Quantization of lattice gauge theories and their continuum limit

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Contents

- General setup for lattice gauge theory
- Refinements of graphs and associated constructions
- Quantization and groupoids
- (Continuum) limit
Discretisation of space

Hamiltonian framework:
Approximate a Cauchy surface $M$ with an oriented, finite, connected graph $\Lambda = (\Lambda^0, \Lambda^1)$:

- Points in $M$ $\rightarrow$ set of vertices $\Lambda^0$
- Paths between points $\rightarrow$ set of oriented edges $\Lambda^1$
Discretising connections

The graph \( \Lambda \) comes with two maps \( s, t : \Lambda^1 \to \Lambda^0 \): The source and target maps.

Let \( G \) be a compact, connected Lie group.

- The space of connections: \( G^{\Lambda^1} \)
- The gauge group: \( G^{\Lambda^0} \)
- Gauge transformations:

\[
\begin{align*}
G^{\Lambda^0} \times G^{\Lambda^1} &\to G^{\Lambda^1}, \\
((g_x)_{x \in \Lambda^0}, (a_e)_{e \in \Lambda^1}) &\mapsto (g_s(e)a_egt^{-1}_e)_{e \in \Lambda^1}.
\end{align*}
\]
Approximating the continuum

One graph is insufficient;
We require a **net** of graphs:
Refinement (1)

Let $\Lambda = (\Lambda^0, \Lambda^1)$ be an oriented graph.
Consider the free category $C_{\Lambda}$ generated by $\Lambda$:

- **Objects:** $\Lambda^0$
- **Let** $x, y \in \Lambda^0$.
  - **Morphisms from** $x$ to $y$: paths from $x$ to $y$ in $\Lambda$
- **Composition:** concatenation of paths
- **Identity element at** $x$: trivial path at $x$

**Notation:**

- **Set of objects:** $C^{(0)}_{\Lambda}$
- **Set of morphisms:** $C^{(1)}_{\Lambda}$
Refinement (2)

Let $\Lambda_i, \Lambda_j$ be oriented graphs. Suppose $\iota_{i,j} : C_i \to C_j$ is a functor. Suppose in addition that:

- The map between objects $\iota_{i,j}^{(0)} : C_i^{(0)} \to C_j^{(0)}$ is an injection;
- $\iota_{i,j}^{(1)} : C_i^{(1)} \to C_j^{(1)}$ maps edges to nontrivial paths;
- for each $e, e' \in \Lambda_i^1$, if $e \neq e'$, then $\iota_{i,j}^{(1)}(e)$ and $\iota_{i,j}^{(1)}(e')$ have no common edges;

Then we call $(\Lambda_i, \Lambda_j, \iota_{i,j})$ a refinement of the graph $\Lambda_i$. 
Examples (1)

Addition of an edge:

![Addition of an edge diagram]

Subdivision of an edge:

![Subdivision of an edge diagram]

Every refinement is a composition of a finite sequence of the above two types of refinements.
Examples (2)

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Category of refinements

Let \textbf{Refine} denote the category with

- **Objects**: finite, connected, oriented graphs $\Lambda$
- **Morphisms**: refinements $(\Lambda_i, \Lambda_j, \nu_{i,j})$

Given the Lie group $G$, there exists a contravariant functor from \textbf{Refine} to the category of spaces with group actions.

- **Objects**: $\Lambda \mapsto (G^{\Lambda_0}, G^{\Lambda_1}, \cdot)$
- **Morphisms**: $(\Lambda_i, \Lambda_j, \nu_{i,j}) \mapsto (\rho_{i,j}, R^{(0)}_{i,j})$

- $\rho_{i,j} : G^{\Lambda_j} \rightarrow G^{\Lambda_i}$ 'restriction map'
- $R^{(0)}_{i,j} : G^{\Lambda_j} \rightarrow G^{\Lambda_i}$
**Definition of $R_{i,j}^{(0)}$**

Addition of an edge:

$$(a_1, a_2, a_3) \leftrightarrow (a_1, a_2, a_3, a_4)$$

Subdivision of an edge:

$$(a_1 a_2) \leftrightarrow (a_1, a_2)$$

- For general refinements, define $R_{i,j}^{(0)}$ by composition of these maps;
- $R_{i,j}^{(0)}$ is independent of the chosen sequence;
Define covariant functors from \textbf{Refine} to the category of Hilbert spaces.

\textbf{Unreduced Hilbert space:}

- **Objects:** \( \Lambda \mapsto L^2(G^{\Lambda^1}) \)
- **Morphisms:**

\[ (\Lambda_i, \Lambda_j, \iota_{i,j}) \mapsto u_{i,j} : L^2(G^{\Lambda^1_i}) \to L^2(G^{\Lambda^1_j}), \quad \psi \mapsto \psi \circ R^{(0)}_{i,j} \]
Reduced Hilbert space

\[ L^2(G^{Λ^1}) \] carries a continuous unitary representation of \( G^{Λ^0} \):

\[(g \cdot \psi)(a) := \psi(g^{-1} \cdot a), \quad a \in G^{Λ^1}, \ g \in G^{Λ^0}.\]

Reduced Hilbert space:

- **Objects:** \( Λ \mapsto L^2(G^{Λ^1})^{G^{Λ^0}} \)
- **Morphisms:**

\[(Λ_i, Λ_j, \iota_{i,j}) \mapsto u_{i,j}|_{L^2(G^{Λ^1}_i)^{G^{Λ^0}_i}} : L^2(G^{Λ^1}_i)^{G^{Λ^0}_i} \to L^2(G^{Λ^1}_j)^{G^{Λ^0}_j}. \]
Define covariant functors from \textbf{Refine} to the category of \(C^*\)-algebras.

**Unreduced observable algebra:**
- **Objects:** \(\Lambda \mapsto B_0(L^2(G^{\Lambda^1}))\)
- **Morphisms:**

\[
(\Lambda_i, \Lambda_j, \iota_{i,j}) \mapsto v_{i,j}: B_0(L^2(G^{\Lambda^1_i})) \rightarrow B_0(L^2(G^{\Lambda^1_j})),
\]

\[
a \mapsto u_{i,j} a u_{i,j}^*.
\]

**Reduced observable algebra:**
- **Objects:** \(\Lambda \mapsto B_0(L^2(G^{\Lambda^1_0} G^{\Lambda^0}))\)
- **Morphisms:**

\[
(\Lambda_i, \Lambda_j, \iota_{i,j}) \mapsto v_{i,j} \bigg|_{B_0(L^2(G^{\Lambda^1_i} G^{\Lambda^0_i}))}: B_0(L^2(G^{\Lambda^1_i} G^{\Lambda^0_i})) \rightarrow B_0(L^2(G^{\Lambda^1_j} G^{\Lambda^0_j})).
\]
The Stone–Von Neumann Theorem

Let $U$ and $V$ be one-parameter groups. The continuous unitary representations of these groups on $L^2(\mathbb{R})$ given by

$$(U(t)\psi)(x) := \psi(x - t), \quad (V(s)\psi)(x) := e^{isx}\psi(x).$$

are jointly irreducible, and satisfy the Weyl form of the CCR

$$U(t)V(s) = e^{-ist}V(s)U(t).$$

Any other pair of continuous unitary jointly irreducible representation of $U$ and $V$ on a Hilbert space $\mathcal{H}$ satisfying the CCR is unitarily equivalent to this one.
Crossed product formulation (1)

Let $G$ be a locally compact group. Then $(C_0(G), G, L)$ is a $C^*$-dynamical system.

$$(L(g)(f))(a) = f(g^{-1} \cdot a).$$

Now form its crossed product $C^*$-algebra $C_0(G) \rtimes_L G$:

- Endow $C_c(G, C_0(G))$ with the structure of a $^*$-algebra by

$$f * g(a) := \int_G f(b)L(b)(g(b^{-1}a)) \, db,$$

$$f^*(a) := \Delta(a^{-1})L(a)(f(a^{-1})).$$
Crossed product formulation (2)

- Complete \( C_c(G, C_0(G)) \) with respect to the **universal norm**:

\[
\|f\| := \sup_{(\pi, U)} \|\pi \rtimes U(f)\|,
\]

where \((\pi, U)\) is a nondegenerate covariant representation of \((C_0(G), G, L)\).

A **covariant representation** \((\pi, U)\) consists of

- a *-representation: \( \pi: C_0(G) \to B(H) \);
- a continuous unitary representation: \( U: G \to U(H) \);
- a covariance condition:

\[
\pi(L(a)(f)) = U(a)\pi(f)U(a)^*, \quad \forall a \in G, \forall f \in C_0(G).
\]

\[
\pi \rtimes U: C_0(G) \rtimes L G \to B(H), \quad f \mapsto \int_G \pi(f(a))U(a)\, da.
\]
Let $G$ be a locally compact group. Then

$$C_0(G) \rtimes_L G \cong B_0(L^2(G)).$$

In particular, the natural covariant representation $(M, L)$ on $L^2(G)$ is irreducible and faithful.
Crossed products and groupoids

Let \( G \) be a compact Lie group. Then

\[
B_0(L^2(G)) \cong C(G) \rtimes_L G \cong C^*(G \ltimes G) \cong C^*(G).
\]

Here

- \( G \ltimes G \) is an action groupoid;
- \( G \) is the pair groupoid associated to \( G \);
- \( C^*(\cdot) \) denotes the groupoid \( C^* \)-algebra of the groupoid.
What is a groupoid?

A **groupoid** is a small category in which each morphism is invertible.

**Examples:**

Any group $H$:
- **Objects**: a set containing a single point \{*$\} 
- **Morphisms**: $H$
- **Composition**: group multiplication in $H$
- **Identity element at $*$**: identity of $H$

Any set $X$:
- **Objects**: $X$
- **Morphisms**: $X$
- **Composition**: $x \circ x = x$
- **Identity element at $x$**: $x$
The action groupoid

Let $X$ be a set, $H$ a group with unit $e$.
Suppose $\cdot : H \times X \to X$ is a group action.
The **action groupoid** $H \ltimes X$ is now defined as follows:

- **Objects:** $X$
- **Morphisms:** $H \times X$, $(h, x) \in \text{Mor}(h^{-1} \cdot x, x)$.
- **Composition:** $(h_1, x_1) \circ (h_2, x_2) = (h_1 h_2, x_1)$ if $x_2 = h_1^{-1} \cdot x_1$
- **Identity element at** $x$: $(e, x)$

Take $X = H = G$, and $\cdot : G \times G \to G$ is the group multiplication
$\Rightarrow G \ltimes G$. 
The pair groupoid

Let $X$ be a set.

The **pair groupoid** $X$ is now defined as follows:

- **Objects**: $X$
- **Morphisms**: $X \times X$, $\text{Mor}(x, y) = \{(x, y)\}$.
- **Composition**: $(x_1, y_1) \circ (x_2, y_2) = (x_2, y_1)$ if $x_1 = y_2$
- **Identity element at** $x$: $(x, x)$

Take $X = G \Rightarrow G$.

$G \ltimes G$ is isomorphic to $G$:

$$(G \ltimes G)^{(1)} \to G^{(1)}, \quad (b, a) \mapsto (b^{-1}a, a).$$

This isomorphism is compatible with the Haar systems on both groupoids $\Rightarrow C^*(G \ltimes G) \cong C^*(G)$. 

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Let \( f \in C_c(G^{(1)}) = C_c(G \times G) \). Let \( \pi : C^*(G) \to B(L^2(G)) \) be the natural representation. Then
\[
\pi(f)(\psi)(a) = \int_G f(b, a)\psi(b) \, db,
\]
where \( \psi \in L^2(G) \) and \( a \in G \).

**Question:** How does this operator behave w.r.t. refinements? Let \((\Lambda_i, \Lambda_j, \iota_{i,j})\) be a refinement. Then
\[
\nu_{i,j}(\pi_i(f)) = \pi_j(f \circ R^{(1)}_{i,j}),
\]
where \( R^{(1)}_{i,j} = R^{(0)}_{i,j} \times R^{(0)}_{i,j} : G^{(1)}_j \to G^{(1)}_i \Rightarrow \) a morphism of groupoids (functor) \( R_{i,j} : G_j \to G_i \).
Direct systems

Let \((I, \leq)\) be a directed set, let \(\mathbf{C}\) be a category. A **direct system** in \(\mathbf{C}\) consists of

- A net of objects \((A_i)_{i \in I}\);
- A collection of morphisms \((\varphi_{i,j}: A_i \to A_j)_{i,j \in I, i \leq j}\) such that
  - \(\varphi_{i,i} = \text{Id}_{A_i}\);
  - \(\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}\) whenever \(i \leq j \leq k\).

Example:
Inverse systems

Let \((I, \leq)\) be a directed set, let \(\mathbf{C}\) be a category. An inverse system in \(\mathbf{C}\) consists of

- A net of objects \((A_i)_{i \in I}\);
- A collection of morphisms \((\varphi_{i,j} : A_j \to A_i)_{i,j \in I, i \leq j}\) such that
  - \(\varphi_{i,i} = \text{Id}_{A_i}\);
  - \(\varphi_{i,k} = \varphi_{i,j} \circ \varphi_{j,k}\) whenever \(i \leq j \leq k\).
More direct and inverse systems

Let \(((\Lambda_i)_{i \in I}, ((\Lambda_i, \Lambda_j, \iota_{i,j}))_{i,j \in I, i \leq j})\) be a direct system in \textbf{Refine}. Contravariant functors to the categories of
- Spaces with group actions
- Groupoids

\implies \text{inverse systems.}

Covariant functors to the categories of
- Hilbert spaces
- \(C^\ast\)-algebras

\implies \text{direct systems.}
Inverse limits

**Inverse limit** of sets:

\[ \lim_{i \in I} A_i := \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i : \varphi_{i,j}(a_j) = a_i \quad \forall i, j \in I, \quad i \leq j \right\}. \]

- Spaces with group actions:

\[ \left( \lim_{i \in I} G^\Lambda_i^0, \lim_{i \in I} G^\Lambda_i^1 \right); \]

- Groupoids:

\[ \lim_{i \in I} G_i \cong \text{Pair groupoid associated to} \lim_{i \in I} G^\Lambda_i^1. \]
Direct limits (1)

**Direct limit** $\lim_{i \in I} A_i$ of Banach spaces with linear contractions: Completion of the space $\bigsqcup_{i \in I} A_i / \sim$, where

$$(i, a_i) \sim (j, a_j) \iff \exists k \in I, k \geq i, j : \varphi_{i,k}(a_i) = \varphi_{j,k}(a_j),$$

with respect to the norm

$$\| [i, a_i]_\sim \| := \lim_{j \in I : j \geq i} \| \varphi_{i,j}(a_i) \|_{A_j}.$$ 

- Hilbert spaces:

$$\lim_{i \in I} L^2(G_i^\Lambda) \cong L^2 \left( \lim_{i \in I} G_i^\Lambda \right);$$
Direct limits (2)

- Observable algebras:

\[
\lim_{i \in I} B_0(L^2(G^\Lambda_i^1)) \cong B_0 \left( \lim_{i \in I} L^2(G^\Lambda_i^1) \right) \cong B_0 \left( L^2 \left( \lim_{i \in I} G^\Lambda_i^1 \right) \right).
\]

It can be shown that

\[
B_0 \left( L^2 \left( \lim_{i \in I} G^\Lambda_i^1 \right) \right) \cong C^* \left( \lim_{i \in I} G_i \right).
\]

Hence

\[
\lim_{i \in I} C^*(G_i) \cong C^* \left( \lim_{i \in I} G_i \right).
\]
Outlook

- More direct proof of correspondence groupoids ↔ observable algebras in the limit, e.g. using induced representations
- Dynamics
- Inverse system of state spaces, renormalization group
Questions?
Selected references (1)

J. C. Baez.
Spin networks in gauge theory.

D. P. Williams.
Crossed products of C\(^\ast\)-algebras, volume 134 of Mathematical surveys and monographs.

J. Renault.
A Groupoid approach to C\(^\ast\)-algebras, volume 793 of Lecture notes in Mathematics.
Selected references (2)

N. P. Landsman.
*Mathematical topics between classical and quantum mechanics.*

A. Ashtekar, J. Lewandowski,
Representation theory of analytic holonomy C*-algebras,

P. S. Muhly, J. N. Renault, D. P. Williams
Equivalence and isomorphism for groupoid C*-algebras,