LQG, the signature-changing Poincaré algebra and spectral dimension

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Outline:

1. Deformed algebras of relativistic symmetries
   - Deformed HDA $\Rightarrow$ deformed Poincaré algebra
   - An abstracted example of the deformation

2. Spectral dimension in the case of deformed symmetries
   - Necessary ingredients to calculate the dimension
   - Definitions and results for our model

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Context and motivation

Deformed relativistic symmetries

- LQG deformation of the hypersurface deformation algebra [M. Bojowald & G. M. Paily (2012), M. Bojowald et al. (2016)]
- The corresponding deformed Poincaré algebra and its relation with the κ-Poincaré deformation [G. Amelino-Camelia et al. (2017)]; exact results in 3d [F. Cianfrani et al. (2016)]
- The dynamical signature change and asymptotic silence scenario in cosmology [J. Mielczarek (2012), M. Bojowald & J. Mielczarek (2015)]

Spectral dimension

- The dimensional flow to 2 in the UV is a common QG prediction
- Different results may indicate separate phases of gravity

Hypersurface deformation algebra (HDA)

In the ADM formalism a covariant field theory has to appropriately transform under local diffeomorphisms on any spatial hypersurface. These diffeos can be parametrized by a lapse function N and shift vector field N^a, $a = 1, 2, 3$, while their generators are the scalar constraint $S[N]$ and diffeomorphism constraint $D[N^a]$, satisfying the HDA

$$\begin{aligned} \{D[N^a], D[\tilde{N}^a]\} &= D\left[N^b \partial_b \tilde{N}^a - \tilde{N}^b \partial_b N^a\right] , \\ \{S[N], D[N^a]\} &= -S\left[N^b \partial_b N\right] , \\ \{S[N], S[\tilde{N}]\} &= D\left[s q^{ab} \left(N \partial_b \tilde{N} - \tilde{N} \partial_b N\right)\right] , \end{aligned} \tag{1}$$

where q_{ab} denotes the spatial metric. Spacetime signature $s = 1$ in the Lorentzian case and $s = -1$ in the Euclidean one.
The prediction of LQG, besides quantization of the constraints, is that several approximate calculations lead to a deformation in the bracket

$$\{S[N], S[\tilde{N}]\} = D\left[s \Omega q^{ab} \left(N \partial_b \tilde{N} - \tilde{N} \partial_b N\right)\right] , \tag{2}$$

with some function Ω of gravitational variables.

Linearization of the (classical) HDA

HDA can be seen as a generalization of the Poincaré algebra. To uncover the underlying Poincaré symmetry one restricts to linear hypersurface deformations, imposing the conditions $q_{ab} = \delta_{ab}$ and

$$N(x) = \delta t + v_a x^a \,, \quad N^a(x) = \delta x^a + R^a{}_b x^b \,, \tag{3}$$

where $R^a{}_b \equiv \epsilon_b{}^{ac} \varphi_c$, v^a, δt and δx^a are infinitesimal parameters of rotations, boosts and translations. Then the scalar and diffeomorphism constraints can be expressed in terms of the latter, with the respective generators K_a, J_a, P_0 and P_a, namely

$$S[N] = -\delta t \, P_0 - v^a K_a \,, \qquad D[N^a] = -\delta x^b P_b - \varphi^b J_b \,. \tag{4}$$

Substituting the above expressions into the HDA brackets, in the classical case (with $\Omega = 1$) we arrive at the Poincaré algebra. For the deformed case we will remain in the semiclassical regime.

Deformed Poincaré algebra

We may follow the analogous approach if Ω is extracted out of the diffeo constraint. To this end we introduce the “effective signature"

$$s_{\text{eff}} := s\tilde{\Omega} = s\,\frac{D\left[\Omega q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\right]}{D\left[q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\right]}\,, \qquad (5)$$

which allows us to rewrite the third bracket of the deformed HDA as

$$\begin{aligned}\{S[N], S[\tilde{N}]\} &= D\left[s\Omega q^{ab}\left(N\partial_b\tilde{N} - \tilde{N}\partial_b N\right)\right] \\ &= s_{\text{eff}} D\left[q^{ab}\left(N\partial_b\tilde{N} - \tilde{N}\partial_b N\right)\right]. \qquad (6)\end{aligned}$$

As the result we obtain the deformed Poincaré (non-Lie) algebra

$$\begin{aligned}&\{J_a, J_b\} = \epsilon_{abc}J^c, \quad &&\{J_a, K_b\} = \epsilon_{abc}K^c, \quad &&\{K_a, K_b\} = -s_{\text{eff}}\epsilon_{abc}J^c,\\ &\{J_a, P_b\} = \epsilon_{abc}P^c, &&\{J_a, P_0\} = 0\,, &&\{K_a, P_b\} = \delta_{ab}P_0\,,\\ &\{K_a, P_0\} = s_{\text{eff}}P_a\,, &&\{P_a, P_b\} = 0\,, &&\{P_a, P_0\} = 0\,, \qquad (7)\end{aligned}$$

where $\tilde{\Omega}$ in general can be some function of the generators.

Changing signature of the metric

In particular, for a perturbed homogeneous and isotropic spacetime configuration with LQG holonomy corrections the deformation factor is

$$\tilde{\Omega} = \Omega = \cos(2\gamma\bar{\mu}\bar{k}) \cong 1 - 2\frac{\rho}{\rho_c} \in [-1, 1]\,, \tag{8}$$

where $\bar{\mu}$, $\bar{k}$ depend on the Ashtekar variables and γ denotes the Immirzi parameter, while ρ is the universe's energy density and ρ_c its critical value.

The effective spacetime's metric $s_{\text{eff}} = s\Omega$ is Lorentzian for $\rho < \rho_c/2$ but becomes Euclidean for $\rho > \rho_c/2$ and at $\rho = \rho_c/2$ is indefinite. The latter can be interpreted as realizing the asymptotic silence scenario, or BKL conjecture, in cosmology.

Deformation factor and Casimir

To derive the form of $\tilde{\Omega}$ (for $s = 1$) in an exact case we may assume:
• all Jacobi identities for the symmetry algebra are satisfied,
• the deformation factor $\tilde{\Omega}$ is rotationally invariant
• and the deformation vanishes in the appropriate limit.
As the final, more specific assumption we take that $\tilde{\Omega} = \tilde{\Omega}(P_0, |\mathbf{P}|) = F(P_0)G(|\mathbf{P}|)$, $\mathbf{P} \equiv (P_1, P_2, P_3)$. As the result we find

$$\tilde{\Omega}(P_0, |\mathbf{P}|) = \frac{P_0^2 - \alpha}{\mathbf{P}^2 - \alpha}, \tag{9}$$

where $\alpha \in \mathbb{R}$ is a free parameter, with $\lim_{|\alpha| \to \infty} \tilde{\Omega} = 1$ and the signature can change for $\alpha > 0$. Furthermore, combining $\tilde{\Omega}$ and the unit element of the algebra we construct the deformed mass Casimir

$$\mathcal{C} = \frac{-P_0^2 + \mathbf{P}^2}{1 - \alpha^{-1}\mathbf{P}^2}. \tag{10}$$

Phase space with deformed symmetries

Let us consider an extension of our deformed Poincaré algebra by the undeformed Heisenberg algebra of phase space coordinates

$$\{x_\mu, x_\nu\} = 0\,, \qquad \{x_\mu, p_\nu\} = \eta_{\mu\nu}\,, \qquad \{p_\mu, p_\nu\} = 0\,, \qquad (11)$$

where $\mu, \nu = 0, 1, 2, 3$ and the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Such an Ansatz can be implemented by using the following realization of the symmetry generators in terms of x_μ and p_μ :

$$\begin{aligned} \epsilon_{abc} J^c &:= x_a p_b - x_b p_a\,, \qquad & K_a &:= x_a p_0 - x_0 p_a \tilde{\Omega}(p_0, |\mathbf{p}|)\,, \\ P_a &:= p_a\,, \qquad & P_0 &:= p_0\,. \end{aligned} \qquad (12)$$

The remaining brackets of the total phase space algebra are

$$\begin{aligned} \{K_a, x_0\} &= x_a - 2x_0 \frac{p_0 p_a}{p_0^2 - \alpha} \tilde{\Omega}\,, \\ \{K_a, x_b\} &= x_0 \left(\delta_{ab} - 2 \frac{p_a p_b}{\mathbf{p}^2 - \alpha} \right) \tilde{\Omega} \end{aligned} \qquad (13)$$

and all Jacobi identities for the x_μ generators are indeed satisfied.

Deformed Lorentz transformations

The result is standard phase space but equipped with deformed symmetries**. Lorentz transformations preserving the Casimir $\mathcal{C} = \frac{-p_0^2+\mathbf{p}^2}{1-\alpha^{-1}\mathbf{p}^2}$ are naturally deformed as well. For example, the boost with a velocity v in the direction of p_1 acting on a four-momentum $(p_0, \mathbf{p})$ gives

$$\begin{aligned} p_0' &= Q\gamma(p_0 - vp_1)\,, \\ p_1' &= Q\gamma(p_1 - vp_0)\,, \\ p_2' &= Qp_2\,, \qquad p_3' = Qp_3\,, \end{aligned} \tag{14}$$

where $\gamma \equiv \sqrt{1-v^2}^{\,-1}$ denotes the Lorentz factor, while the deformation is contained in the function

$$Q \equiv \sqrt{1 + \frac{\gamma^2 v^2}{\alpha}\left(p_0^2 + p_1^2 - \frac{2}{v}p_0 p_1\right)}^{\,-1}. \tag{15}$$

**Since the full Hopf algebra structure for the symmetry algebra is unknown here, this construction is ambiguous.

Characteristic energy scale

Solving the equation $p_0'(p_0) = p_0$ we discover that for $\alpha > 0$ there exists a boost-invariant energy scale[†] $p_0 = \pm\sqrt{\alpha}$. Consequently, taking any vector $(p_0 = \varepsilon\sqrt{\alpha}, p_1, p_2, p_3)$, $\varepsilon \in (-1, 1)$ we find that

$$p_0' = \frac{\sqrt{\alpha}\,(\varepsilon\sqrt{\alpha} - vp_1)}{\sqrt{(\varepsilon\sqrt{\alpha} - vp_1)^2 + (1 - v^2)(1 - \varepsilon^2)\alpha}} \tag{16}$$

and hence $-\sqrt{\alpha} < p_0' < \sqrt{\alpha}$. It means there are three unconnected momentum sectors, with $p_0 \in (-\infty, -\sqrt{\alpha})$, $p_0 \in (-\sqrt{\alpha}, \sqrt{\alpha})$ or $p_0 \in (\sqrt{\alpha}, \infty)$. Moreover, both $\tilde{\Omega}$ and $\mathcal{C}$ become divergent at $|\mathbf{p}| = \sqrt{\alpha}$, while $\tilde{\Omega}$ changes its sign at $|\mathbf{p}| = \sqrt{\alpha}$ and $|p_0| = \sqrt{\alpha}$.
Meanwhile, for $\alpha < 0$ there is no such a special scale.

[†]Similar to DSR models, especially J. Magueijo & L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).

Measure on momentum space

The standard volume element of momentum space transforms into

$$d^4p' = \det\left(\frac{\partial p'_\mu}{\partial p_\nu}\right) d^4p = Q^6 d^4p\,. \tag{17}$$

In order to find the measure that is boost-invariant we look for a function $f(p)$ satisfying the condition $f(p')d^4p' = f(p)d^4p$ and derive

$$d\mu(p) \equiv f(p)d^4p = \left(1 - \frac{\mathbf{p}^2}{\alpha}\right)^{-3} d^4p\,. \tag{18}$$

On the other hand, introducing a metric on momentum space via the Casimir $\mathcal{C} := g^{\mu\nu}(p)p_\mu p_\nu$ one obtains the non-invariant measure

$$\sqrt{|\det g(p)|}\, d^4p = \left(1 - \frac{\mathbf{p}^2}{\alpha}\right)^2 d^4p\,, \quad g^{\mu\nu}(p) \equiv \frac{\eta^{\mu\nu}}{1-\alpha^{-1}\mathbf{p}^2}\,. \tag{19}$$

Euclidean domain of the model

The Lorentzian and Euclidean versions of our symmetry algebra can be connected by the usual Wick rotation: $P_0 \longrightarrow -iP_0$, $K_a \longrightarrow -iK_a$. The only brackets of the algebra that become (implicitly) modified are

$$\{K_a, K_b\} = -s_{\mathrm{eff}}\epsilon_{abc}J^c, \qquad \{K_a, P_0\} = s_{\mathrm{eff}}P_a\,, \tag{20}$$

where we already set $s = -1$ and now $s_{\mathrm{eff}} = s\tilde{\Omega}^E$, $\tilde{\Omega}^E = -\frac{P_0^2+\alpha}{\mathbf{P}^2-\alpha}$, $\lim_{|\alpha|\to\infty}\tilde{\Omega}^E = 1$. Similarly, the Euclidean Casimir in the momentum representation is obtained via $p_0 \longrightarrow -ip_0$, which gives

$$\mathcal{C}^E = \frac{p_0^2 + \mathbf{p}^2}{1 - \alpha^{-1}\mathbf{p}^2}\,. \tag{21}$$

If $\alpha > 0$, it is negative for $|\mathbf{p}| > \sqrt{\alpha}$. The same $\tilde{\Omega}^E$, $\mathcal{C}^E$ can also be derived starting from the deformed Euclidean symmetry algebra.

Diffusion and the dimension of space(time)

On a manifold of d topological dimensions and with the Riemannian metric g one considers a diffusion process described by the equation

$$\frac{\partial}{\partial\sigma}\mathcal{K}(x,x_0;\sigma) = \Delta_x\mathcal{K}(x,x_0;\sigma)\,, \quad \mathcal{K}(x,x_0;0) = \frac{\delta^{(d)}(x-x_0)}{\sqrt{|\det g(x)|}}\,, \qquad (22)$$

with the Laplacian Δ_x (not necessarily $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$) and auxiliary time σ. The solution of (22) can be written as the Fourier transform

$$\mathcal{K}(x,x_0;\sigma) = (2\pi)^{-4}\int d\mu(p)\, e^{ip_\mu(x-x_0)^\mu}\, e^{-\sigma\Delta_p}\,. \qquad (23)$$

For a flat g the trace of $\mathcal{K}(x,x_0;\sigma)$ is the average return probability

$$\mathcal{P}(\sigma) \equiv \mathcal{K}(x,x;\sigma) = (2\pi)^{-4}\int d\mu(p)\, e^{-\sigma\Delta_p} \qquad (24)$$

and the spectral dimension of the manifold is defined as

$$d_S(\sigma) := -2\frac{\partial\log\mathcal{P}(\sigma)}{\partial\log\sigma}\,. \qquad (25)$$

Spacetime's dimension in our model: $\alpha > 0$

In general, the Laplacian on momentum space may be given by

$$\Delta_p = \sum_{n=1} c_n \alpha^{n-1} \left(\mathcal{C}^E\right)^n , \tag{26}$$

with some coefficients c_n , but we assume the simple $\Delta_p := \mathcal{C}^E$. Using the measure $d\mu(p)$ we then write the average return probability[‡]

$$\mathcal{P}(\sigma) = \frac{4\pi}{(2\pi)^4} \int \frac{dp\, p^2}{(1-\frac{p^2}{\alpha})^3} \int dp_0\, e^{-\sigma \frac{p_0^2+p^2}{1-\alpha^{-1}p^2}} . \tag{27}$$

In the case of $\alpha > 0$ the integration range of 3-momenta has to become $[0, \sqrt{\alpha}) \ni |\mathbf{p}| \equiv p$ (so that Δ_p is positive). Then we calculate that $\mathcal{P}(\sigma) = \frac{1}{16\pi^2\sigma^2}$ and the spectral dimension of spacetime is constant

$$d_S(\sigma) = 4 . \tag{28}$$

[‡]It may be compared with the case of noncommutative κ-Minkowski space, see M. Arzano & **T. T.**, Phys. Rev. D **89**, 124024 (2014).

Spacetime's dimension in our model: $\alpha < 0$

On the other hand, for $\alpha < 0$ we obtain the dimensional reduction:

$$\mathcal{P}(\sigma) = \frac{\mathrm{erf}(\sqrt{|\alpha|\sigma})}{16\pi^2\sigma^2} - \frac{\sqrt{|\alpha|}\, e^{-|\alpha|\sigma}}{8\pi^{5/2}\sigma^{3/2}}\,, \tag{29}$$

$$d_S(\sigma) = 4 - \frac{4(|\alpha|\sigma)^{\frac{3}{2}}}{\sqrt{\pi}\, e^{|\alpha|\sigma}\mathrm{erf}(\sqrt{|\alpha|\sigma}) - 2\sqrt{|\alpha|\sigma}}\,. \tag{30}$$

In particular, in the IR $\lim_{\sigma\to\infty} d_S(\sigma) = 4$, while in the UV $\lim_{\sigma\to 0} d_S(\sigma) = 1$.

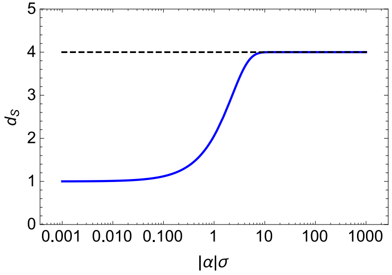

Figure: Spectral dimension (30) as a function of $|\alpha|\sigma$

The Carrollian limit at high energies

The UV value of $d_S(\sigma)$ agrees with that the symmetry algebra becomes the Carroll[§] algebra at $|\mathbf{p}| \to \infty$ (for $\alpha < 0$). The Carrollian (or ultralocal) limit is defined as vanishing speed of light and for spacetime it leads to the collapse of lightcones into a congruence of null worldlines. In contrast to the asymptotic silence scenario, here it happens not in the early universe but at the smallest scales.

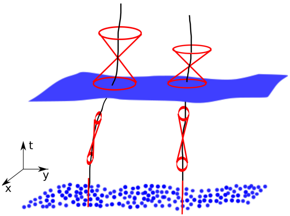

Figure: The asymptotic silence scenario

[§] L. Carroll, *Through the Looking Glass and what Alice Found There*: "Now, here, you see, it takes all the running you can do, to keep in the same place."

Summary

- LQG-deformed HDA in the linear limit reduces to a certain deformed Poincaré algebra
- Using some assumptions we obtain a model example of the deformation, which exhibits the variable signature
- Such a deformed symmetry algebra can be tentatively attached to commutative phase space
- It leads to deformed Lorentz transformations, invariant energy scale and nontrivial measure on momentum space
- Finally, we calculate the spectral dimension of spacetime
- For $\alpha > 0$ the dimension stays unchanged, while for $\alpha < 0$ it reduces to 1 in the UV, corresponding to the Carrollian limit