On the semiclassical limit of quantum fields on quantum cosmological spacetimes

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February 14, 2017

Tux

word based on [Stottmeister, Thiemann, 2016]

DOI:10.1063/1.4954228, 10.1063/1.4954803, 10.1063/1.4960823

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Outline

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Aims & Motivation

- Formulate the semiclassical limit of quantum fields of quantum spacetimes

  $\rightsquigarrow$ Extract QFT on curved spacetimes

  $\rightsquigarrow$ Control the QFT and spacetime dynamics in this limit

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- Semiclassical formalism given by space-adiabatic perturbation theory [Panati, Spohn, Teufel, 2003]

  $\rightsquigarrow$ Formalises the splitting of quantum system into fast and slow variables

  $\rightsquigarrow$ Requires a suitable Weyl-Moyal calculus

Section outline

Prerequisites

Consider a quantum dynamical system, $(\mathfrak{H}, (\hat{H}, D(\hat{H})))$, such that:

(a) The Hilbert space, $\mathfrak{H}$, splits into slow, $\mathfrak{H}_s$, and fast, $\mathfrak{H}_f$, degrees of freedom

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$$\widehat{\cdot}^{\,\varepsilon} : S^\infty(\varepsilon; \Gamma, \mathcal{B}(\mathfrak{H}_f)) \subset C^\infty(\Gamma, \mathcal{B}(\mathfrak{H}_f)) \longrightarrow L(\mathfrak{H}), \tag{1}$$

of the (classical) phase space, Γ, of the slow variables with values in linear operators, $L(\mathfrak{H})$

$\rightsquigarrow$ The operator product in $L(\mathfrak{H})$ corresponds to $\star_\varepsilon$-product on $S^\infty(\varepsilon; \Gamma, \mathcal{B}(\mathfrak{H}_f))$

$\rightsquigarrow$ Asymptotic expansion in ε. $\mathcal{O}(\varepsilon^\infty)$-elements in $S^\infty(\varepsilon; \Gamma, \mathcal{B}(\mathfrak{H}_f))$ are "small" bounded operators (*smoothing operators*)

$\rightsquigarrow$ $\widehat{H_\varepsilon}^{\,\varepsilon} = \hat{H}$ with asymptotic expansion

$$H_\varepsilon \sim \sum_{k=0}^{\infty} \varepsilon^k H_k, \;\; \forall k \in \mathbb{N}_0 : H_k \in C^\infty(\Gamma, \mathcal{B}(\mathfrak{H}_f)). \tag{2}$$

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(c) There is a relevant part, $\sigma_*(H_0)$, of the (point-wise) spectrum of the *principal symbol* H_0, that is isolated by a finite gap (global over Γ).

The algorithm I

Space-adiabatic perturbation theory consists of four steps:

1. Construct an almost invariant projection $\hat{\Pi}^{\varepsilon}$, i.e.

$$[\hat{H}, \hat{\Pi}^{\varepsilon}] = \mathcal{O}_0(\varepsilon^{\infty}) \tag{3}$$

$\rightsquigarrow$ Use the spectral projection π_0 of H_0 onto the relevant part σ_*

$\rightsquigarrow$ $\hat{\Pi}^{\varepsilon} = \widehat{\pi_{\varepsilon}}^{\varepsilon} + \mathcal{O}_0(\varepsilon^{\infty})$

$\rightsquigarrow$ π_{ε} has an asymptotic expansion with principal symbol π_0,

$$\pi_{\varepsilon} \sim \sum_{k=0}^{\infty} \varepsilon^k \pi_k, \tag{4}$$

that qualifies as an invariant projection relative to Moyal product:

$$\pi_{\varepsilon} \star_{\varepsilon} \pi_{\varepsilon} = \pi_{\varepsilon}, \qquad \pi_{\varepsilon}^* = \pi_{\varepsilon}, \qquad [H_{\varepsilon}, \pi_{\varepsilon}]_{\star_{\varepsilon}} = 0. \tag{5}$$

The subspace $\hat{\Pi}^{\varepsilon}\,\mathfrak{H} \subset \mathfrak{H}$ is called almost invariant subspace, as it remains approximately invariant w.r.t. the dynamics (*Duhamel's formula*):

$$[e^{-i\hat{H}s}, \hat{\Pi}^{\varepsilon}] = \mathcal{O}_0(|s|\varepsilon^{\infty}). \tag{6}$$

The algorithm II

2. Construct a unitary operator $\hat{U}^{\varepsilon} \in \mathcal{B}(\mathfrak{H})$

   ⇝ Identify the almost invariant subspace $\hat{\Pi}^{\varepsilon}\,\mathfrak{H}$ with an ε-independent reference (sub)space $\hat{\Pi}_r\,\mathfrak{H}$

   ⇝ Quantisation of a semiclassical symbol $u_{\varepsilon} \in S^{\infty}(\varepsilon;\Gamma,\mathcal{B}(\mathfrak{H}_f))$:

$$u_{\varepsilon} \sim \sum_{k=0}^{\infty} \varepsilon^k u_k, \tag{7}$$

   ⇝ u_0 defines a reference projection $\pi_r \in \mathcal{B}(\mathfrak{H}_f)$ by

$$u_0(\gamma)\pi_0(\gamma)u_0(\gamma)^* = \pi_r \tag{8}$$

   u_0 trivialises the *adiabatic bundle* $\pi_0\,\mathfrak{H} \to \Gamma$

   ⇝ The quantisation $\hat{\Pi}_r = \mathbb{1}_{\mathfrak{H}_s} \otimes \pi_r$ defines the reference space $\hat{\Pi}_r\,\mathfrak{H}$

$$\hat{\Pi}_r = \hat{U}^{\varepsilon}\hat{\Pi}^{\varepsilon}(\hat{U}^{\varepsilon})^*. \tag{9}$$

   ⇝ Characterise u_{ε} (not uniquely) by:

$$u_{\varepsilon} \star_{\varepsilon} u_{\varepsilon}^* = 1 = u_{\varepsilon}^* \star_{\varepsilon} u_{\varepsilon}, \qquad u_{\varepsilon} \star_{\varepsilon} \pi_{\varepsilon} \star_{\varepsilon} u_{\varepsilon}^* = \pi_r. \tag{10}$$

The algorithm III

3. Map the dynamics of $\hat{H}$ (almost) inside $\hat{\Pi}^{\varepsilon}\,\mathfrak{H}$ to the reference space $\hat{\Pi}_r\,\mathfrak{H}$

   $\rightsquigarrow$ Effective Hamiltonian $\hat{h}$

   $\rightsquigarrow$ quantisation of a self-adjoint semiclassical symbol $h_\varepsilon \in S^\infty(\varepsilon;\Gamma,\mathcal{B}(\mathfrak{H}_f))$, the effective Hamiltonian symbol:

$$h_\varepsilon \sim u_\varepsilon \star_\varepsilon H_\varepsilon \star_\varepsilon u_\varepsilon^*. \quad (11)$$

   $\rightsquigarrow$ Computation of the $\mathcal{O}(\varepsilon^n)$-truncations $h_{\varepsilon,(n)}$ possible

   $\rightsquigarrow$ The effective Hamiltonian $\hat{h}$ satisfies:

$$[\hat{h},\hat{\Pi}_r] = 0, \qquad e^{-i\hat{H}s} - (\hat{U}^\varepsilon)^* e^{-i\hat{h}s}\hat{U}^\varepsilon = \mathcal{O}_0((1+|s|)\varepsilon^\infty), \quad (12)$$

   which entails the space adiabatic theorem with time scale $t > 0$:

$$e^{-i\hat{H}s}\hat{\Pi}^\varepsilon - (\widehat{u_{\varepsilon,(n)}}^{\varepsilon})^* e^{-i\widehat{h_{\varepsilon,(n+k)}}^{\varepsilon}s}\hat{\Pi}_r\widehat{u_{\varepsilon,(n)}}^{\varepsilon} = \mathcal{O}_0((1+|t|)\varepsilon^{n+1}), \quad (13)$$

   for large enough $n,k \in \mathbb{N}_0,\ |s| \le \varepsilon^{-k}t$

The algorithm IV

4. Consider an isolated eigenvalue $\sigma_*(H_0) = \{E_*\}$ (possibly degenerate)

$\rightsquigarrow$ semiclassical approximation to the dynamics inside $\hat{\Pi}^\varepsilon\,\mathfrak{H}$

$$\widehat{O_\varepsilon}^\varepsilon(t) = e^{\frac{i}{\varepsilon}\hat{h}t}\widehat{O_\varepsilon}^\varepsilon e^{-\frac{i}{\varepsilon}\hat{h}t}, \quad \left(\partial_t\widehat{O_\varepsilon}^\varepsilon\right)(t) = \frac{i}{\varepsilon}[\hat{h}, \widehat{O_\varepsilon}^\varepsilon(t)] \tag{14}$$

$\rightsquigarrow$ Expansion on the level of symbols (Egorov's hierachy):

$$(\partial_t O_0)(t) = \{E_*, O_0(t)\} + i[h_1, O_0(t)]\ldots \tag{15}$$

$\rightsquigarrow$ Solve iteratively for $O_n,\ n \in \mathbb{N}_0$

$\rightsquigarrow$ Time evolution of the principal symbol O_0:

$$O_0(\gamma,t) = V(\gamma,t)^* O_0(\Phi_t(\gamma))V(\gamma,t), \qquad O_0(\gamma,0) = O_0(\gamma),\ \gamma \in \Gamma, \tag{16}$$

where

$$\begin{aligned} \partial_t\Phi_t(\gamma) &= X_{E^*}(\gamma), & \partial_t V(\gamma,t) &= -ih_1(\Phi_t(\gamma))V(\gamma,t) \\ \Phi_0(\gamma) &= \gamma, & V(\gamma,0) &= \mathbb{1}_{\pi_r\,\mathfrak{H}_f}, \end{aligned} \tag{17}$$

with X_{E_*} denoting the Hamiltonian vector field of E_* w.r.t. the symplectic structure on Γ. For scalar principal symbols $O_0 = o_0\mathbb{1}_{\pi_r\,\mathfrak{H}_f}$ (16) reduces to

$$o_0(\gamma,t) = o_0(\Phi_t(\gamma)), \qquad o_0(\gamma,0) = o_0(\gamma),\ \gamma \in \Gamma, \tag{18}$$

hence the name semiclassical approximation.

The algorithm V – final remarks

- First order *Egorov's theorem* for the principal part O_0 and its quantisation:

$$\forall T \in \mathbb{R}_{\geq 0} : \exists C_T > 0 : \forall t \in [-T, T] : \left\| e^{\frac{i}{\varepsilon}\hat{h}t}\widehat{O_0}^{\varepsilon} e^{-\frac{i}{\varepsilon}\hat{h}t} - \widehat{O_0(t)}^{\varepsilon} \right\|_{\mathcal{B}(\hat{\Pi}_r\,\mathfrak{H})} \leq \varepsilon\, C_T. \quad (19)$$

- Semiclassical observables w.r.t. $\hat{\Pi}^{\varepsilon}\,\mathfrak{H}$ are modelled by operators $\hat{O} \in L(\mathfrak{H})$ that are almost diagonal w.r.t. $\hat{\Pi}^{\varepsilon}$:

$$[\hat{O}, \hat{\Pi}^{\varepsilon}] = \mathcal{O}_0(\varepsilon^{\infty}). \quad (20)$$

  The dynamics of general observables $\hat{O} \in L(\mathfrak{H})$ can be considered in the weak sense:

$$\hat{O}_{|\hat{\Pi}^{\varepsilon}\,\mathfrak{H}} = \hat{\Pi}^{\varepsilon}\hat{O}\hat{\Pi}^{\varepsilon}. \quad (21)$$

Section outline

Almost-periodic pseudo-differential operators

A suitable Weyl-Moyal formalism to handle cosmological scenarios was introduced by [Shubin, 1978] and elaborated on in [Stottmeister, Thiemann, 2016]:

$$(A_\sigma \Psi)(x) = \frac{1}{2\pi\varepsilon} \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} dx' \, \sigma(\tfrac{1}{2}(x+x'), \lambda) e_{-\lambda}(\tfrac{x-x'}{\varepsilon}) \Psi(x'), \tag{22}$$

for $\Psi \in \mathrm{Trig}(\mathbb{R})$ or $CAP^\infty(\mathbb{R})$.

$$\sigma \in APS^m_{\rho,\delta}(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2) \;:\Leftrightarrow\; \mathbb{R} \ni \lambda \mapsto \sigma(\,.\,,\lambda) \in CAP(\mathbb{R}) \text{ is continuous} \tag{23}$$

$$m \in \mathbb{R},\; 0 \le \delta \le \rho \le 1$$

$$\&\; \forall \alpha, \beta \in \mathbb{N}_0 : \forall (x,\lambda) \in \mathbb{R}^2 : \exists C_{\alpha\beta} > 0 : |(\partial_x^\alpha \partial_\lambda^\beta \sigma)(x,\lambda)| \le C_{\alpha\beta} \langle \lambda \rangle^{m-\rho\beta+\delta\alpha}, \tag{24}$$

$$APS^{-\infty}(\mathbb{R}^2) := \bigcap_{m\in\mathbb{R}} APS^m_{\rho,\delta}(\mathbb{R}^2), \;\; APS^\infty_{\rho,\delta}(\mathbb{R}^2) := \bigcup_{m\in\mathbb{R}} APS^m_{\rho,\delta}(\mathbb{R}^2).$$

Section outline

Definition of toy models I

- Consider models given by (FLRW cosmologies deparametrised w.r.t. a dust field)

$$\{p_\omega, \omega\} = 1, \qquad |\omega| = a^3 l^3, \tag{25}$$

$$H_{\mathsf{G}} = -\frac{3}{4}\kappa' |\omega| p_\omega^2 + \frac{2\Lambda}{3\kappa'}|\omega| - \frac{2l^2 k}{\kappa'}|\omega|^{\frac{1}{3}}, \tag{26}$$
$$H_\phi = \frac{\lambda'}{2|\omega|}\sum_{s\in\hat{\Sigma}}\left(p_s^2 + \frac{1}{\lambda'^2}\left((ls)^2|\omega|^{\frac{4}{3}} + m^2|\omega|^2\right)q_s^2\right),$$

with the canonical pairs $(p_\omega, \omega) \in \mathbb{R}^2,\ (p_s, q_s) \in \mathbb{R}^2,\ s \in \hat{\Sigma} = \frac{2\pi}{l}\mathbb{Z}^3$

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- The canonical pair (p_ω, ω) is related to the LQC variables (b, ν)

$$\{b, \nu\} = 2, \qquad \omega = \frac{\kappa'\gamma}{4}\nu, \qquad p_\omega = \frac{2}{\kappa'\gamma}b, \tag{27}$$

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- The ratio $\frac{\kappa'}{\lambda'} = \epsilon^2 \sim \frac{m_\Phi^2}{m_{\mathsf{Planck}}^2}$ serves as adiabatic parameter

Definition of toy models II

- Setting $\lambda' = 1$ and specialising to $\Lambda = 0, k = 0$

$$H = H_{\mathrm{G}} + H_{\phi} = -\frac{3}{4}|\omega|(\varepsilon p_{\omega})^2 + \frac{1}{2|\omega|}\sum_{s\in\hat{\Sigma}}\left(p_s^2 + \underbrace{\left((ls)^2|\omega|^{\frac{4}{3}} + m^2|\omega|^2\right)}_{=\Omega_s(|\omega|;l,m)^2} q_s^2\right) \tag{28}$$

- Using an LQC-type quantisation for (p_ω, ω) allows to define regularised Hamiltonians

$$\hat{H}_{\ell}^{f,\mp} = \mp\frac{3}{4}\widehat{\frac{\sin(\omega_0 p_\omega)}{\omega_0}}\,\widehat{|\omega|_\ell}\,\widehat{\frac{\sin(\omega_0 p_\omega)}{\omega_0}} + \frac{1}{2}\widehat{|\omega|_\ell^{-1}}\sum_{s\in\hat{\Sigma}}\left(\left(\hat{p}_s^f\right)^2 + \Omega_s(\widehat{|\omega|_\ell}; l, m)^2\left(\hat{q}_s^f\right)^2\right), \tag{29}$$

on $D(\hat{H}_{\ell}^{f,\mp}) := C_c^\infty(\mathbb{R}) \otimes F_s(l^2(\hat{\Sigma})) \subset L^2(\mathbb{R}, d\omega) \otimes \mathcal{F}_s(l^2(\hat{\Sigma}))$ or

$$d(\hat{H}_{\ell}^{f,\mp}) := d(\mathbb{R}) \otimes F_s(l^2(\hat{\Sigma})) \subset l^2(\mathbb{R}) \otimes \mathcal{F}_s(l^2(\hat{\Sigma}))$$

- $\hat{p}_s^f := \frac{-i}{\sqrt{2}}(\overline{f}_s a_s - f_s a_s^*)$, $\hat{q}_s^f := \frac{1}{\sqrt{2}}(\overline{f}_s a_s + f_s a_s^*)$ for $f \in h_{\frac{1}{2}}^1(\hat{\Sigma}) \subset l^2(\hat{\Sigma})$

The perturbative expansion I

- $H_\ell^{f,\mp}$ is the ε-Weyl quantisation of the $L(F_s(l^2(\hat{\Sigma})), \mathcal{F}_s(l^2(\hat{\Sigma})))$-valued symbol

$$\begin{aligned} H_\ell^{f,\mp}(\omega, p_\omega) &= \pm H_{\mathsf{G},\ell}(\omega, p_\omega) + H^f_{\pi,\ell}(\omega) \\ &= \pm \frac{3}{8\omega_0^2}\left(\cos(2\omega_0 p_\omega)|\omega|_\ell - \frac{1}{2}\left(|\omega + \varepsilon\omega_0|_\ell + |\omega - \varepsilon\omega_0|_\ell\right)\right)\mathbb{1}_{\mathcal{F}(l^2(\hat{\Sigma}))} \\ &\quad + H^f_{\phi,\ell}(\omega) \\ &= \mp\frac{3}{4\omega_0^2}\sin^2(\omega_0 p_\omega)|\omega|_\ell \mathbb{1}_{\mathcal{F}(l^2(\hat{\Sigma}))} + H^f_{\phi,\ell}(\omega) \\ &\quad \underbrace{\mp\frac{3}{8\omega_0^2}\left(\frac{1}{2}\left(|\omega + \varepsilon\omega_0|_\ell + |\omega - \varepsilon\omega_0|_\ell\right) - |\omega|_\ell\right)}_{:=\sigma_\ell^{(1)}(\omega,\varepsilon\omega_0),\ \sigma_\ell^{(1)}(\ .\ ,\varepsilon\omega_0)\in C_c^\infty(\mathbb{R})}\mathbb{1}_{\mathcal{F}(l^2(\hat{\Sigma}))}, \end{aligned} \tag{30}$$

- The leading order in the semi-classical expansion of $\sigma_\ell^{(1)}$ is $\mathcal{O}(\varepsilon^2)$:

$$\sigma_\ell^{(1)}(\omega, \varepsilon\omega_0) \sim -\varepsilon^2\frac{3}{8}\delta_\ell(\omega) + \mathcal{O}(\varepsilon^4), \tag{31}$$

The perturbative expansion II

- For simplicity, we further restrict ourselves to Hamiltonians $\hat{H}^{f,\mp}_{\ell}$ with $f \in d(\hat{\Sigma})$, s.t. $f_s = 1$ for $|s| \leq s_c$ and $f_s = 0$ for $|s| > s_c$
- The (low lying) spectrum of the ε^0-order symbol, $H^{s_c,\mp}_{\ell,0}(\omega, p_\omega)$ has the following properties:
  1. The minimal distance between the discrete eigenvalues,
  $$E^{s_c,\mp}_{n}(|\omega|_\ell, p_\omega; l, m) = \mp\frac{3}{4\omega_0^2}\sin^2(\omega_0 p_\omega)|\omega|_\ell + \sum_{|s|\leq s_c}\frac{\Omega_s(|\omega|_\ell; l, m)}{|\omega|_\ell}(n_s + \tfrac{1}{2}), \quad (32)$$
  is $d_{\min} = \frac{\Omega_0(|\omega|_\ell; l, m)}{|\omega|_\ell} = m$
  2. The ground state energy, $E^{s_c,\mp}_{0}(|\omega|_\ell, p_\omega; l, m) = \mp\frac{3}{4\omega_0^2}\sin^2(\omega_0 p_\omega)|\omega|_\ell + \frac{1}{2}\sum_{|s|\leq s_c}\frac{\Omega_s(|\omega|_\ell; l, m)}{|\omega|_\ell}$, and the lowest eigenvalue above it, $E^{s_c,\mp}_{1}(|\omega|_\ell, p_\omega; l, m) = m + E^{s_c,\mp}_{0}(|\omega|_\ell, p_\omega; l, m)$, are non-degenerate.
  3. For sufficiently large volumes, $|\omega|_\ell^{\frac{2}{3}} > \frac{4\pi^2}{3m^2}$, the second lowest eigenvalue,
  $$E^{s_c,\mp}_{2}(|\omega|_\ell, p_\omega; l, m) = \sqrt{m^2 + 4\pi^2|\omega|_\ell^{-\frac{2}{3}}} + E^{s_c,\mp}_{0}(|\omega|_\ell, p_\omega; l, m), \quad (33)$$
  is sixfold degenerate.
  4. As $\forall s \in \hat{\Sigma} : \lim_{|\omega|\to\infty}\frac{\Omega_s(|\omega|_\ell; l, m)}{|\omega|_\ell} = m$, we have $\lim_{|\omega|\to\infty}|E^{s_c,\mp}_{n}(|\omega|_\ell, p_\omega; l, m) - E^{s_c,\mp}_{n'}(|\omega|_\ell, p_\omega; l, m)| = 0$ for all n, n' such that $\sum_{|s|\leq s_c} n_s = \sum_{|s|\leq s_c} n'_s$.

The perturbative expansion III

- Applying the algorithm to $H^{s_c,\mp}_{\ell} = H^{s_c,\mp}_{\ell,0} + \varepsilon^2 H^{s_c,\mp}_{\ell,2} + \mathcal{O}(\varepsilon^4) \in S^1_{1,0}(\varepsilon)$, and its uniformly isolated ground state band $\{E^{s_c,\mp}_0(|\omega|_\ell, p_\omega; l, m)\}_{(\omega,p_\omega)\in\mathbb{R}^2}$:

$$
\begin{aligned}
H^{s_c,\mp}_{\ell,0}(\omega,p_\omega) &= U^{s_c}(|\omega|_\ell; l, m)\Big(E^{s_c,\mp}_0(|\omega|_\ell, p_\omega; l, m)\mathbb{1}_{\mathcal{F}_s(\mathbb{C}^{n_{s_c}})} \\
&\qquad + \sum_{|s|\leq s_c}\frac{\Omega_s(|\omega|_\ell; l, m)}{|\omega|_\ell}a^*_s a_s\Big)U^{s_c}(|\omega|_\ell; l, m)^*, \qquad (34)\\
H^{s_c,\mp}_{\ell,2}(\omega,p_\omega) &= -\frac{3}{8}\delta_\ell(\omega)\mathbb{1}_{\mathcal{F}_s(\mathbb{C}^{n_{s_c}})},\\
\pi^{s_c,\mp}_{\ell,0}(\omega) &= \Omega_{s_c}(|\omega|_\ell; l, m)\otimes(\Omega_{s_c}(|\omega|_\ell; l, m),\ .\ )_{\mathcal{F}_s(\mathbb{C}^{n_{s_c}})},\\
u^{s_c,\mp}_{\ell,0}(\omega) &= U^{s_c}(|\omega|_\ell; l, m)^*\\
&= \prod_{|s|\leq s_c} e^{-\frac{1}{2}\xi^{s_c}_s(|\omega|_\ell; l, m)((a^*_s)^2 - a^2_s)},\ \pi_r = \Omega\otimes(\Omega,\ .\ )_{\mathcal{F}_s(\mathbb{C}^{n_{s_c}})}.
\end{aligned}
$$

- Reference unitaries are squeezings of the reference vacuum

The perturbative expansion IV

- The effective Hamiltonian on the reference space takes the form

$$\hat{h}^{s_c,\mp}_{\ell,(1)} = \widehat{\left(\mp\frac{3}{4\omega_0^2}|\omega|_\ell \sin^2(\omega_0 p_\omega) + \left(\frac{1}{2}\sum_{|s|\leq s_c}\frac{\Omega_s(|\omega|_\ell; l, m)}{|\omega|_\ell}\right)\right)} \otimes \pi_r \tag{35}$$

$$= \left(\mp\frac{3}{4\omega_0^2}\left(W_\varepsilon(0,\omega_0)\widehat{|\omega|_\ell}W_\varepsilon(0,\omega_0) - 2\widehat{|\omega|_\ell} + W_\varepsilon(0,-\omega_0)\widehat{|\omega|_\ell}W_\varepsilon(0,-\omega_0)\right)\right.$$

$$\left. + \frac{1}{2}\sum_{|s|\leq s_c}\widehat{|\omega|_\ell}^{-1}\Omega(\widehat{|\omega|_\ell}; l, m)\right)\otimes\pi_r.$$

- Time evolution in the gravitation sector is affected by the eigenvalue corresponding to the almost invariant subspace (Peierl's substitution)
- Effective spacetime depends on the spectral band of the quantum field

Section outline

Summary & final remarks

- Approximation method independent of specific states, e.g. coherent states
- Control on the error terms is possible, though difficult
- Inclusion of back reaction is possible
- Treatment of quantum fields on quantum spacetimes requires regularisation (existence of reference unitaries)
- Classical phase space and dynamics are extracted from the quantum system

Thank you!