Perturbation theory in loop quantum cosmology with separate universes

Edward Wilson-Ewing

Albert Einstein Institute
Max Planck Institute for Gravitational Physics

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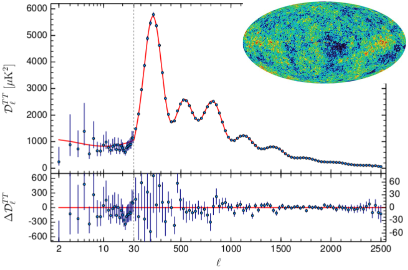
Motivation

High precision observations of the cosmic microwave background (CMB) have taught us a lot about the early universe. In particular, they have ruled out a number of cosmological models, including some of the simplest models for inflation, as well as alternatives to inflation.



[Planck2015+BICEP2/Keck]

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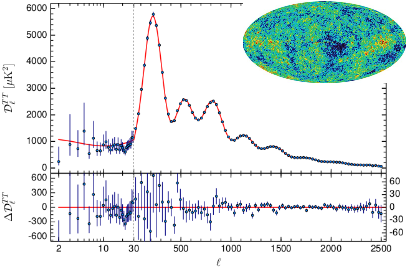

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Can they teach us anything about quantum gravity effects in the early universe? More specifically, can we test loop quantum cosmology (LQC)? To do this, it is necessary to develop a framework for cosmological perturbation theory in LQC.

Outline

1. Standard Cosmological Perturbation Theory
2. Separate Universe Framework
3. Effective Equations for Long-Wavelength Modes
4. Results and Outlook

The Standard Hamiltonian Treatment

In the usual treatment of linear perturbations in cosmology, one treats the background and the perturbations separately.

The background is typically taken to be a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, with

$$\mathrm{d}s^2 = a(\eta)^2 \left[-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right],$$

and a matter content consisting of perfect fluids with energy density ρ, pressure P and sound speed c_s and/or scalar fields ϕ.

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In the Hamiltonian framework, the symplectic structure splits nicely into a background part for the background variables and a perturbation part for the variables describing the perturbations around the background.

Then, there is a Hamiltonian constraint H_o for the background and a separate Hamiltonian δH for the perturbations.

The Mukhanov-Sasaki Equation

The dynamics for scalar perturbations in general relativity (those mainly responsible for the temperature anisotropies in the CMB) are given by the Mukhanov-Sasaki equation,

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0, \qquad f' := \frac{\mathrm{d}f}{\mathrm{d}\eta},$$

where the Fourier modes of the Mukhanov-Sasaki variable v_k are related to the co-moving curvature perturbation $\mathcal{R}_k$ by $v_k = z\mathcal{R}_k$, and

$$z = \frac{a^2\sqrt{\rho + P}}{c_s \mathcal{H}}, \qquad \mathcal{H} = \frac{a'}{a},$$

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depends on the dynamics of the background space-time.

Importantly, in cosmology it is almost always a good approximation to entirely neglect either k^2 or z''/z.

What About Loop Quantum Cosmology?

In loop quantum cosmology (LQC), we study cosmological space-times—like the FLRW and Bianchi models—following loop quantum gravity as closely as possible.

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But now if we want to follow the standard treatment of linear perturbations in LQC, we must perform a loop quantization of the Hamiltonian for the perturbations δH, which contains terms corresponding to perturbations in the connection δA_a^i.

How can you build a holonomy out of a perturbation of a connection? The whole connection is necessary for this...

The Separate Universe Framework

For long-wavelength modes—Fourier modes that satisfy $k^2 \ll z''/z$—gradients can entirely be neglected. So, if one splits the space-time into patches with size $\sim \lambda_{long}$, then the dynamics of the perturbation in that patch are entirely determined by the background dynamics: interactions between patches are negligible [Salopek, Bond; Wands, Malik, Lyth, Liddle; ...].

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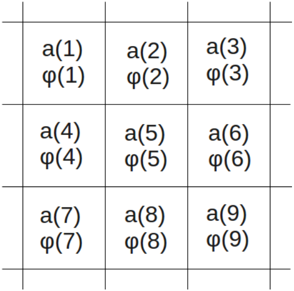

Furthermore, it is reasonable to approximate each of these patches to be homogeneous. This corresponds to a discretization of the space-time with a lattice spacing of $\sim \lambda_{long}$.

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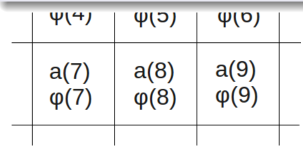
Application to LQC:

So, to study long-wavelength perturbations, it is sufficient to consider a collection of uninteracting homogeneous patches. Importantly, since each patch is homogeneous, the standard LQC quantization techniques for homogeneous space-times can safely be used in each patch.



discretization of the space-time with a lattice spacing of $\sim \lambda_{long}$.

Aside: Other Approaches to Perturbations in LQC

There are two other approaches to study perturbations in LQC:

- **Effective constraints:** Start with the classical constraints for cosmological perturbations, and add in 'correction functions' encoding holonomy or inverse triad corrections, while requiring an anomaly-free constraint algebra [Bojowald, Kagan, Hussein, Shankaranarayanan; Cailleteau, Mielczarek, Barrau, Grain; Ben Achour, Brahma, Grain, Marciano; ...].
- **Hybrid quantization:** Do a loop quantization of the background degrees of freedom, and a Fock quantization of the perturbations [Fernandez-Mendez, Mena Marugan, Olmedo; Agullo, Ashtekar, Nelson; ...].

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The three approaches each try to bypass the difficulties of performing a loop quantization of all inhomogeneous degrees of freedom. Note that while these other two approaches can be used for both short- and long-wavelength perturbations, neither provides a loop quantization of the perturbations.

Choice of Gauge and Variables

For scalar perturbations, in the longitudinal gauge (assuming the matter field has zero anisotropic stress), the metric is

$$\mathrm{d}s^2 = -a^2(1+2\psi)\mathrm{d}\eta^2 + a^2(1-2\psi)\mathrm{d}\vec{x}^2,$$

where ψ encodes the perturbations in the lapse and the scale factor.

Breaking the spatial manifold into large patches n which are each approximated to be homogeneous, following the separate universe approach, in this gauge the metric in each patch is that of a spatially flat FLRW space-time, with

$$a_n = a(1-\psi), \qquad N_n = a(1+\psi).$$

This greatly simplifies the Hamiltonian treatment and the loop quantization.

The Variables in Each Patch

Since each patch is a spatially flat FLRW space-time, the basic variables in each patch are, as usual in LQC,

$$(E_i^a)_n = p_n \left(\frac{\partial}{\partial x^i}\right)^a, \qquad \left(A_a^i\right)_n = c_n \left(\mathrm{d}x^i\right)_a,$$

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and so $a_n := \sqrt{p_n}$. The mean scale factor is

$$a = \frac{1}{n_{tot}} \sum_{n=1}^{n_{tot}} a_n, \qquad \Rightarrow \qquad \psi_n = \frac{a - a_n}{a},$$

and it follows that

$$N_n = a(1 + \psi_n) = 2a - a_n.$$

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From this discussion, it is clear that the usual LQC quantization in each patch is possible, the only modification being a different lapse function.

Loop Quantization

The Hilbert space H is given by the tensor product of the Hilbert spaces for each cell n in the discretization,

$$H = \bigotimes_n H_n^{(LQC)},$$

where $H_n^{(LQC)}$ is the usual LQC Hilbert space for FLRW space-times.

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The next step is to impose the constraints. Given that the patches do not interact in the separate universe approximation, this is simply obtained by requiring

$$\widehat{(\mathcal{C}_S)_n}\Psi(p_i) = 0,$$

in each patch. Here $\widehat{\mathcal{C}_S}$ is the usual scalar constraint operator of LQC for a spatially flat FLRW space-time.

Quantum Dynamics

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Finally, for small perturbations the different patches should be 'similar' to each other: for non-constraint operators $\widehat{\mathcal{O}}_n$,

$$\left| \langle \widehat{\mathcal{O}_i} \rangle - \langle \widehat{\mathcal{O}_j} \rangle \right| \ll \left| \frac{1}{n_{tot}} \sum_n \langle \widehat{\mathcal{O}_n} \rangle \right| .$$

The Quantum Theory and Effective Dynamics

Following these steps gives a loop quantization for the background and long-wavelength perturbative degrees of freedom. However, the constraints and quantum dynamics are cumbersome and difficult to work with.

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On the other hand, for states in homogeneous LQC that are sharply-peaked, there exist effective equations that provide an excellent approximation to the full quantum dynamics so long as the spatial volume is much larger than $\ell_{\rm Pl}^3$ [Ashtekar, Pawłowski, Singh; Taveras; Rovelli, WE; ...]

The same is true patch by patch in the separate universe framework: so long as the volume of each patch remains much larger than $\ell_{\rm Pl}^3$, sharply-peaked states will remain sharply-peaked and the effective equations in each patch can be trusted.

Effective Equations

The effective equations in each patch are the usual LQC Friedmann equations in conformal time,

$$\mathcal{H}_n^2 = \frac{8\pi G}{3} N_n^2 \rho_n \left(1 - \frac{\rho_n}{\rho_c}\right),$$

$$\mathcal{H}_n' = -4\pi G N_n^2 (\rho_n + P_n) \left(1 - \frac{2\rho_n}{\rho_c}\right) + \frac{N_n' a_n'}{N_n a_n},$$

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Expressing the above equations in terms of a, $\bar{\rho}$ and $\bar{P}$ and perturbations away from the average values, the background variables clearly follow the usual LQC effective dynamics.

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In the longitudinal gauge, (assuming a scalar field as matter content) the Mukhanov-Sasaki variable v has the form

$$v_n = z\psi_n + a\,\delta\phi_n,$$

and with some work, the perturbative part of the effective equations above can be shown to imply that

$$v_n'' - \frac{z''}{z} v_n = 0.$$

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While the equation's form is the same as in general relativity, the dynamics of z will be different in LQC.

Evolution Through the LQC Bounce

Using the LQC effective long-wavelength Mukhanov-Sasaki equation, it is possible to calculate the evolution of long-wavelength scalar perturbations through the LQC bounce.

Assuming a constant equation of state ω in the matter field for simplicity, before the bounce when GR holds (recall $\mathcal{R} = v/z$)

$$\mathcal{R}_k = A_k + B_k |t|^{(\omega-1)/(1+\omega)},$$

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there is a constant and a growing mode (assuming $-1 < \omega < 1$).

Then, as a result of the LQC dynamics, after the bounce

$$\mathcal{R}_k = A_k - B_k \alpha^{(1-\omega)/2(1+\omega)} \left(\frac{\omega - 1}{1+\omega}\right) \frac{\sqrt{\pi}\,\Gamma\left(\frac{2+\omega}{1+\omega} - \frac{3}{2}\right)}{2\,\Gamma\left(\frac{2+\omega}{1+\omega}\right)} + \text{decay},$$

where $\alpha = 6\pi G \rho_c (1+\omega)^2$.

Application to Cosmological Models

This result is particularly interesting when considering alternatives to inflation in LQC, like the matter bounce scenario and the ekpyrotic universe, where scale-invariant perturbations are generated in a contracting pre-bounce epoch: it shows precisely how these perturbations propagate to the post-bounce era.

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In particular, this result shows that both scenarios—combined with an LQC bounce—are viable alternatives to inflation, although these two alternatives are now quite strongly constrained by observational bounds on non-Gaussianities.

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This framework could also be used to study the behaviour of long-wavelength modes during the bounce in inflationary models, which would correspond to super-horizon perturbations today.

Comments on Coarse-Graining & Renormalization

If one is interested in physics at very large scales, then it is possible to consider coarse-graining patches, for example coarse-graining

$$8 \text{ patches} \rightarrow 1 \text{ large patch.}$$

This is similar to the block-spin renormalization procedure used in, e.g., the Ising model. Is there any running of coupling constants?

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Also, note that here the RG flow occurs with respect to a length scale, not an energy scale. Is this an artifact of the setting, or should we expect this more generally for quantum gravity?

Conclusions

- The separate universe framework in LQC provides a loop quantization of long-wavelength scalar perturbations;
- The effective equations allow us to calculate the dynamics of long-wavelength perturbations through the bounce;
- This is particularly important for alternatives to inflation, like the matter bounce scenario and the ekpyrotic universe, where scale-invariant fluctuations are generated in a contracting pre-bounce epoch, and could also be used to study super-horizon modes in inflation.

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Thank you for your attention!