Loop quantum gravity coupled to a scalar field

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joint work with Jerzy Lewandowski

Motivation

- Simplest matter field
- Comparison to symmetry reduced models (Gambini + Pullin, …)
- Comparison with deparametrized models (Rovelli + Smolin, …)
- may also learn something about the vacuum case

Main Results

J Lewandowski, HS:

- Phys.Rev. D91 (2015) 4, 044022
- Phys.Rev. D93 (2016) 2, 024042

- Non-standard scalar
- New diffeo invariant operators
- a well defined Hamiltonian constraint
- part of the hypersurface deformation algebra

Classical theory

Hamilton formulation (+time gauge).

Kinematic phase space coordinates (ϕ, π, A, E)

$$\{F[\phi,\pi],\ G[\phi,\pi]\} = \int \frac{\delta F}{\delta\phi(x)}\frac{\delta G}{\delta\pi(x)} - \frac{\delta G}{\delta\phi(x)}\frac{\delta F}{\delta\pi(x)}$$

$$\{F[A,E],\ G[A,E]\} = 8\Pi G\beta \int \frac{\delta F}{\delta A^i_a(x)}\frac{\delta G}{\delta E^a_i(x)} - \frac{\delta G}{\delta A^i_a(x)}\frac{\delta F}{\delta E^a_i(x)}$$

Gauge, diffeo constraint.

Scalar constraint

$$H(x) = \frac{\pi^2(x) + \phi_{,a}(x)\phi_{,b}(x)E^a_i(x)E^b_i(x)}{2\sqrt{|\det E(x)|}} + V(\phi(x))\sqrt{|\det E(x)|} + C_{\mathrm{GR}}(A,E)(x)$$

In the following, will use equivalent constraint (from solving H=0 for **π**):

$$C(x) = \pi(x) \mp \widetilde{\pi}(x)$$

where

$$\widetilde{\pi}(x) \;=\; \sqrt{-\phi_{,a}\phi_{,b}E^a_i E^{bi} - 2V(\phi)|\det E| - 2\sqrt{|\det E|}C_{\mathrm{GR}}(A,E)(x)}$$

For later we note commutator of constraint:

$$\{C(N), C(M)\} \;=\; \int d^3x\,(N_{,a}M - M_{,a}N)\,E^a_i E^b_i \left(\frac{\pi\phi_{,b}}{\pi\tilde{\pi}} + \frac{E^c_i F^i_{ac}}{8\Pi G\beta\tilde{\pi}^2}\right)$$

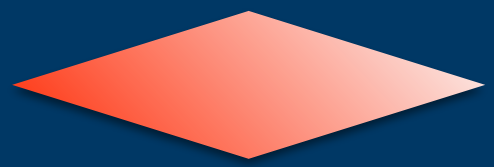
Diffeo generator which depends on the phase space point.

Kinematic quantization

Gravity

Usual LQG treatment.

$$h_{\alpha,j}[A] = \mathrm{tr}_j \left[\mathcal{P} \exp \int_\alpha -A \right] \qquad E_i[S] = \int_S *E_i$$

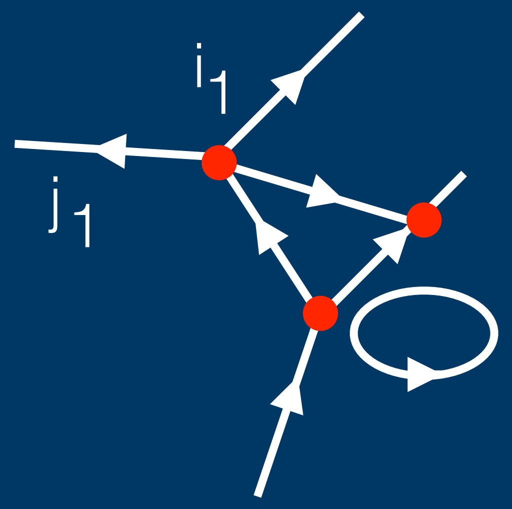


Hilbert space (Ashtekar+Lewandowski 1993): Spanned by

$$\sum_{\underline{\alpha},j} \prod_I h_{\alpha_I, j_I} \, |0\rangle$$

has spin net basis

$$|\gamma, \underline{j}, \underline{\iota}\rangle$$



Scalars: Previous work

- Thiemann (1997): Diffeo invariant representation of group valued scalar fields. In particular

$$\widehat{e^{in\phi(x)}}, \qquad \widehat{\pi}(f)$$

well defined operators, and form an orthonormal basis

$$|\,\underline{x},\underline{n}\rangle := \prod_k \widehat{e^{in_k\phi(x_k)}}\,|\,0\rangle$$

$$\langle \underline{x},\underline{n}\,|\,\underline{x}',\underline{n}'\rangle = \delta_{\underline{n},\underline{n}'}\delta_{\underline{x},\underline{x}'}$$

- Diffeo invariant representations for this algebra are unique in 3+1d (Bobienski,Kaminski,Lewandowski, Okolow 2005).

- Domagala, Giesel, Kaminski, Lewandowski (2010): use diffeo constraint to eliminate Φ from $\widetilde{\pi}$. Constraint takes Schrödinger form.

- Gambini and Pullin: Quantum scalar on quantum spacetime (2013-now)

Non-standard scalar

Start with states dual to polymer states

$$\langle \varphi |, \qquad \varphi \in C^\infty(\Sigma) \qquad\qquad \langle \varphi \,|\, \underline{x}, \underline{n} \rangle := \exp \frac{i}{\hbar} \sum_{x \in \underline{x}} n_x \varphi(x)$$

Dual action

$$\widehat{\phi}(x) | \varphi \rangle = \varphi(x) | \varphi \rangle \qquad\qquad \exp \left[\frac{1}{i\hbar} \int \mathrm{d}^3 x \; f(x) \widehat{\pi}(x) \right] \, | \varphi \rangle \; = \; | \varphi + f \rangle$$

Can turn into Hilbert space by inner product

$$\langle \varphi \,|\, \varphi' \rangle = \begin{cases} 1 & \text{if } \varphi = \varphi' \\ 0 & \text{otherwise} \end{cases}$$

Previously employed in cosmology by Hossain, Husain, Seaha (2010), Barbero, Pawlowski, Villasenor (2014). Also: Campiglia, Varadarajan (2013,14)!

Can go to the dual of the dual: $(\Psi | = \sum_{\varphi} \Psi[\varphi] \langle \varphi |$

No inner product on these states anymore, in general. However, for some states dual action of field and momentum

$$[(\Psi | \widehat{\pi}(f)] | \phi \rangle := i \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} (\Psi | e^{i\epsilon\widehat{\pi}(f)} | \phi \rangle$$
$$= i \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \Psi[\phi + \epsilon f] = \delta_f \Psi[\phi]$$

So

$$(\Psi | \widehat{\pi}(f) = (\delta_f \Psi |$$

Note: If

$$\Psi[\varphi] = \prod_k e^{i n_k \varphi(x_k)}$$

recover the standard polymer states.

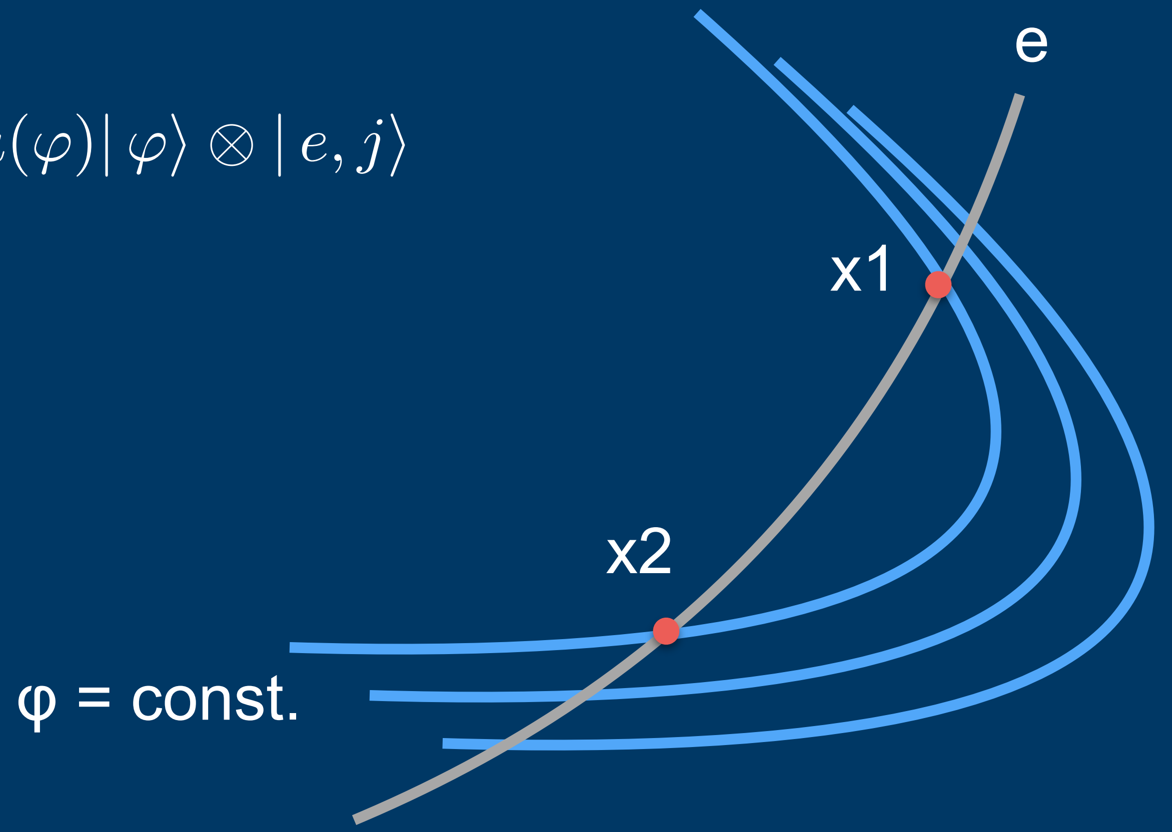
New geometric operator

Ingredient in quantisation of C: $\sqrt{\phi_{,a}\phi_{,b}E^a_i E^b_i}(x)$

Closely related to to Ma and Ling's (2008) $\widehat{Q}(\omega) = \int \mathrm{d}^3x \, \sqrt{\widehat{E}^a_i \widehat{E}^{bi}\omega_a\omega_b(x)}$

Single edge, φ monotone along e:

$$\int \mathrm{d}^3x \; f\sqrt{\phi_{,a}\phi_{,b}E^a_i E^{bi}}\; |\varphi\rangle \otimes |e,j\rangle = \int \mathrm{d}\varphi \; a(\varphi)|\varphi\rangle \otimes |e,j\rangle$$



with

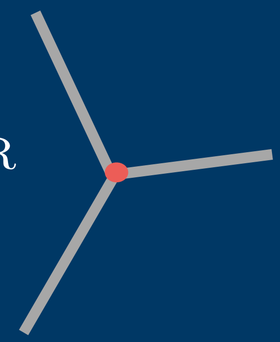
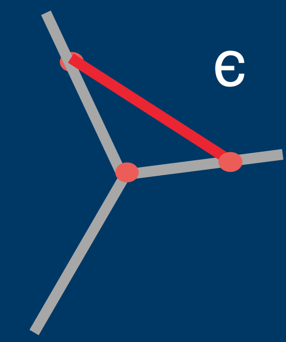
$$a(\varphi) = l_{\mathrm{P}}^2 \sum_{x_I} \sqrt{j(j+1)} f(x_I, \varphi)$$

general case: similar.

Constraints

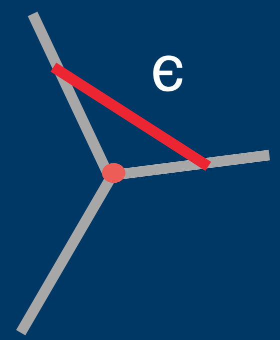
Quantisation of C_GR

In LQG we introduce a regulator

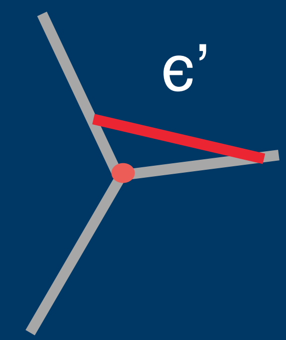
$\widehat{C}_\epsilon^{\mathrm{GR}}$  $=$ 

$+\ldots$

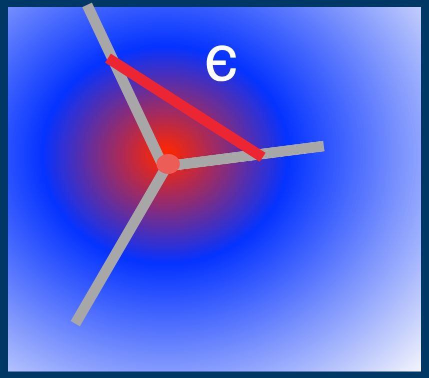
Thiemann's quantisation of C_GR relies on



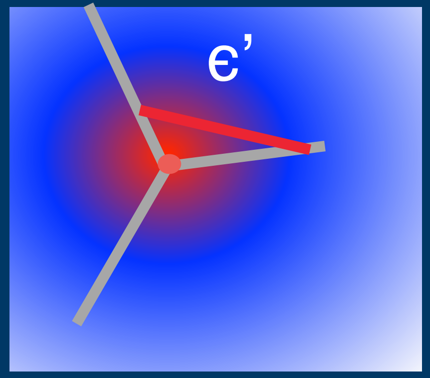
diffeo equivalent to



Not true anymore with scalar field:



not diffeo equivalent to



Our solution: scalar field constant near vertices, differ average only near vertices

$$\mathcal{H}_{\text{diff}} = \eta\left(\text{Cyl}_{\text{SF}} \otimes \text{Cyl}_{\text{GR}}\right) = \bigoplus_{\varphi,\{v_1,\ldots,v_m\}\subset\Sigma} \mathcal{H}_{\varphi,\{v_1,\ldots,v_m\}}$$

On this Hilbert space, can obtain operator version of

$$\widetilde{\pi}(N) \; = \; \int N(x)\sqrt{-\phi_{,a}\phi_{,b}E^a_i E^{bi} - 2V(\phi)|\det E| - 2\sqrt{|\det E|}C_{\text{GR}}(A,E)}(x)$$

Full constraint can not be defined because of $\boldsymbol{\pi}$. Action of full constraint

$$C(x) = \pi(x) \mp \widetilde{\pi}(x)$$

well defined on (double) dual.

Commutators

Constraint only defined on dual, but commutator defined on H_diff. Explicit calculation:

$$[\widehat{C}(M), \widehat{C}(N)]\, \eta(\langle\, \varphi| \otimes \langle\, \gamma, j, \iota|) = 8\Pi\beta\ell_{\mathrm{P}}^2 \left(\sum_{e\in\gamma} \sqrt{j_e(j_e+1)} \int_e \mathrm{sgn}(d\varphi)\,(N dM - M dN) \right)\, \eta(\langle\, \varphi| \otimes \langle\, \gamma, j, \iota|)$$

where for the derivation we assumed M,N constant in neighbourhood of vertices.

We will argue: This is a quantisation of

$$\{C(N), C(M)\} \;=\; \int d^3x\, (N_{,a}M - M_{,a}N)\, E^a_i E^b_i \left(\frac{\pi\phi_{,b}}{\pi\tilde{\pi}} + \frac{E^c_i F^i_{ac}}{8\Pi G\beta\tilde{\pi}^2} \right)$$

Comparison:

$$[\widehat{C}(M), \widehat{C}(N)]\; \eta(\langle\, \varphi| \otimes \langle\, \gamma, j, \iota|)$$

$$= 8\Pi\beta\ell_{\mathrm{P}}^2 \left(\sum_{e\in\gamma} \sqrt{j_e(j_e+1)} \int_e \mathrm{sgn}(d\varphi)\,(N dM - M dN) \right)\; \eta(\langle\, \varphi| \otimes \langle\, \gamma, j, \iota|)$$

$$\{C(N), C(M)\} \;=\; \int d^3x\, (N_{,a}M - M_{,a}N)\, E^a_i E^b_i \left(\frac{\pi\phi_{,b}}{\pi\tilde{\pi}} + \frac{E^c_i F^i_{ac}}{8\Pi G\beta\tilde{\pi}^2} \right)$$

- inverse metric supported on edges ✔
- do not see the contribution of C_GR, V because they would act at the vertices, where N,M constant ✔
- do not see contribution of EF/**π**^2 because

$$(N_{,a}M - M_{,a}N)\, \widehat{E^a_i E^b_i}$$

acting on state would give vector density tangent to edges, vanishing near vertices ✔

Further remarks, and outlook

Cautionary remarks

- $\widehat{C}(M)\eta((\,\Phi\,|\otimes\langle\,\gamma,j,\iota|) = 0, \qquad \widehat{C}(N)\eta((\,\Phi\,|\otimes\langle\,\gamma,j,\iota|) = 0$

  $$\Leftrightarrow [\widehat{C}(M),\widehat{C}(N)]\,\eta((\,\Phi\,|\otimes\langle\,\gamma,j,\iota|) = 0$$

  The latter will require a solution of the constraint to vanish in the current setup.

- No inner product on the double dual

- Restricting the scalar field to be constant around vertices is somewhat ad hoc

Nice things

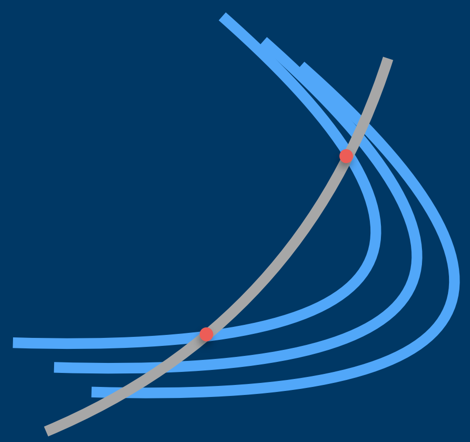
- Non-standard quantum scalar: position rep

$$\widehat{\phi}(x)|\,\varphi\rangle = \varphi(x)|\,\varphi\rangle \qquad \exp\left[\frac{1}{i\hbar}\int \mathrm{d}^3x\; f(x)\widehat{\pi}(x)\right]\,|\,\varphi\rangle \;=\; |\,\varphi + f\rangle$$

- relation to standard rep

- New operators coupling gravity and scalar

$$\int \mathrm{d}^3x\,\sqrt{\phi_{,a}\phi_{,b}E^a_iE^{bi}}\;|\,\varphi\rangle\otimes|\,e,j\rangle = \int \mathrm{d}\varphi\; a(\varphi)|\,\varphi\rangle\otimes|\,e,j\rangle$$



- Have good home for $\widetilde{\pi}$
- Intriguing commutator

$$[\widehat{C}(M),\widehat{C}(N)]\;\eta(\langle\,\varphi|\otimes\langle\,\gamma,j,\iota|)$$

$$= 8\Pi\beta\ell_{\mathrm{P}}^2\left(\sum_{e\in\gamma}\sqrt{j_e(j_e+1)}\int_e \mathrm{sgn}(d\varphi)\,(NdM - MdN)\right)\;\eta(\langle\,\varphi|\otimes\langle\,\gamma,j,\iota|)$$

Outlook

- Have worked out solutions for toy models of the full constraint
- position representation for gravity (—> Bahr, Dittrich, Geiller: BF vacuum)
- Further analysis of the commutator
- Physics?