Reduced quantisation versus quantum reduction for spherical symmetry

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Motivation - Black hole in LQG

Traditional approach
- Isolated horizons
- Chern-Simons on boundary
- Gas of punctures
  [Ashtekar, Gambini, Kuchar, Perez, Pullin, ...]

Recent approaches
- Spherical symmetric QG
  [Gambini, Pullin, Olemedo, ...]
- GFT condensates
  [Oriti, Pranzetti, Sindoni '15]
  ....

Open questions
- Fluctuations around symmetry reduced sector
- State of a black hole in Loop Quantum Gravity
  ....
Motivation - What we are doing

- Midisuperspace model
- Quantum reduced model
- GR in the radial gauge
Motivation - What we are doing

Midisuperspace model

Quantum reduced model

Phase space $R, \ldots$

GR in the radial gauge

Phase space $q, \ldots$
Motivation - What we are doing

Midisuperspace model

Phase space $R, \cdots$

Quantise

Quantum algebra $\hat{R}, \cdots$

Reduce

GR in the radial gauge

Quantum reduced model

Phase space $q, \cdots$

Quantise

Quantum algebra $\hat{q}, \cdots$
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GR in the radial gauge

Quantum reduced model

Phase space
$q, \cdots$

Phase space functions
$R(q, \cdots), \cdots$

Identify

Quantise

Quantum algebra
$\hat{q}, \cdots$
Motivation - What we are doing

Midisuperspace model

Quantum reduced model

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GR in the radial gauge

Phase space $q, \cdots$

Phase space functions $R(q, \cdots), \cdots$

Identify

Quantise

Quantum algebra $\hat{q}, \cdots$

Operators $\hat{R}, \cdots$

Quantise and reduce
Motivation - What we are doing

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Phase space $q, \cdots$

Phase space functions $R(q, \cdots), \cdots$

Quantise

Quantum algebra $\hat{q}, \cdots$

Operators $\hat{R}, \cdots$

Reduce

Compare

Quantise and reduce
Core assumption
The midisuperspace model is an effective model of the finer quantum reduced one, the latter of which captures more quantum degrees of freedom.

We do not aim at...
... a mathematical exact embedding of the midisuperspace model into the quantum reduced model. Such embedding might even be not possible.

What we find
- Qualitative match in sub-sector of quantum reduced model
- But: Identification of sub-sector needs to be refined
Plan of the talk

1. Classical Preparation
   - The radial gauge - full system
   - The symmetry reduced system
   - Relation of full and reduced system

2. Quantum theory
   - Quantisation of the reduced system
   - Quantisation of the full system
   - Relation of full and reduced system

3. Summary and outlook
Classical Background - Radial Gauge

[Duch, Kaminski, Lewandowski, Świeżewski '14], [Bodendorfer, Lewandowski, Świeżewski '15]

Step I: Choose adapted coordinates and gauge fix spherical diffeomorphisms

\[
q_{ab} = \begin{pmatrix}
q_{rr} & q_{rA} \\
q_{rA} & q_{AB}
\end{pmatrix}
\xrightarrow{\text{gauge fix } C^A}

q_{ab} = \begin{pmatrix}
q_{rr} & 0 \\
0 & q_{AB}
\end{pmatrix}
\]

$q_{ab}$ 3-metric, $A = 1, 2$ label angular coordinates, $C^A$ generator of spherical diffeomorphisms

Reduced phase space

\[
\{q_{rr}, P_{rr}\} = \delta \cdots \text{ and } \{q_{AB}, P^{AB}\} = \delta \cdots
\]

Step II: Solve spherical diffeomorphisms constraint $C^A$ for momenta $P^{rA}$

\[
C^A = 0 \quad \Rightarrow \quad P^{rA}(q_{AB}, P^{AB}, q_{rr}, P_{rr})
\]

Reduction to spherical symmetry

\[
P^{rA} = 0 \quad \Leftrightarrow \quad \text{spherical diffeomorphisms preserving sphere } S_r^2
\]
Classical Background - Connection formulation I

Canonical transformation

\[ q_{rr} \rightarrow \Lambda := \sqrt{q_{rr}} \quad \text{and} \quad P^{rr} \rightarrow P^{\Lambda} \]
\[ q_{AB} \rightarrow \tilde{q}_{AB} := \Lambda^2 q_{AB} \quad \text{and} \quad P^{AB} \rightarrow \tilde{P}^{AB} := \Lambda^{-2} P^{AB} \]

Partial gauge fixing of radial diffeomorphisms

\[ \Lambda = \Lambda(r) \sim P_{\Lambda}(r) = \int_{S^2_r} P_{\Lambda}(r, \Theta) \]

Constraints

- Hamiltonian
- Radial diffeomorphisms \{ not considered \}
- Reduction constraint \( P^{rA} = 0 \)
\( \tilde{q}_{AB}, \tilde{P}^{AB} \) can now be replaced by triads and SU(2)-connections.

**Triads**

\[ E^A_i E^B_j := (\text{det } \tilde{q}) \tilde{q}^{AB} \]

**SU(2)-connections**

\[ A^A_j := \tilde{K}^A_j + \tilde{\Gamma}^A_j \]

“curvature” of \( \tilde{q} \)

metric connection

**Trap:** \( \tilde{K} \) is not the extrinsic curvature of the sphere.

+ Gauss-constraint on \( S^2 \)
Midisuperspace model

Quantum reduced model

Phase space $R, \cdots$

Quantise

Quantum algebra $\hat{R}, \cdots$

Compare

Identify

Phase space functions $R(q, \cdots), \cdots$

Quantise

Quantum algebra $\hat{q}, \cdots$

GR in the radial gauge

Reduce

Operators $\hat{R}, \cdots$

Quantise and reduce

Motivation - What we are doing
Classical Background - Spherical symmetric model

Spherical symmetric metric

\[ q_{AB} = R^2(r)\Omega_{AB} \]

\( \Omega_{AB} \) standard metric on \( S^2 \)

Symplectic structure

\[ \{ \Lambda, P_\Lambda \} = \delta \quad \text{and} \quad \{ R, P_R \} = \delta \]

Constraints

- radial diffeomorphism
- Hamiltonian

\[ \text{not considered} \]
Motivation - What we are doing

Midisuperspace model

- ✔ Phase space $R, \cdots$
- Quantise
- Quantum algebra $\hat{R}, \cdots$
- Reduce

Quantum reduced model

- Phase space $q, \cdots$
- Phase space functions $R(q, \cdots), \cdots$
- Identify?
- Quantise
- Quantum algebra $\hat{q}, \cdots$
- Operators $\hat{R}, \cdots$
- Quantise and reduce

GR in the radial gauge
Classical Background - Relation to full theory

- $\Lambda, P_\Lambda$ are defined as in full theory
- Canonical Transformation: $R \rightarrow \tilde{R}$, $P_R \rightarrow \tilde{P}_R$
- $\tilde{R}, \tilde{P}_R$ related to Area($S^2$) and scalar “curvature” $\tilde{K} = \tilde{K}_A E^A$

\[
\tilde{R}^2(r) = \frac{1}{4\pi} \int_{S^2} \sqrt{\det \tilde{q}_{AB}}
\]

and

\[
\tilde{P}_R(r) = -\frac{1}{\tilde{R}(r)} \int_{S^2} \sqrt{\det \tilde{q}_{AB}} \tilde{K}
\]

Trap: $\tilde{K}$ is not the extrinsic curvature of the sphere
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3. Summary and outlook
Quantisation of the reduced system

Point holonomies

\[ h_\rho^P(P_\Lambda) = e^{-i\rho P_\Lambda} \]

\( \rho \in \mathbb{R}^+ \)

“Fluxes”

\[ \Lambda(e_r) := \int_{r_1}^{r_2} \Lambda(r) \quad \text{with} \quad e_r = [r_1, r_2] \]

“Spin networks”

\[ |\rho_1, \ldots, \rho_n\rangle = h_{r_1}^{\rho_1} \cdots h_{r_n}^{\rho_n} \]

Operators

\( \hat{h}^\rho \) acts by multiplication:

\[ h^\rho' |\rho\rangle = |\rho + \rho'\rangle \]

\( \hat{\Lambda} \) acts by derivation:

\[ \hat{\Lambda} |\rho\rangle = \rho |\rho\rangle \]
Quantisation of the reduced system

Point holonomies

\[ h_ρ^ρ(\tilde{P}_R) = e^{-i ρ \tilde{P}_R} \quad ρ \in \mathbb{R}^+ \]

“Fluxes”

\[ \tilde{R}(e_r) := \int_{r_1}^{r_2} \tilde{R}(r) \quad \text{with} \quad e_r = [r_1, r_2] \]

“Spin networks”

\[ |ρ_1, \ldots, ρ_n⟩ = h_{ρ_1}^{ρ_1} \cdots h_{ρ_n}^{ρ_n} \]

Operators

\[ \hat{h}^ρ \text{ acts by multiplication:} \quad h^{ρ'} |ρ⟩ = |ρ + ρ'⟩ \]

\[ \hat{R} \text{ acts by derivation:} \quad \hat{R} |ρ⟩ = ρ |ρ⟩ \]
Quantisation of the full system

- $(\Lambda, P_{\Lambda})$ quantised as in reduced theory
- $(A^i_A, E^A_i)$ usual loop-quantisation
- Holonomies $h(A)$ along edges in $S^2$
- Fluxes $E(S)$, where the surface $S$ intersects sphere in a line
- Scalar product as usual

**Reduction $Pr^A = 0$**

Equivalent to imposing spherical diffeomorphisms
$
\leadsto \text{in the same way as diffeo’s are imposed in 3+1}
$
Motivation - What we are doing

Midisuperspace model

- Phase space $R, \cdots$
  - Quantise
  - Quantum algebra $\hat{R}, \cdots$
  - GR in the radial gauge
  - Reduce
- Identify

Quantum reduced model

- Phase space $q, \cdots$
  - Quantise
  - Quantum algebra $\hat{q}, \cdots$
  - Operators
  - Quantise and reduce?
- Phase space functions $R(q, \cdots), \cdots$
  - Identify
  - Compare
  - Quantise
Reduced operators in full theory - $\tilde{R}$

Recall: $\tilde{R}^2(r) = \frac{1}{4\pi} \int_{S^2} \sqrt{\text{det} \tilde{q}_{AB}}$

Can be quantised in the same way as volume $V^{(2d)}$ in 2+1

- Rewrite $\text{det} \tilde{q}_{AB}$ in terms of triads
- Choose appropriate regularisation

**Trap:** $V^{(2d)}$ is not the 2d-Volume of the sphere

Simplest non trivial action

$$\tilde{R} \hat{j} \hat{j} = \sqrt{V^{(2d)}} \hat{j} \hat{j} \approx \sqrt{j} \hat{j} \hat{j}$$
Reduced operators in full theory - $\tilde{P}_R$

Classical expression

$$\tilde{P}_R = -\frac{1}{R} \int_{S^2} \sqrt{\det \tilde{q}} \, \tilde{K}$$

$\tilde{K}$ scalar “curvature” of $\tilde{q}$, not the geometric curvature!

Thiemann Trick - define “Hamiltonian” $H^{(2d)}$ s.t.

$$\sqrt{\det \tilde{q}} \, \tilde{K} \propto \{H^{(2d)}, V^{(2d)}\}$$

Trap: $H^{(2d)}$ is not the physical Hamiltonian

Another Thiemann Trick

$$\tilde{P}_R \propto \{H^{(2d)}, \sqrt{V^{(2d)}}\}$$
Reduced operators in full theory - $\tilde{\mathcal{P}}_R$

- Regularisation of $H^{(2d)}$ along the lines of [Thiemann]
- But here: graph-preserving
- Curvature $F$ of $A$ in $H^{(2d)}$ changes representation on links
- Evaluation via graphical calculus [Alesci, Thiemann, AZ'13]

Simplest non trivial action

$$\tilde{\mathcal{P}}_R \chi_j = \cdots \chi_{j+1/2} + \cdots \chi_{j-1/2}$$
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- Phase space $R, \cdots$
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- Quantise
- Identify $\checkmark$
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- Reduce

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- Phase space $q, \cdots$
- Phase space functions $R(q, \cdots), \cdots$
- Operators $\hat{q}, \cdots$
- Quantum algebra $\hat{R}, \cdots$
- Quantise
- Quantise and reduce
- Compare $\,$?
- Identify $\checkmark$
# Comparison

**Sub sector - kink space**

\[ |j\rangle = \begin{array}{c} \text{(Diagram)} \end{array} \]

**Note:** \( \tilde{P}_R \) preserves kink space only if the regularisation is graph preserving

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<table>
<thead>
<tr>
<th></th>
<th>Midisuperspace model</th>
<th>Kink space</th>
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<tbody>
<tr>
<td>states on ( S^2 )</td>
<td>(</td>
<td>\rho\rangle )</td>
</tr>
<tr>
<td>Action of ( \tilde{R} )</td>
<td>( \tilde{R}</td>
<td>\rho\rangle = \rho</td>
</tr>
<tr>
<td>Action of ( \tilde{P}_R )</td>
<td>( \langle \rho \pm \mu</td>
<td>\tilde{P}_R</td>
</tr>
<tr>
<td>( [\tilde{P}_R, \tilde{R}] )</td>
<td>( \langle \rho \pm \mu</td>
<td>[\tilde{P}_R, \tilde{R}]</td>
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</table>
Comments

We found

- Qualitative match for $\mu = 1/2$ and $\rho \leftrightarrow j$ on the kinematical level between midisuperspace model and quantum reduced model in kink sector

Midisuperspace model can be seen as effective theory of the full model

But

- Quantitative mismatch in commutators
  $\rightarrow$ strongly dependent on choice of regularisation and choice of state
- Different scaling behaviours of spectra of $\tilde{R}$ and $\tilde{P}_R$
  $\rightarrow$ strongly dependent on choice of state

We need to improve identification of the sub-sector

$\rightsquigarrow$ coarse-graining?
Summary and outlook

Core assumption
The classical reduced model is an effective theory of the quantum reduced.

What we found
- Qualitative match in sub-sector of quantum reduced model
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Outlook
Can we construct a coarse graining procedure such that the effective action of the coarse-grained full theory operators on the coarsest possible state matches with the action in the midisuperspace model?
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Thank you!

Have fun on the slopes!