation

No clear link between Spacetime Noncommutativity and QG:

- Just heuristic arguments but no rigorous derivation
- Some hints from Strings[6] and 2+1 gravity[7]
- What about LQG?

Minkowski regime of LQG poorly understood:

- Quantum spacetime? Noncommutative?
- Fate of Lorentz symmetries? Exact, broken or deformed?
- Experimental tests?

To answer such questions we must look at the SYMMETRIES: how are they encoded?

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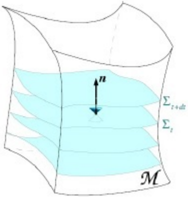
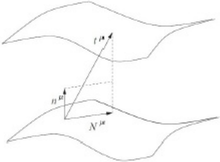
[6] N. Seiberg, E. Witten, JHEP 09, 032 (1999).

[7] L. Freidel, E. R. Livine, Phys. Rev. Lett. 96, 221301 (2006).

Hamiltonian formulation of General Relativity (ADM)

..let us start from the classical theory!
Based on the 3+1 foliation of spacetime[8]





- $g_{\mu\nu}(x) \leftrightarrow h_{ij}(x), N^k(x), N(x)$
- Phase-space variables: $\{\pi^{ij}(x), h_{kl}(y)\} = -\frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k)\delta^{(3)}(x-y)$

Does it break general covariance?..of course NOT!
(just not manifest..)

[8] R. L. Arnowitt, S. Deser, C. W. Misner, Gen. Rel. Grav. 40, 1997 (2008).

Constraints

Diffeomorphism invariance is implemented by means of constraints:

$$H[N] = \int d^3x \quad N(x)(\frac{\pi_{lk}\pi^{lk}}{\sqrt{-h}} - \frac{\pi^2}{2\sqrt{-h}} -^3 R\sqrt{-h})$$

$$D[N^k] = -2\int d^3x \quad N^k(x)h_{kj}(x)D_l\pi^{lj}(x)$$

the Hojman-Kuchař-Teitelboim theorem tells they are the only possible generators of the hypersurface-deformation algebra (HDA) [9]:

$$\{D[N^i], D[N'^j]\} = D[N'^j\partial_j N^i - N^j\partial_j N'^i]$$
$$\{D[N^i], H[N']\} = H[N^j\partial_j N']$$
$$\{H[N], H[N']\} = D[h^{ij}(N\partial_j N' - N'\partial_j N)]$$

In which sense does the HDA encode the principle of general covariance?

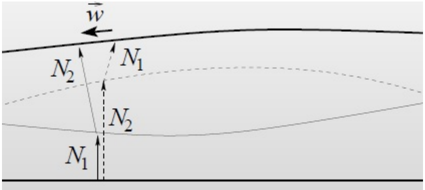
[9] P. A.M. Dirac, Proc. Roy. Soc. Lond. A246, 333 (1958).

Hypersurface deformations

Hypersurface-deformation algebra ensures that theory respects:

1. gauge transformations for coordinate changes:
   $\{f, H[N] + D[N^k]\} = \delta f$
2. slicing independence: algebra amounts to deformations of hypersurfaces

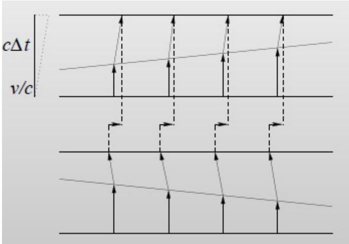
Non-linear coordinate changes translates into non-linear deformations of space (example: $\{H, H\} = D$).



What if we restrict to linear deformations?

Minkowski limit

If we restrict to linear coordinate changes of flat slices $h_{ij} = \delta_{ij}$:



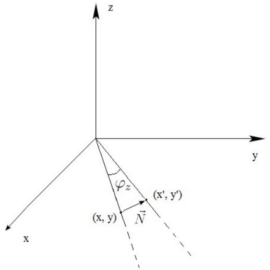
that means choosing:

$$N^k(x) = \delta^k + \epsilon^{kij}\varphi_i x_j, \quad N(x) = \delta + \alpha_i x^i$$

which are the Killing vectors of the Minkowski spacetime.
Then we show that the Poincaré symmetries are contained into the HDA.

Rotations from spatial diffeomorphims of flat slices

Let us see how are rotations encoded if $h_{ij} = \delta_{ij}$:



It is generated by the momentum constraint $D[N^k]$ with $N^k = \epsilon^{k3j}\varphi_3 x_j$:

$$x'_3 = x_3, \quad x'_1 = x_1 - \varphi_3 x_2, \quad x'_2 = x_2 + \varphi_3 x_1$$

Poincaré algebra from linear hypersurface deformations

Let us start from $\{D[N^l], D[M^j]\} = D[\mathcal{L}_{N^i} M^j]$ with $N^i = \epsilon^{ilk}\varphi_{l1}x_k$ and $M^j = \epsilon^{jmn}\varphi_{m2}x_n$ and compute $\mathcal{L}_{N^i} M^j$:

$$\begin{aligned}\mathcal{L}_{N^i} M^j = N^i\partial_i M^j - M^i\partial_i N^j = \epsilon^{ilk}\varphi_{l1}x_k\epsilon^{jmn}\varphi_{m2}\delta_{ni} + \\ -\epsilon^{imn}\varphi_{m2}x_n\epsilon^{jlk}\varphi_{l1}\delta_{ki} = (\delta_{lj}\delta_{km} - \delta_{lm}\delta_{kj})\varphi_{l1}\varphi_{m2}x_k + \\ -(\delta_{mj}\delta_{nl} - \delta_{ml}\delta_{nj})\varphi_{l1}\varphi_{m2}x_n = \varphi_{j1}\varphi_{k2}x_k - \varphi_{l1}\varphi_{j2}x_l = \\ = -\epsilon^{jlk}\epsilon_{lts}\varphi_{t1}\varphi_{s2}x_k = -\epsilon^{jlk}\varphi_{l3}x_k\end{aligned}$$

thus we have that the Poisson bracket between two generators of rotations gives another rotation i.e. $\{J_l, J_j\} = \epsilon_{ljk}J_k$.
Similar arguments lead to identify the other Poisson brackets of the Poincaré algebra.

Poincaré algebra from linear hypersurface deformations

In particular $N = \delta$ corresponds to time translations, $N = \alpha_i x^i$ to boosts and $N^j = \delta^j$ to spatial translations.
Thus in the Minkowski limit the HDA reduces to the Poincaré algebra[10]:

$$\{P_\mu, P_\nu\} = 0 \quad \{M_{\mu\nu}, P_\rho\} = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu$$
$$\{M_{\mu\nu}, M_{\rho\sigma}\} = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}$$

Is the HDA affected by quantum effects? [11] If so, is there a corresponding deformation of the Poincaré algebra? [12]

[10] T. Regge, C. Teitelboim, Annals Phys. 88, 286 (1974). M. Bojowald, Canonical Gravity and Applications (2010).
[11] A. Barrau, M. Bojowald, G. Calcagni, J. Grain, M. Kagan, JCAP 1505 (2015) 051.
[12] M. Bojowald, G. M. Paily, Phys. Rev. D87, 044044 (2013).

Loop quantization

First step is Ashtekar's formulation[13] (use $SU(2)$ fields!):

$$(h_{ij}, \pi^{ij}) \longrightarrow (A_i^a, E_a^i)$$

where:

$$A_i^a = \Gamma_i^a + \gamma K_i^a$$
$$E_a^i = \sqrt{det(h)} e_a^i, \quad h^{ij} = e_a^i e_b^j \delta_{ab}$$

To quantize the theory one must turn to[14]

1. *Holonomies*: $h_e(A) = exp(\int_e ds A_i^a \dot{e}^i \tau_a) \Rightarrow$ holonomy corrections!
2. *Fluxes*: $F_S(E) = \int_S d^2 y n_i E_a^i \Rightarrow$ inverse-triad corrections!

$\hookrightarrow$ Do they affect the spacetime structure?

[13] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).

[14] C. Rovelli and L. Smolin, Nucl. Phys. B331, 80 (1990).

Weak closure VS strong closure

Canonical quantum gravity:

- $H, D, G \rightarrow \widehat{H}, \widehat{D}, \widehat{G}$
- $\mathcal{H}_{kin} = ker(\widehat{D}, \widehat{G})$
- $\mathcal{H}_{phys} = ker(\widehat{H}, \widehat{D}, \widehat{G})$

weak closure

$\langle\Psi_{kin}| [\widehat{H}, \widehat{H}] |\Psi_{kin}\rangle \approx 0$ [Thiemann]

strong closure

$[\widehat{H}, \widehat{H}] = \widehat{D}$ $\Rightarrow$ not yet available!

$\Rightarrow$ off-shell HDA needed to assure anomaly freedom!!

Strategy

use effective methods: $[\widehat{H}, \widehat{H}] \rightarrow \{H^Q, H^Q\}$

Spherically symmetric phase space in a nutshell

Introduce rotationally invariant densitized triads[15] by solving the equation $\mathcal{L}_{L_b} E^i_a = -[T_b, E^i_a] = -\epsilon_{abc}\Lambda_b E^i_a$:

$$E = E^i_a \tau^a \frac{\partial}{\partial x^i} = E^r(r)\tau_3 \sin\theta \frac{\partial}{\partial r} + E^\phi(r)\tau_1 \sin\theta \frac{\partial}{\partial \theta} + E^\phi(r)\tau_2 \frac{\partial}{\partial \phi},$$

which are conjugate to:

$$K = K^a_i \tau_a dx^i = K_r(r)\tau_3 dr + K_\phi(r)\tau_1 d\theta + K_\phi(r)\tau_2 \sin\theta d\phi$$

In fact the symplectic structure is:

$$\{K_r(r), E^r(r')\} = 2G\delta(r - r'), \quad \{K_\phi(r), E^\phi(r')\} = G\delta(r - r')$$

The scalar constraint is then given by:

$$H[N] = -\frac{1}{2G}\int_B drN[K_\phi^2 E^\phi + 2K_r K_\phi E^r + (1 - \Gamma_\phi^2)E^\phi + 2\Gamma'_\phi E^r]$$

[15] I. Bengtsson, Class. Quantum Grav. 8, 1847 (1991).

Deformed hypersurface deformations: spherical symmetry

Considering only the point-wise holonomy of the homogeneous connections:

$$K_\phi \to \frac{\sin(\delta K_\phi)}{\delta}$$

requiring anomaly freedom (quantization preserves degrees of freedom):

$$\{D[N^r], D[N'^r]\} = D[N^r \partial_r N'^r - N'^r \partial_r N^r]$$
$$\{D[N^r], H^Q[N']\} = H^Q[N^r \partial_r N']$$
$$\{H^Q[N], H^Q[N']\} = D[\cos(2\delta K_\phi) \frac{E^r}{(E^\phi)^2} (N \partial_r N' - N' \partial_r N)]$$

Covariance is NOT violated, BUT deformed! $\rightarrow$ correspondingly deformed relativistic symmetries? consequences for spacetime?

Deformed Poincaré algebra: a bridge to noncommutativity?

Restricting to linear hypersurface deformations[16]:

$$[B_r, P_0] = iP_r\cos(\lambda P_r) \quad [B_r, P_r] = iP_0 \quad [P_r, P_0] = 0$$

thanks to the relation $\lambda P_r = -\frac{1}{G}\frac{K_\phi}{\sqrt{E^r}} = 2\delta K_\phi$ ($\lambda \approx 10^{-35}m$) between K_ϕ and the Brown-York (ADM) momentum:

$$P = 2\int_{\partial\Sigma} d^2 z \upsilon_i (n_j \pi^{ji} - \overline{n}_j \overline{\pi}^{ji})$$

The Minkowski limit of the HDA with quantum corrections produces a deformation of the Poincaré algebra! (see works by Mielczarek, Trześniewski)

⇒ Deformed Special Relativity[17] (DSR) derived from LQG!

⇒ Which is the underlying quantum Minkowski spacetime?

[16] M. Bojowald, G. M. Paily, Phys. Rev. D87, 044044 (2013).

[17] G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35 (2002).

Proposal: κ-Minkowski

κ-Minkowski noncommutative spacetime[18] is defined by:

$$[\widehat{X}_r, \widehat{X}_0] = -i\lambda \widehat{X}_r$$

To prove that a deformed Poincaré algebra generates its symmetries we need to:

- check the fulfilment of all the Jacobi identities
- compute the coproducts: ΔB_r, ΔP_r and ΔP_0

Non-linear deformations of the Poincaré algebra are Hopf (rather than Lie) algebraic structures!

WARNING: violation of Leibniz's rule: $G \triangleright (fg) \neq (G \triangleright f)g + f(G \triangleright g)$ which must involve only the generators of the Hopf algebra: $\{B_r, P_r, P_0\}$

[18] G. Amelino-Camelia and S. Majid, Int. J. Mod. Phys. A15, 4301 (2000).

Representations

We propose a general *ansatz* for the representations:

$$B_r = F(p_0, p_r)\widehat{X}_r p_0 - G(p_0, p_r)\widehat{X}_0 p_r, \quad P_r = Z(p_r), \quad P_0 = p_0$$

where p_r, p_0 are standard momenta that act on κ-Minkowski coordinates as follows:

$$[p_r, \widehat{X}_0] = i\lambda p_r, \quad [p_r, \widehat{X}_r] = -i, \quad [p_0, \widehat{X}_0] = i, \quad [p_0, \widehat{X}_r] = 0$$

We find that $F(p_0, p_r), G(p_0, p_r), Z(p_r)$ must obey:

$$\lambda Z(p_r)\sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 {p_r}^2}{2} + 1,$$

$$F(p_0, p_r) = G(p_r)e^{\lambda p_0} = \frac{Z(p_r)\cos(\lambda Z(p_r))e^{\lambda p_0}}{p_r},$$

$$G(p_r) = \frac{Z(p_r)\cos(\lambda Z(p_r))}{p_r}$$

Jacobi identities

Then we verify that all the Jacobi identities involving $\{\widehat{X}_r, \widehat{X}_0, B_r, P_r, P_0\}$ are satisfied.

By a way of example we have that:

$$
\begin{aligned}
&[[B_r, \widehat{X}_r], \widehat{X}_0] + [[\widehat{X}_0, B_r], \widehat{X}_r] + [[\widehat{X}_r, \widehat{X}_0], B_r] = \\
&-i[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_r p_0, \widehat{X}_0] + \\
&+i[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_0 p_r, \widehat{X}_0] + \\
&-[i\frac{Z\cos(\lambda Z)}{p_r} x_r - \lambda[B_r, \widehat{X}_r]p_r - i\lambda \frac{Z\cos(\lambda Z)}{p_r} x_r p_0, \widehat{X}_r] + \\
&+[i\frac{Z\cos(\lambda Z)}{p_r} x_0, \widehat{X}_0] + i\lambda[B_r, \widehat{X}_r] = 0
\end{aligned}
$$

Perturbative explicit solution

We are not able to solve analytically the equation for $Z(p_r)$:

$$\lambda Z(p_r)\sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 {p_r}^2}{2} + 1,$$

Thus we provide a perturbative solution up to the quartic order:

$$Z(p_r) \simeq p_r + \frac{1}{8}\lambda^2 {p_r}^3 + \frac{55}{1152}\lambda^4 {p_r}^5 + \bigcirc(\lambda^5)$$

This allow us to calculate also the coproducts of the generators!

Coproducts

A way to compute coproducts is to act with generators on the product of two plane waves:

$$G \triangleright (e^{ik_r\widehat{X}_r} e^{ik_0\widehat{X}_0} e^{iq_r\widehat{X}_r} e^{iq_0\widehat{X}_0}) = G \triangleright (e^{i(k_r + e^{-\lambda k_0} q_r)\widehat{X}_r} e^{i(k_0+q_0)\widehat{X}_0})$$

We find for the boost:

$$\begin{aligned}
\Delta B_r = &\ B_r \otimes \mathbf{1} + \mathbf{1} \otimes B_r - \lambda P_0 \otimes B_r + \frac{1}{8}\lambda^2 {P_r}^2 \otimes B_r + \frac{1}{2}\lambda^2 {P_0}^2 \otimes B_r - \frac{3}{8}\lambda^2 B_r \otimes {P_r}^2 - \frac{3}{4}\lambda^2 P_r B_r \otimes P_r \\
& - \frac{3}{4}\lambda^2 P_r \otimes P_r B_r - \frac{5}{8}\lambda^3 P_0 {P_r}^2 \otimes B_r + \frac{3}{4}\lambda^3 P_0 P_r \otimes P_r B_r - \frac{3}{4}\lambda^3 {P_r}^2 B_r \otimes P_0 - \frac{3}{4}\lambda^3 {P_r}^2 \otimes P_0 B_r \\
& - \frac{3}{4}\lambda^3 P_r B_r \otimes P_0 P_r - \frac{3}{4}\lambda^3 P_r \otimes P_0 P_r B_r + \frac{67}{1152}\lambda^4 P_r^4 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 \otimes {P_r}^2 B_r - \frac{1}{8}\lambda^4 {P_0}^4 \otimes B_r \\
& + \frac{9}{16}\lambda^4 {P_0}^2 {P_r}^2 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 B_r \otimes {P_r}^2 - \frac{167}{288}\lambda^4 {P_r}^3 B_r \otimes P_r - \frac{59}{288}\lambda^4 P_r \otimes {P_r}^3 B_r \\
& - \frac{97}{144}\lambda^4 {P_r}^3 \otimes P_r B_r - \frac{3}{8}\lambda^4 {P_0}^2 P_r \otimes P_r B_r + \frac{3}{4}\lambda^4 P_0 {P_r}^2 \otimes P_0 B_r + \frac{3}{4}\lambda^4 P_0 P_r \otimes P_0 P_r B_r \\
& + \frac{11}{144}\lambda^4 P_r B_r \otimes {P_r}^3 - \frac{3}{4}\lambda^4 {P_r}^2 B_r \otimes {P_0}^2 - \frac{3}{4}\lambda^4 {P_r}^2 \otimes {P_0}^2 B_r - \frac{5}{1152}\lambda^4 B_r \otimes {P_r}^4 \\
& - \frac{3}{8}\lambda^4 P_r B_r \otimes {P_0}^2 P_r - \frac{3}{8}\lambda^4 P_r \otimes {P_0}^2 P_r B_r
\end{aligned}$$

Coproducts

The coproduct of the momentum is:

$$\Delta P_r = P_r \otimes 1 + 1 \otimes P_r + \lambda P_r \otimes P_0 + \frac{1}{2}\lambda^2 P_r \otimes {P_0}^2 - \frac{1}{8}\lambda^2 Pr \otimes {P_r}^2 + \frac{3}{8}\lambda^2 {P_r}^2 \otimes P_r + \frac{1}{4}\lambda^3 {P_r}^3 \otimes P_0$$
$$+\frac{3}{8}\lambda^3 P_r \otimes P_0 {P_r}^2 + \frac{3}{4}\lambda^3 {P_r}^2 \otimes P_0 P_r + \frac{1}{2}\lambda^4 {P_r}^3 \otimes {P_0}^2 - \frac{49}{1152}\lambda^4 P_r \otimes {P_r}^4 + \frac{11}{36}\lambda^4 {P_r}^3 \otimes {P_r}^2$$
$$-\frac{1}{8}\lambda^4 P_r \otimes {P_0}^4 + \frac{7}{16}\lambda^4 P_r \otimes {P_0}^2 {P_r}^2 + \frac{1}{18}\lambda^4 {P_r}^2 \otimes {P_r}^3 + \frac{167}{1152}\lambda^4 {P_r}^4 \otimes P_r + \frac{3}{4}\lambda^4 {P_r}^2 \otimes {P_0}^2 P_r$$

The coproduct of the energy is:

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 + \lambda P_r \otimes P_r + \frac{1}{2}\lambda^2 P_0 \otimes {P_0}^2 + \frac{1}{2}\lambda^2 {P_0}^2 \otimes P_0 - \frac{1}{2}\lambda^2 P_0 \otimes {P_r}^2 - \lambda^2 P_0 P_r \otimes P_r$$
$$+\frac{1}{2}\lambda^2 {P_r}^2 \otimes P_0 - \frac{1}{8}\lambda^3 P_r \otimes {P_r}^3 + \frac{3}{8}\lambda^3 {P_r}^3 \otimes P_r + \frac{1}{2}\lambda^3 {P_0}^2 P_r \otimes P_r - \lambda^3 P_0 {P_r}^2 \otimes P_0 + \frac{1}{4}\lambda^4 {P_r}^4 \otimes P_0$$
$$-\frac{1}{8}\lambda^4 P_0 \otimes {P_0}^4 - \frac{1}{8}\lambda^4 {P_0}^4 \otimes P_0 + \frac{1}{8}\lambda^4 P_0 P_r \otimes {P_r}^3 + \frac{1}{4}\lambda^4 P_0 \otimes {P_0}^2 {P_r}^2$$
$$+\frac{3}{4}\lambda^4 {P_0}^2 {P_r}^2 \otimes P_0 - \frac{7}{8}\lambda^4 P_0 {P_r}^3 \otimes P_r$$

B_r, P_r, P_0 generate the relativistic symmetries of κ-Minkowski!

κ-Poincaré: bicrossproduct basis

Most used basis[19,20] defined as:

$$[M_r, Q_r] = i\frac{1-e^{-2\lambda Q_0}}{2\lambda} - i\frac{\lambda}{2}Q_r^2 \quad [M_r, Q_0] = iQ_r \quad [Q_0, Q_r] = 0$$

with coproducts given by:

$$\Delta M_r = M_r \otimes 1 + e^{-\lambda Q_0} \otimes M_r$$
$$\Delta Q_0 = Q_0 \otimes 1 + 1 \otimes Q_0$$
$$\Delta Q_r = Q_r \otimes 1 + e^{-\lambda Q_0} \otimes Q_r$$

The generators $\{B_r, P_r, P_0\}$ satisfy different commutation relations! $\Rightarrow$ However different bases of the SAME Hopf algebra can be related through NON-LINEAR maps! If so, this guarantees they act on the SAME noncommutative spacetime!

[19] S. Majid, H. Ruegg, Phys. Lett. B334 (1994) 348-354.

[20] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. 243 (1995) 90.

New basis: compatibility with κ-Minkowski

The desired map from $\{M_r, Q_r, Q_0\}$ to $\{B_r, P_r, P_0\}$ is simply:

$$B_r = \frac{Z(Q_r e^{\lambda Q_0}) \cos \lambda Z(Q_r e^{\lambda Q_0})}{Q_r e^{\lambda Q_0}} M_r \quad P_r = Z(\lambda Q_r e^{\lambda Q_0})$$

$$P_0 = \frac{\sinh(\lambda Q_0)}{\lambda} + \frac{\lambda}{2} Q_r^2 e^{\lambda Q_0}$$

where:

$$\lambda Z(\lambda Q_r e^{\lambda Q_0}) \sin(\lambda Z(\lambda Q_r e^{\lambda Q_0})) + \cos(\lambda Z(\lambda Q_r e^{\lambda Q_0})) = \frac{\lambda^2 Q_r^2 e^{2\lambda Q_0}}{2} + 1$$

$\Rightarrow$ COMPATIBLE WITH κ-MINKOWSKI!!

Perturbative map

Expanding again to the quartic order:

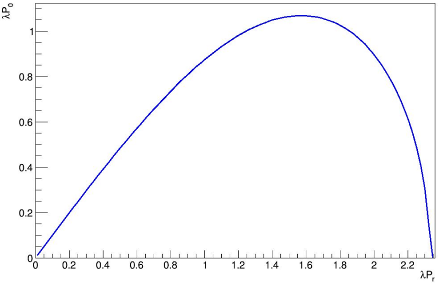
$$B_r = (1 + \lambda^2 Q_0^2 - \frac{3}{8}\lambda^2 Q_r^2 - \frac{3}{4}\lambda^3 Q_0 Q_r^2 - \frac{5}{4}\lambda^4 Q_r^2 Q_0^2 - \frac{113}{1152}\lambda^4 Q_r^4) M_r$$

$$P_r = Q_r + \lambda Q_r Q_0 + \frac{\lambda^2}{2} Q_r Q_0^2 + \frac{\lambda^2}{8} Q_r^3 + \frac{3}{8}\lambda^3 Q_0 Q_r^3 + \\ + \frac{\lambda^3}{6} Q_r Q_0^3 + \frac{9}{16}\lambda^4 Q_0^2 Q_r^3 + \frac{\lambda^4}{24} Q_r Q_0^4 + \frac{55}{1152}\lambda^4 Q_r^5$$

$$P_0 = Q_0 + \frac{\lambda}{2} Q_r^2 + \frac{\lambda^2}{6} Q_0^3 + \frac{\lambda^2}{2} Q_0 Q_r^2 + \frac{\lambda^3}{4} Q_0^2 Q_r^2 + \frac{\lambda^4}{120} Q_0^5 + \frac{\lambda^4}{12} Q_0^3 Q_r^2$$

which are the redefinitions of our generators in terms of bicrossproduct operators.

Modified dispersion relation



Experimentally interesting
to look at the mass Casimir:

$$P_0^2 = 2\left(\frac{\lambda P_r \sin \lambda P_r + \cos \lambda P_r - 1}{\lambda^2}\right)$$

Non-monotonic function
$\rightarrow P_r^{max} = \frac{\pi}{2\lambda}$ maybe a cutoff! $\rightarrow$
Does it spoil relativistic invariance?

The second order
expansion of the Casimir gives:

$$P_0^2 \simeq P_r^2 - \frac{\lambda^2}{4} P_r^4 \rightarrow v(E) \simeq 1 - \frac{3}{8}\lambda^2 E^2$$

Comparison with previous results

In-vacuo dispersion from quantum geometry:

- photons: $E_\pm^2 \approx p^2c^2 \mp \frac{pc}{E_P} p^2c^2$ [Gambini, Pullin]
- neutrinos: $E_\pm^2 \approx m^2\tilde{c}_\pm^4 + p^2\tilde{c}_\pm^2 + \frac{p\tilde{c}_\pm}{E_P} p^2\tilde{c}_\pm^2$ [Alfaro, Morales-Tecotl, Urrutia]
- scalars: $E^2 \approx m^2\tilde{c}^4 + p^2\tilde{c}^2$ [Assanioussi, Dapor, Lewandowski]

Common limitation: no dynamical information on spacetime!

$\Rightarrow$ quantum spacetime structure assumed not derived!

Advantages of effective HDA analysis:

- self-consistency: checked (no anomaly)
- spacetime dynamics: encoded in $\{H^Q, H^Q\}$

A fresh look at complex connections

Original Ashtekar variables: $A_a^i = \Gamma_a^i \pm iK_a^i\,, \quad \gamma = \pm i$

pros

- Ashtekar variables transform as space-time connections [Samuel, Alexandrov]
- Bekenstein–Hawking formula without fine tuning [Perez, Noui, Engle, Geiller, Ben Achour]

cons

- Secondary constraints: reality conditions
- Ill-defined Hilbert space: non-compact measure

- use $SU(1,1)$ holonomies [Noui, Ben Achour]
- use generalized holonomies [Wilson-Ewing]

$SL(2,\mathbb{C})$ holonomies

Holonomy along angular direction

$$h_\phi(r,\mu) = \exp(\mu\gamma K_\phi \Lambda^A_\phi) = \cosh(\mu K_\phi)\mathbb{I} - 2\sinh(\mu K_\phi)\overline{\Lambda}\,,$$

then holonomy-corrections are: $K_\phi \to \sinh(K_\phi\bar{\delta})/\bar{\delta}$ and H^Q is

$$\begin{aligned} H^Q[N] \quad = \quad & -\frac{1}{2G}\int_B drN\left[-\frac{\sinh^2(K_\phi\bar{\delta})}{\bar{\delta}^2}E^\phi \right. \\ & \left. +2K_r\frac{\sinh(K_\phi\bar{\delta})}{\bar{\delta}}E^r + (\Gamma_\phi^2 - 1)E^\phi - 2\Gamma'_\phi E^r\right] \end{aligned}$$

$\Rightarrow$ quantum modifications of hypersurface deformations?

MDR from $SL(2,\mathbb{C})$ holonomy corrections

Modified HDA: $\{H^Q[N], H^Q[N']\} = D[\cosh(2\bar{\delta}K_\phi)g^{rr}(N\partial_r N' - N'\partial_r N)]$
In the Minkowski limit translates into

$$[B_r, P_0] = iP_r\cosh(\lambda P_r) \Rightarrow P_0^2 = 2(\frac{\lambda P_r \sinh\lambda P_r - \cosh\lambda P_r + 1}{\lambda^2})$$

second order effects

- $E^2 \approx p^2 + \frac{\lambda^2}{4}p^4$
- $v(E) := \frac{dH}{dp} \approx 1 + \frac{3}{8}\lambda^2 E^2$

$\Rightarrow$ Different than the real case with: $v(E) \approx 1 - \frac{3}{8}\lambda^2 E^2$!!

$SU(1,1)$ holonomies

Non-compact gauge choice: $e^i_3 \equiv (0,0,0,1) \Rightarrow$ from $SL(2,\mathbb{C})$ to $SU(1,1)$!

Motivation: preserve reality of area operator

$$a_I = 8\pi l_P^2 \gamma \sqrt{j_I(j_I+1)} \quad \textit{discrete}$$

$$j_I \to \frac{1}{2}(-1+is) \;\to\; a_I = 4\pi l_P^2 \sqrt{s_I^2+1} \quad \textit{continuous}$$

Angular holonomy corrections become

$$K_\phi \to \frac{\sinh(\delta K_\phi)}{\delta} \sqrt{\frac{-3}{s(s^2+1)\sinh(\theta_\phi)} \frac{\partial}{\partial \theta_\phi} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right)}$$

$$\sinh\left(\frac{\theta_\phi}{2}\right) = \sinh^2\left(\frac{\delta K_\phi}{2}\right)$$

MDR from $SU(1,1)$ holonomy corrections

Modified HDA:

$$\{H^Q[N], H^Q[N']\} = \frac{-3}{s(s^2+1)} D \Bigg[\cosh(2\delta K_\phi)\left(\frac{1}{\sinh(\theta_\phi)}\frac{\partial}{\partial\theta_\phi}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right)\right)$$
$$+ \frac{\sinh(2\delta K_\phi)}{\delta}\frac{\partial\theta_\phi}{\partial K_\phi}\left(-\frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)}\frac{\partial}{\partial\theta_\phi}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right) + \frac{1}{\sinh(\theta_\phi)}\frac{\partial^2}{\partial\theta_\phi^2}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right)\right)$$
$$+ \frac{\sinh^2(\delta K_\phi)}{\delta^2}\frac{\partial^2\theta_\phi}{2\partial K_\phi^2}\left(-\frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)}\frac{\partial}{\partial\theta_\phi}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right) + \frac{1}{\sinh(\theta_\phi)}\frac{\partial^2}{\partial\theta_\phi^2}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right)\right)$$
$$+ \frac{\sinh^2(\delta K_\phi)}{\delta^2}\left(\frac{\partial\theta_\phi}{\partial K_\phi}\right)^2\left(\frac{1}{\sinh^3(\theta_\phi)}\frac{\partial}{\partial\theta_\phi}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right) - \frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)}\frac{\partial^2}{\partial\theta_\phi^2}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right)\right)$$
$$+ \frac{1}{2\sinh(\theta_\phi)}\frac{\partial^3}{\partial\theta_\phi^3}\left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)}\right) g^{rr}(N\partial_r N' - N'\partial_r N)\Bigg]$$

Ansatz for MDR: $P_0^2 = f(P_r)$ with $f(P_r) = 2\int \beta(P_r)P_r dP_r$

second order effects

- $E^2 \approx p^2 + \frac{\lambda^2}{4}p^4 \to$ higher orders to differentiate from $SL(2,\mathbb{C})$!
- $v(E) := \frac{dH}{dp} \approx 1 + \frac{3}{8}\lambda^2 E^2$

Generalized holonomies

Motivation: consistent self-dual cosmology!

definition of generalized holonomies

$h_e(A) = \mathcal{P}\exp(\alpha \int_e \dot{e}^a A_a^I \tau_I) \rightarrow h_j(c) = \cosh(\frac{\alpha\tilde{\mu}c}{2i})\mathbb{I} + \sinh(\frac{\alpha\tilde{\mu}c}{2i})\sigma_j$

shift operator: $e^{\mu c}|p\rangle = |p+\mu\rangle = |p + (\alpha\tilde{\mu})/2i\rangle \Rightarrow \alpha \equiv i$

$\Rightarrow$ well-defined measure and solvable reality conditions in FRW spacetimes!

Angular holonomy :

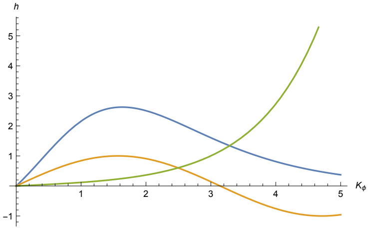
$$h_\phi(r,\mu) = \cosh(\mu i K_\phi)\mathbb{I} + \sinh(\mu i K_\phi)\sigma_\phi = \cos(\mu K_\phi)\mathbb{I} + \sin(\mu K_\phi)\Lambda$$

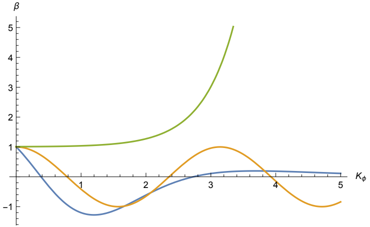
$\Rightarrow$ holonomy corrections same as $SU(2)$ ones!!

MDR is: $P_0^2 = 2(\frac{\lambda P_r \sin\lambda P_r + \cos\lambda P_r - 1}{\lambda^2})$

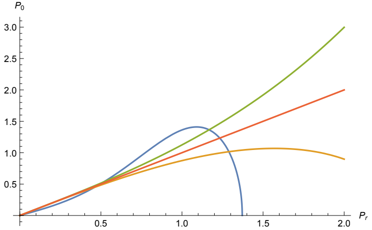
$\Rightarrow$ indistinguishable from real connection case!!

Physical predictions can restrict ambiguities





1. Holonomy corrections $h(K)$
2. Beta correction functions $\beta(K)$
3. In-vacuo dispersion $P_0^2(P)$



Signature change

General modified HDA: $\{H^Q[N], H^Q[M]\} = D[\beta h^{jk}(N\partial_j M - M\partial_j N)]$
$\Rightarrow$ β gives the signature of spacetime! [Hojman, Kuchař, Teitelboim]
$\beta = \beta(K)$ becomes negative in the high-curvature regime $K \sim 1/\lambda$!
$\rightarrow$ signature change!

- $SU(2)$, $SU(1,1)$, generalized holonomy corrections: signature change
- $SL(2,\mathbb{C})$ holonomy corrections: no signature change (depending on the choice of variables) [Ben Achour, Brahma, Marciano]

physical predictions?

- consequences for CMB power spectrum [Barrau, Bolliet, Grain, Mielczarek]
- unstable cosmological solutions [Bojowald, Mielczarek]
- Euclidean phase $\longleftrightarrow$ singularity resolution [Bojowald, Brahma,

Testable?

An (incomplete) state of the art of Lorentz violation/deformation:

$$E^2 \simeq m^2 + p^2 + \eta_1 \frac{E}{E_{QG}} p^2 + \eta_2 \frac{E^2}{E_{QG}^2} p^2 + \dots$$

- gravitational Cherenkov radiation $\rightarrow \Delta c/c < (10^{-15} - 10^{-19})$ if $c_g < c$ [Moore, Nelson, Coleman, Glashow]
- flux of Cherenkov photons from Mrk 501 $\rightarrow E_{QG} > 10^{19}$ GeV [HESS collaboration]
- stochastic in-vacuo dispersion with Fermi-LAT $\rightarrow E_{QG} > 10^{19}$ GeV [Amelino-Camelia, Piran, Granot, Vasileiou]
- higher dimension operators in effective field theory $\rightarrow E_{QG} > 10^{19}$ GeV [Meyers, Pospelov]
- flux of synchrotron radiation from Crab Nebula $\rightarrow E_{QG} > 10^{26}$ GeV [Jacobson, Liberati, Mattingly]

Testable?

Best messengers

gamma ray bursts:

- large distances (effects add up)
- rapidly varying emission (almost simultaneous)
- high-energy extent (tens of GeV)

$\Rightarrow$ preliminary evidence of first order modifications:

- with Fermi-LAT photons: $E_{QG} \simeq 10^{17}$ GeV [Ma, Xu]
- with IceCube neutrinos: $E_{QG} \simeq 10^{18}$ GeV [Amelino-Camelia, Barcaroli, D'Amico, Rosati, Loret]

second order effects: $n = 2$

typical constraints are $E_{QG} > (10^9 - 10^{11})$ GeV !

$\Rightarrow$ NOT YET TESTABLE!

Conclusions and outlook

Summary:

- Effective models to infer spacetime structure
- DSR models derived from semi-classical canonical gravity
- Room for κ-Minkowski in LQG
- Dynamical derivation of MDR
- Different quantum corrections give different predictions (sensitive to gauge choices)

Developments:

- Try to include matter (difficulties to overcome) [Bojowald, Brahma, Reyes]
- Study the effects of higher spin representations on MDR
- Derive quasi-local charges from effective quantum constraints
- Link modifications of HDA to dimensional flow [Ronco, Mielczarek, Trześniewski]
- Quantum HDA in other approaches [Calcagni, Ronco]

Thanks for your patience and attention!