Quantization of lattice gauge theories and their continuum limit

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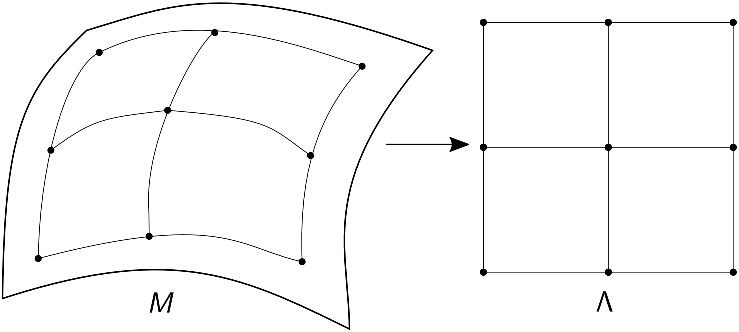
Contents

- General setup for lattice gauge theory
- Refinements of graphs and associated constructions
- Quantization and groupoids
- (Continuum) limit

Discretisation of space

Hamiltonian framework:
Approximate a Cauchy surface M with an oriented, finite, connected graph $\Lambda = (\Lambda^0, \Lambda^1)$:

- Points in $M \to$ set of vertices Λ^0
- Paths between points $\to$ set of oriented edges Λ^1



Discretising connections

The graph Λ comes with two maps $s, t \colon \Lambda^1 \to \Lambda^0$:
The source and target maps.

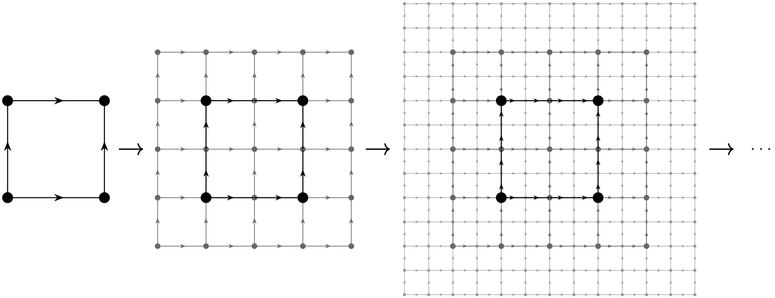
Let G be a compact, connected Lie group.

- The space of connections: G^{Λ^1}
- The gauge group: G^{Λ^0}
- Gauge transformations:

$$G^{\Lambda^0} \times G^{\Lambda^1} \to G^{\Lambda^1},$$
$$((g_x)_{x\in\Lambda^0}, (a_e)_{e\in\Lambda^1}) \mapsto (g_{s(e)} a_e g_{t(e)}^{-1})_{e\in\Lambda^1}.$$

Approximating the continuum

One graph is insufficient;
We require a **net** of graphs:



Refinement (1)

Let $\Lambda = (\Lambda^0, \Lambda^1)$ be an oriented graph.
Consider the **free category** C_Λ **generated by** Λ:

- Objects: Λ^0
- Let $x, y \in \Lambda^0$.
  Morphisms from x to y: paths from x to y in Λ
- Composition: concatenation of paths
- Identity element at x: trivial path at x

Notation:

- Set of objects: $C_\Lambda^{(0)}$
- Set of morphisms: $C_\Lambda^{(1)}$

Refinement (2)

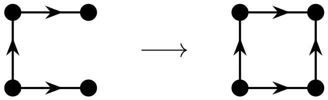
Let Λ_i, Λ_j be oriented graphs.
Suppose $\iota_{i,j} \colon \mathsf{C}_i \to \mathsf{C}_j$ is a functor.
Suppose in addition that:

- The map between objects $\iota_{i,j}^{(0)} \colon \mathsf{C}_i^{(0)} \to \mathsf{C}_j^{(0)}$ is an injection;
- $\iota_{i,j}^{(1)} \colon \mathsf{C}_i^{(1)} \to \mathsf{C}_j^{(1)}$ maps edges to nontrivial paths;
- for each $e, e' \in \Lambda_i^1$, if $e \neq e'$, then $\iota_{i,j}^{(1)}(e)$ and $\iota_{i,j}^{(1)}(e')$ have no common edges;

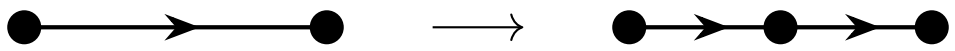
Then we call $(\Lambda_i, \Lambda_j, \iota_{i,j})$ a **refinement of the graph** Λ_i.

Examples (1)

Addition of an edge:

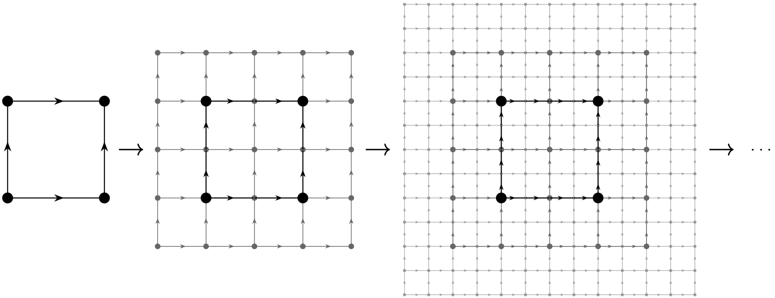


Subdivision of an edge:



Every refinement is a composition of a finite sequence of the above two types of refinements.

Examples (2)



Category of refinements

Let **Refine** denote the category with

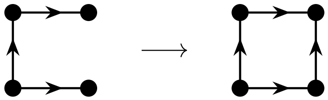
- Objects: finite, connected, oriented graphs Λ
- Morphisms: refinements $(\Lambda_i, \Lambda_j, \iota_{i,j})$

Given the Lie group G, there exists a contravariant functor from **Refine** to the category of spaces with group actions.

- Objects: $\Lambda \mapsto (G^{\Lambda^0}, G^{\Lambda^1}, \cdot)$
- Morphisms: $(\Lambda_i, \Lambda_j, \iota_{i,j}) \mapsto (\rho_{i,j}, \mathrm{R}^{(0)}_{i,j})$
- $\rho_{i,j} \colon G^{\Lambda_j^0} \to G^{\Lambda_i^0}$ 'restriction map'
- $\mathrm{R}^{(0)}_{i,j} \colon G^{\Lambda_j^1} \to G^{\Lambda_i^1}$

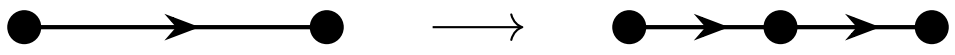
Definition of $R_{i,j}^{(0)}$

Addition of an edge:



$(a_1, a_2, a_3) \;\leftarrow\!\shortmid\; (a_1, a_2, a_3, a_4)$

Subdivision of an edge:



$(a_1 a_2) \;\leftarrow\!\shortmid\; (a_1, a_2)$

- For general refinements, define $R_{i,j}^{(0)}$ by composition of these maps;
- $R_{i,j}^{(0)}$ is independent of the chosen sequence;

Hilbert space

Define covariant functors from **Refine** to the category of Hilbert spaces.

Unreduced Hilbert space:

- Objects: $\Lambda \mapsto L^2(G^{\Lambda^1})$
- Morphisms:

$$(\Lambda_i, \Lambda_j, \iota_{i,j}) \quad \mapsto \quad \begin{aligned} u_{i,j} \colon L^2(G^{\Lambda_i^1}) &\to L^2(G^{\Lambda_j^1}), \\ \psi &\mapsto \psi \circ \mathrm{R}_{i,j}^{(0)}. \end{aligned}$$

Reduced Hilbert space

$L^2(G^{\Lambda^1})$ carries a continuous unitary representation of G^{Λ^0}:

$$(g \cdot \psi)(a) := \psi(g^{-1} \cdot a), \quad a \in G^{\Lambda^1},\ g \in G^{\Lambda^0}.$$

Reduced Hilbert space:

- Objects: $\Lambda \mapsto L^2(G^{\Lambda^1})^{G^{\Lambda^0}}$
- Morphisms:

$$(\Lambda_i, \Lambda_j, \iota_{i,j}) \quad \mapsto \quad u_{i,j}\big|_{L^2(G^{\Lambda_i^1})^{G^{\Lambda_i^0}}} : L^2(G^{\Lambda_i^1})^{G^{\Lambda_i^0}} \to L^2(G^{\Lambda_j^1})^{G^{\Lambda_j^0}}.$$

Observable algebra

Define covariant functors from **Refine** to the category of C*-algebras.

Unreduced observable algebra:

- Objects: $\Lambda \mapsto B_0(L^2(G^{\Lambda^1}))$
- Morphisms:

$$(\Lambda_i, \Lambda_j, \iota_{i,j}) \quad \mapsto \quad \begin{aligned} v_{i,j} \colon B_0(L^2(G^{\Lambda_i^1})) &\to B_0(L^2(G^{\Lambda_j^1})), \\ a &\mapsto u_{i,j} a u_{i,j}^*. \end{aligned}$$

Reduced observable algebra:

- Objects: $\Lambda \mapsto B_0(L^2(G^{\Lambda^1})^{G^{\Lambda^0}})$
- Morphisms: $(\Lambda_i, \Lambda_j, \iota_{i,j}) \mapsto$

$$v_{i,j}\big|_{B_0(L^2(G^{\Lambda_i^1})^{G^{\Lambda_i^0}})} \colon B_0(L^2(G^{\Lambda_i^1})^{G^{\Lambda_i^0}}) \to B_0(L^2(G^{\Lambda_j^1})^{G^{\Lambda_j^0}}).$$

The Stone–Von Neumann Theorem

Let U and V be one-parameter groups.
The continuous unitary representations of these groups on $L^2(\mathbb{R})$ given by

$$(U(t)\psi)(x) := \psi(x-t), \quad (V(s)\psi)(x) := e^{isx}\psi(x).$$

are jointly irreducible, and satisfy the Weyl form of the CCR

$$U(t)V(s) = e^{-ist}V(s)U(t).$$

Any other pair of continuous unitary jointly irreducible representation of U and V on a Hilbert space $\mathcal{H}$ satisfying the CCR is unitarily equivalent to this one.

Crossed product formulation (1)

Let G be a locally compact group.
Then $(C_0(G), G, L)$ is a **C*-dynamical system**.

$$(L(g)(f))(a) = f(g^{-1} \cdot a).$$

Now form its **crossed product C*-algebra** $C_0(G) \rtimes_L G$:

- Endow $C_c(G, C_0(G))$ with the structure of a *-algebra by

$$f * g(a) := \int_G f(b) L(b)(g(b^{-1}a))\, db,$$
$$(f^*)(a) := \Delta(a^{-1}) L(a)(f(a^{-1}));$$

Crossed product formulation (2)

- Complete $C_c(G, C_0(G))$ with respect to the **universal norm**:

$$\|f\| := \sup_{(\pi,U)} \|\pi \rtimes U(f)\|,$$

where (π, U) is a nondegenerate covariant representation of $(C_0(G), G, L)$.

A **covariant representation** (π, U) consists of

- a *-representation: $\pi \colon C_0(G) \to B(H)$;
- a continuous unitary representation: $U \colon G \to U(H)$;
- a **covariance condition**:

$$\pi(L(a)(f)) = U(a)\pi(f)U(a)^*, \quad \forall a \in G, \forall f \in C_0(G).$$

$$\pi \rtimes U \colon C_0(G) \rtimes_L G \to B(H), \quad f \mapsto \int_G \pi(f(a))U(a)\, da.$$

The Stone–Von Neumann theorem for crossed products

Let G be a locally compact group. Then

$$C_0(G) \rtimes_L G \cong B_0(L^2(G)).$$

In particular, the natural covariant representation (M, L) on $L^2(G)$ is irreducible and faithful.

Crossed products and groupoids

Let G be a compact Lie group.
Then

$$B_0(L^2(G)) \cong C(G) \rtimes_L G \cong C^*(G \ltimes G) \cong C^*(\mathbf{G}).$$

Here

- $G \ltimes G$ is an **action groupoid**;
- $\mathbf{G}$ is the **pair groupoid** associated to G;
- $C^*(\cdot)$ denotes the **groupoid C*-algebra** of the groupoid.

What is a groupoid?

A **groupoid** is a small category in which each morphism is invertible.

Examples:

Any group H:

- Objects: a set containing a single point $\{*\}$
- Morphisms: H
- Composition: group multiplication in H
- Identity element at $*$: identity of H

Any set X:

- Objects: X
- Morphisms: X
- Composition: $x \circ x = x$
- Identity element at x: x

The action groupoid

Let X be a set, H a group with unit e.
Suppose $\cdot\colon H \times X \to X$ is a group action.
The **action groupoid** $H \ltimes X$ is now defined as follows:

- Objects: X
- Morphisms: $H \times X$, $(h, x) \in \mathrm{Mor}(h^{-1} \cdot x, x)$.
- Composition: $(h_1, x_1) \circ (h_2, x_2) = (h_1 h_2, x_1)$ if $x_2 = h_1^{-1} \cdot x_1$
- Identity element at x: (e, x)

Take $X = H = G$, and $\cdot\colon G \times G \to G$ is the group multiplication
$\Rightarrow G \ltimes G$.

The pair groupoid

Let X be a set.

The **pair groupoid X** is now defined as follows:

- Objects: X
- Morphisms: $X \times X$, $\mathrm{Mor}(x, y) = \{(x, y)\}$.
- Composition: $(x_1, y_1) \circ (x_2, y_2) = (x_2, y_1)$ if $x_1 = y_2$
- Identity element at x: (x, x)

Take $X = G \Rightarrow \mathbf{G}$.

$G \ltimes G$ is isomorphic to $\mathbf{G}$:

$$(G \ltimes G)^{(1)} \to \mathbf{G}^{(1)}, \quad (b, a) \mapsto (b^{-1}a, a).$$

This isomorphism is compatible with the Haar systems on both groupoids $\Rightarrow C^*(G \ltimes G) \cong C^*(\mathbf{G})$.

Groupoids and refinements

Let $f \in C_c(\mathbf{G}^{(1)}) = C_c(G \times G)$.
Let $\pi\colon C^*(\mathbf{G}) \to B(L^2(G))$ be the natural representation.
Then

$$\pi(f)(\psi)(a) = \int_G f(b,a)\psi(b)\,db,$$

where $\psi \in L^2(G)$ and $a \in G$.
Question: How does this operator behave w.r.t. refinements?
Let $(\Lambda_i, \Lambda_j, \iota_{i,j})$ be a refinement. Then

$$v_{i,j}(\pi_i(f)) = \pi_j(f \circ R^{(1)}_{i,j}),$$

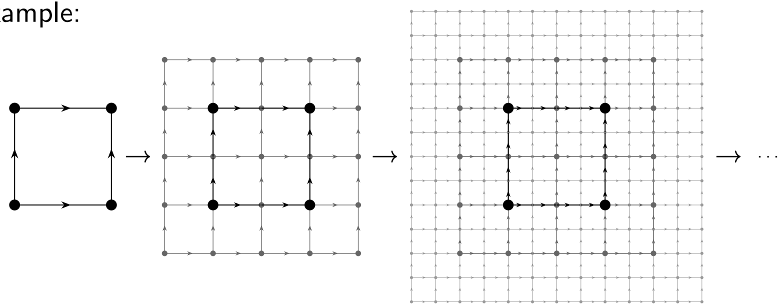
where $R^{(1)}_{i,j} = R^{(0)}_{i,j} \times R^{(0)}_{i,j} \colon \mathbf{G}^{(1)}_j \to \mathbf{G}^{(1)}_i$
$\Rightarrow$ a morphism of groupoids (functor) $R_{i,j}\colon \mathbf{G}_j \to \mathbf{G}_i$.

Direct systems

Let $(I, \leq)$ be a directed set, let $\mathbf{C}$ be a category.
A **direct system** in $\mathbf{C}$ consists of

- A net of objects $(A_i)_{i \in I}$;
- A collection of morphisms $(\varphi_{i,j} \colon A_i \to A_j)_{i,j \in I,\, i \leq j}$ such that
  - $\varphi_{i,i} = Id_{A_i}$;
  - $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ whenever $i \leq j \leq k$.

Example:



Inverse systems

Let $(I, \leq)$ be a directed set, let $\mathbf{C}$ be a category.
An **inverse system** in $\mathbf{C}$ consists of

- A net of objects $(A_i)_{i \in I}$;
- A collection of morphisms $(\varphi_{i,j} \colon A_j \to A_i)_{i,j \in I,\, i \leq j}$ such that
  - $\varphi_{i,i} = Id_{A_i}$;
  - $\varphi_{i,k} = \varphi_{i,j} \circ \varphi_{j,k}$ whenever $i \leq j \leq k$.

More direct and inverse systems

Let $((\Lambda_i)_{i \in I}, ((\Lambda_i, \Lambda_j, \iota_{i,j}))_{i,j \in I,\, i \leq j})$ be a direct system in **Refine**.
Contravariant functors to the categories of

- Spaces with group actions
- Groupoids

$\Rightarrow$ inverse systems.
Covariant functors to the categories of

- Hilbert spaces
- C^*-algebras

$\Rightarrow$ direct systems.

Inverse limits

Inverse limit of sets:

$$\varprojlim_{i\in I} A_i := \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i : \varphi_{i,j}(a_j) = a_i \ \forall i,j \in I,\ i \leq j \right\}.$$

- Spaces with group actions:

$$\left(\varprojlim_{i\in I} G^{\Lambda_i^0}, \varprojlim_{i\in I} G^{\Lambda_i^1} \right);$$

- Groupoids:

$$\varprojlim_{i\in I} \mathbf{G_i} \cong \text{Pair groupoid associated to } \varprojlim_{i\in I} G^{\Lambda_i^1}.$$

Direct limits (1)

Direct limit $\varinjlim_{i \in I} A_i$ of Banach spaces with linear contractions: Completion of the space $\coprod_{i \in I} A_i / \sim$, where

$$(i, a_i) \sim (j, a_j) \Leftrightarrow \exists k \in I,\ k \geq i, j \colon \varphi_{i,k}(a_i) = \varphi_{j,k}(a_j),$$

with respect to the norm

$$\|[i, a_i]_\sim\| := \lim_{j \in I \colon j \geq i} \|\varphi_{i,j}(a_i)\|_{A_j}.$$

- Hilbert spaces:

$$\varinjlim_{i \in I} L^2(G^{\Lambda_i^1}) \cong L^2\left(\varprojlim_{i \in I} G^{\Lambda_i^1}\right);$$

Direct limits (2)

- Observable algebras:

$$\varinjlim_{i \in I} B_0(L^2(G^{\Lambda_i^1})) \cong B_0\left(\varinjlim_{i \in I} L^2(G^{\Lambda_i^1})\right) \cong B_0\left(L^2\left(\varprojlim_{i \in I} G^{\Lambda_i^1}\right)\right).$$

It can be shown that

$$B_0\left(L^2\left(\varprojlim_{i \in I} G^{\Lambda_i^1}\right)\right) \cong C^*\left(\varprojlim_{i \in I} \mathbf{G}_i\right).$$

Hence

$$\varinjlim_{i \in I} C^*(\mathbf{G}_i) \cong C^*\left(\varprojlim_{i \in I} \mathbf{G}_i\right).$$

Outlook

- More direct proof of correspondence groupoids $\leftrightarrow$ observable algebras in the limit, e.g. using induced representations
- Dynamics
- Inverse system of state spaces, renormalization group

Questions?

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